On J-Self-Adjoint Operators with $\mathcal{C}$-Symmetries: Extension Theory Approach.

Sergii Kuzhel

A linear densely defined operator A acting in a Krein space $(\mathfrak{H}, [\cdot,\cdot]_J)$ with fundamental symmetry J and indefinite metric $[\cdot,\cdot]_J = (J\cdot,\cdot)$ is called *J-self-adjoint* if $A^*J = JA$.

In contrast to self-adjoint operators in Hilbert spaces (which necessarily have a purely real spectrum), J-self-adjoint operators, in general, have a spectrum which is only symmetric with respect to the real axis. However, one can ensure the reality of spectrum by imposing an extra condition of symmetry. In particular, a J-self-adjoint operator A has the property of *$\mathcal{C}$-symmetry* if there exists a bounded linear operator $\mathcal{C}$ in $\mathfrak{H}$ such that: (i) $\mathcal{C}^2 = I$; (ii) $J\mathcal{C} > 0$; (iii) $A\mathcal{C} = \mathcal{C}A$.

The investigation of J-self-adjoint operators with $\mathcal{C}$-symmetries is strongly motivated by an intensive development of $\mathcal{PT}$-symmetric quantum mechanics achieved in the last decade.

The properties of $\mathcal{C}$ are nearly identical to those of the charge conjugation operator in quantum field theory and the existence of $\mathcal{C}$ provides an inner product $(\cdot,\cdot)_{\mathcal{C}} = [\mathcal{C}\cdot,\cdot]_J$ whose associated norm is positive definite and the dynamics generated by A is therefore governed by a unitary time evolution. However, the operator $\mathcal{C}$ depends on the choice of A and its finding is a nontrivial problem.

The report deals with the construction of $\mathcal{C}$-symmetries for J-self-adjoint extensions of a symmetric operator A_{sym} with finite deficiency indices $< n, n >$. We present a general method allowing us: (i) to describe the set of J-self-adjoint extensions A of A_{sym} with $\mathcal{C}$-symmetries; (ii) to construct the corresponding $\mathcal{C}$-symmetries in a simple explicit form which is closely related to the Clifford algebra structures; (iii) to establish a Krein-type resolvent formula for J-self-adjoint extensions A with $\mathcal{C}$-symmetries.

The results are exemplified on 1D pseudo-Hermitian Schrödinger and Dirac Hamiltonians with complex point-interaction potentials.