NEW METHOD of STUDYING STABILITY of LURE' SYSTEM under PARABOLIC REGULARITY

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1 Introduction

1.1 Motivating example: $\mathfrak{RC}$–transmission line

An unloaded electric $\mathfrak{RC}$ – transmission line (or W. Thomson's cable of finite length) is used in modeling of humidity sensors (Weremczuk et al, 2012), coaxial cables up to 1 MHz, connecting wires in the MOS integrated circuits, carbon nanotubes (Esen et al, 2007) or organic semiconductors (Lenski et al, 2009).

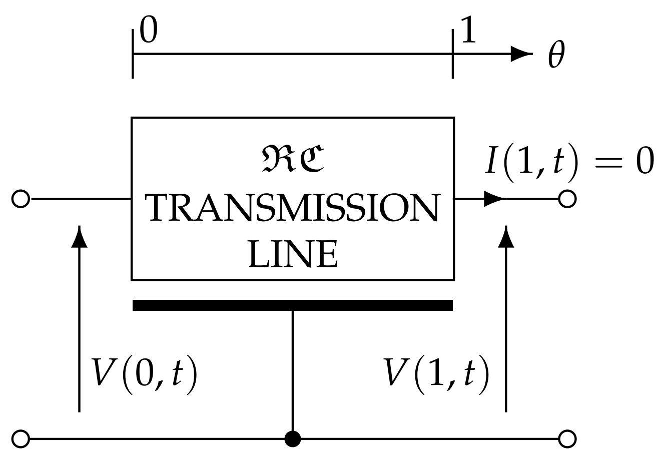

Figure 1.1: Unloaded $\mathfrak{RC}$–transmission line

The system is governed by the partial differential equations

$$\left\{\begin{array}{rcll} \cancelto{0}{\mathfrak{L}}\, I_t(\theta,t) & = & -V_\theta(\theta,t) - \mathfrak{R} I(\theta,t), & t \geq 0, \quad 0 \leq \theta \leq 1 \\ \mathfrak{C} V_t(\theta,t) & = & -I_\theta(\theta,t) - \cancelto{0}{\mathfrak{G}}\, V(\theta,t), & t \geq 0, \quad 0 \leq \theta \leq 1 \\ I(1,t) & = & 0, & t \geq 0 \\ u(t) & = & V(0,t), & t \geq 0 \\ y(t) & = & V(1,t), & t \geq 0 \end{array}\right\}.$$

Time rescaling $x(\theta,t) = v(\theta, \mathfrak{R}\mathfrak{C}t)$ reduces the dynamics to the form:

$$\left\{\begin{array}{rcll} x_t(\theta,t) & = & x_{\theta\theta}(\theta,t) & t \geq 0, \quad 0 \leq \theta \leq 1 \\ x_\theta(1,t) & = & 0, & t \geq 0 \\ u(t) & = & x(0,t), & t \geq 0 \\ y(t) & = & x(1,t), & t \geq 0 \end{array}\right\}. \tag{1.1}$$

In the Hilbert space $\mathrm{H} = \mathrm{L}^2(0,1)$ with standard scalar product, the dynamics (1.1) can be written in the preliminary abstract form

$$\left\{ \begin{array}{rcl} \dot{x} & = & \sigma x \\ \tau x & = & u \\ y & = & c^{\#} x \end{array} \right\}$$

where

$$\sigma x = x'', \qquad D(\sigma) = \left\{ x \in \mathrm{H}^2(0,1) : \; x'(1) = 0 \right\} \; ,$$
$$\tau x = x(0), \qquad D(\tau) = \mathrm{C}[0,1] \supset D(\sigma)$$

and σ is a closed linear operator;

$$c^{\#} x = x(1), \qquad D(c^{\#}) = \mathrm{C}[0,1] \; . \tag{1.2}$$

To obtain the final model of boundary control

$$\left\{ \begin{array}{rcl} \dot{x}(t) & = & \mathcal{A}[x(t) + du(t)] \\ y(t) & = & c^{\#}x(t) \end{array} \right\} \ . \tag{1.3}$$

we take $\mathcal{A} = \sigma|_{\ker \tau}$ and find the factor control vector $d \in D(\sigma)$ satisfying $\sigma d = 0$, $\tau d = -1$. The idea is then that

$$\dot{x}(t) = \sigma x(t) + \sigma du(t) = \sigma[x(t) + du(t)]$$

where, with $x(t)$ and d necessarily in $D(\sigma)$, $[x(t) + du(t)] \in D(\mathcal{A})$ because

$$\tau[x(t) + du(t)] = \tau x(t) + \tau du(t) = \tau x(t) - u(t) = 0 \ .$$

Hence $\dot{x}(t) = \mathcal{A}[x(t) + du(t)]$. Elementary calculations yield

$$d = -\mathbf{1} \in \mathrm{L}^2(0,1), \qquad \mathbf{1}(\theta) = 1, \quad 0 \leq \theta \leq 1 \ , \tag{1.4}$$

$d \in D(c^\#)$ with $c^\# d = -1$, whilst $A = \sigma|_{\ker \tau}$,

$$\mathcal{A}x = x'', \quad D(\mathcal{A}) = \{x \in \mathrm{H}^2(0,1) : \ x'(1) = 0, \ x(0) = 0\} \ . \quad (1.5)$$

1.2 Properties of $\mathcal{A}$, $c^\#$ and d

Since $\mathcal{A} = \mathcal{A}^* < 0$ with the resolvent:

$$\left((\lambda I - \mathcal{A})^{-1} v\right)(\theta) =$$
$$\frac{1}{\cosh\sqrt{\lambda}} \int_0^1 \left\{ \begin{array}{ll} \dfrac{\sinh\sqrt{\lambda}\theta \cosh\sqrt{\lambda}(1-\vartheta)}{\sqrt{\lambda}}, & \theta < \vartheta \\ \dfrac{\sinh\sqrt{\lambda}\vartheta \cosh\sqrt{\lambda}(1-\theta)}{\sqrt{\lambda}}, & \theta > \vartheta \end{array} \right\} v(\vartheta)\mathrm{d}\vartheta, \quad (1.6)$$

where the kernel of the last integral operator is in $\mathrm{C}([0,1]^2) \subset \mathrm{L}^2((0,1)^2)$. Hence the resolvent is a compact (even a Hilbert-Schmidt) operator. By discrete version of the spectral theorem, the spectrum of $\mathcal{A}$ is purely point, i.e., it consists of eigenvalues $\{\lambda_n\}_{n=0}^{\infty}$ and there exists a system of corresponding eigenvectors $\{e_n\}_{n=0}^{\infty}$ being an ONB of H,

$$\left\{ \begin{array}{rcll} e_n(\theta) & = & \sqrt{2}\sin\left(\frac{\pi}{2}+n\pi\right)\theta \ , \quad 0 \le \theta \le 1, & n \ge 0 \\ \lambda_n & = & -\left(\frac{\pi}{2}+n\pi\right)^2, & n \ge 0 \end{array} \right\} .$$

$\mathcal{A}$ generates H an analytic, self–adjoint semigroup $\{S(t)\}_{t\ge 0}$,

$$S(t)x_0 = \sum_{n=0}^{\infty} e^{\lambda_n t}\langle x_0, e_n\rangle_{\mathrm{H}} e_n \qquad \forall x_0 \in \mathrm{H}, \quad \forall t \ge 0 \ .$$

This semigroup is *exponentially stable* (**EXS**), i.e., there exist $M \ge 1$ and $\alpha > 0$ such that

$$\|S(t)x_0\|_{\mathrm{H}} \le Me^{-\alpha t}\|x_0\|_{\mathrm{H}} \qquad \forall x_0 \in \mathrm{H}, \quad \forall t \ge 0 \ .$$

Here $M = 1$ and $\alpha = -\lambda_0 = \frac{\pi^2}{4}$ (by Parseval's identity).

The *fractional powers* of $(-\mathcal{A})$ are defined as

$$(-\mathcal{A})^\alpha x = \sum_{n=0}^{\infty} (-\lambda_n)^\alpha \langle x, e_n \rangle_{\mathrm{H}} e_n \quad ,$$

$$D[(-\mathcal{A})^\alpha] = \left\{ x \in \mathrm{H} : \sum_{n=0}^{\infty} (-\lambda_n)^{2\alpha} \left| \langle x, e_n \rangle_{\mathrm{H}} \right|^2 < \infty \right\}$$

and its is well-known that there exist $c_\alpha > 0$ and $\delta > 0$ such that

$$\|(-\mathcal{A})^\alpha S(t) x_0\|_{\mathrm{H}} \leq c_\alpha \frac{e^{-\delta t}}{t^\alpha} \|x_0\|_{\mathrm{H}} \qquad \forall t > 0, \quad \forall x_0 \in \mathrm{H}.$$

$$c^{\#} e_n = e_n(1) = (-1)^n \sqrt{2} \ .$$

Let $h(\theta) = -\theta$, $\theta \in [0, 1]$. Then

$$x \in D(\mathcal{A}) \implies \quad \langle \mathcal{A}x, h \rangle_{\mathrm{H}} = -\int_0^1 \theta x''(\theta) \mathrm{d}\theta = \int_0^1 x'(\theta) \mathrm{d}\theta = x(1),$$

$$\langle \mathcal{A}x, d \rangle_{\mathrm{H}} = -\int_0^1 x''(\theta) \mathrm{d}\theta = x'(0),$$

whence $c^{\#}\big|_{D(\mathcal{A})} = h^*\mathcal{A}, h^* = c^{\#}\mathcal{A}^{-1}$ and

$$d^*\mathcal{A}e_n = e_n'(0) = \sqrt{2}\sqrt{-\lambda_n} \ .$$

Lemma 1.1. $h \in D[(-\mathcal{A})^\kappa]$ for $\kappa \in [0, \frac{3}{4})$ and $d \in D[(-\mathcal{A})^\alpha]$ for $\alpha \in [0, \frac{1}{4})$.

This is an elementary result (Grabowski, 1990, p. 334).

2 Balakrishnan–Washburn estimates

We shall give sharper estimates than those following from Lemma 1.1.

Lemma 2.1. There holds for $t > 0$

$$\|AS(t)h\|_{\mathrm{H}} \le \sqrt{2}\,\sqrt{\frac{\sqrt{t}+1}{\sqrt{t}}\,e^{-\pi^2 t/4}} \ , \qquad (2.1)$$

$$\|AS(t)d\|_{\mathrm{H}} \leq \frac{\pi}{\sqrt{2}} \sqrt{\frac{t\sqrt{t}+1}{t\sqrt{t}}}\, e^{-\pi^2 t/4} \ . \qquad (2.2)$$

Proof. To prove (2.1) we use successively

$$\lambda_n - \lambda_0 \leq -\pi^2 n^2, \quad n \in \mathbb{N}; \qquad \int_0^\infty e^{-y^2} \mathrm{d}y = \frac{\sqrt{\pi}}{2} \ ,$$

getting

$$\|AS(t)h\|_{\mathrm{H}}^2 = \sum_{n=0}^{\infty} |\langle AS(t)h, e_n\rangle_{\mathrm{H}}|^2 = \sum_{n=0}^{\infty} e^{2\lambda_n t} |c^{\#} e_n|^2 = 2 \sum_{n=0}^{\infty} e^{2\lambda_n t}$$

$$= 2e^{2\lambda_0 t} \left[1 + \sum_{n=1}^{\infty} e^{2(\lambda_n - \lambda_0)t}\right] \leq 2e^{2\lambda_0 t} \left[1 + \sum_{n=1}^{\infty} e^{-2\pi^2 n^2 t}\right] \leq$$

$$2e^{2\lambda_0 t} \left[1 + \int_0^\infty e^{-2\pi^2 n^2 t} \mathrm{d}n\right] = \frac{2\sqrt{2\pi t}+1}{\sqrt{2\pi t}}\, e^{2\lambda_0 t} \leq 2 \frac{\sqrt{t}+1}{\sqrt{t}}\, e^{2\lambda_0 t}.$$

It follows from (2.1) that $\mathcal{A}S(\cdot)h \in \mathrm{L}^p(0, \infty; \mathrm{H})$ for $p \in [1, 4)$.

To prove (2.2) we need, in addition

$$\frac{\lambda_n}{\lambda_0} \le 9n^2, \quad n \in \mathbb{N}; \qquad xe^{-x} \le e^{-1}, \quad x \ge 0 \;,$$

$$\begin{aligned}
\|AS(t)d\|_{\mathrm{H}}^2 = \;& \sum_{n=0}^{\infty} \left|\langle AS(t)d, e_n\rangle_{\mathrm{H}}\right|^2 = -\sum_{n=0}^{\infty} 2\lambda_n e^{2\lambda_n t} = \\
= \;& -2\lambda_0 e^{2\lambda_0 t}\left[1 + \sum_{n=1}^{\infty} \frac{\lambda_n}{\lambda_0}\, e^{2(\lambda_n - \lambda_0)t}\right] \le \\
\le \;& -2\lambda_0 e^{2\lambda_0 t}\left[1 + \frac{9}{\pi^2 t} \sum_{n=1}^{\infty} \pi^2 n^2 t\, e^{-2\pi^2 n^2 t}\right] \le \\
\le \;& -2\lambda_0 e^{2\lambda_0 t}\left[1 + \frac{9}{\pi^2 e t} \sum_{n=1}^{\infty} e^{-\pi^2 n^2 t}\right] \le \\
\le \;& -2\lambda_0 e^{2\lambda_0 t}\left[1 + \frac{9}{\pi^2 e t} \int_0^{\infty} e^{-\pi^2 n^2 t}\,\mathrm{d}n\right] =
\end{aligned}$$

$$= e^{2\lambda_0 t}\ \frac{2\pi^2 et\sqrt{\pi t}+9}{4et\sqrt{\pi t}} \le \frac{\pi^2}{2}\ \frac{t\sqrt{t}+1}{t\sqrt{t}}\ e^{2\lambda_0 t}\quad .$$

It follows from (2.2) that $\mathcal{A}S(\cdot)d \in \mathrm{L}^p(0,\infty;\mathrm{H})$ for $p \in [1, \frac{4}{3})$. $\square$

3 General facts

The following general fact will be important.

Lemma 3.1.

$$\begin{aligned}
\mathcal{A} \text{ generates an analytic } \mathbf{EXS} \text{ semigroup } \{S(t)\}_{t\ge 0} \Longrightarrow \\
\mathcal{A}(sI-\mathcal{A})^{-1} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{H})) \iff \\
\iff f \longmapsto \mathcal{A}S(\cdot)\star f \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{H})) \iff \\
\iff f \longmapsto S(\cdot)\star f \in \mathbf{L}(\mathrm{W}^{1,2}(0,\infty;\mathrm{H}))\quad ,
\end{aligned}$$

where the last fact is known as the *maximal* $\mathrm{L}^2(0,\infty;\mathrm{H})$ - *parabolic regularity*; it means that for $f \in \mathrm{L}^2(0,\infty;\mathrm{H})$ the nonhomogeneous abstract initial value problem $\dot{z} = \mathcal{A}z + f$, $z(0) = 0$ has a strong solution.

Thanks to Lemma 3.1 we have

$$x \in \mathrm{L}^2(0,\infty;\mathrm{H}), \quad x(t) := \quad S(t)x_0 + \mathcal{A}\int_0^t S(t-\tau)du(\tau)\mathrm{d}\tau =$$

$$= \quad S(t)x_0 + \int_0^t \mathcal{A}S(t-\tau)du(\tau)\mathrm{d}\tau$$

and for every $w \in D(\mathcal{A}^*)$ the function $t \longmapsto \langle x(t), w\rangle_{\mathrm{H}}$ is in $\mathrm{W}^{1,2}(0,\infty)$, it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), w\rangle_{\mathrm{H}} = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle S(t)x_0 + \mathcal{A}\int_0^t S(t-\tau)du(\tau)\mathrm{d}\tau, w\right\rangle_{\mathrm{H}} =$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\langle x_0, S^*(t)w\rangle_{\mathrm{H}} + \frac{\mathrm{d}}{\mathrm{d}t}\left\langle \int_0^t S(t-\tau)du(\tau)\mathrm{d}\tau, \mathcal{A}^*w\right\rangle_{\mathrm{H}} =$$

$$= \langle x_0, S^*(t)\mathcal{A}^* w\rangle_{\mathrm{H}} + \left\langle \mathcal{A}\int_0^t S(t-\tau)du(\tau)\mathrm{d}\tau + du(t), \mathcal{A}^* w \right\rangle_{\mathrm{H}} =$$

$$= \left\langle S(t)x_0 + \mathcal{A}\int_0^t S(t-\tau)du(\tau)\mathrm{d}\tau + du(t), \mathcal{A}^* w \right\rangle_{\mathrm{H}} =$$

$$= \langle x(t) + du(t), \mathcal{A}^* w\rangle_{\mathrm{H}} \ ,$$

and

$$\lim_{t\to 0}\langle x(t), w\rangle_{\mathrm{H}} = \lim_{t\to 0}\left\langle S(t)x_0 + \mathcal{A}\int_0^t S(t-\tau)du(\tau)\mathrm{d}\tau, w \right\rangle_{\mathrm{H}} =$$

$$= \langle x_0, w\rangle_{\mathrm{H}} + \lim_{t\to 0}\left\langle \int_0^t S(t-\tau)du(\tau)\mathrm{d}\tau, \mathcal{A}^* w \right\rangle_{\mathrm{H}} = \langle x_0, w\rangle_{\mathrm{H}} \ ,$$

whence x is regarded to be a *weak solution in Balakrishnan's sense* and this solution is unique (Balakrishnan, 1976, Theorem 4.8.3 and Corollary 4.8.1, pp. 255-257).

If, in addition, x is *continuous* x can be named a *weak solution in Ball's sense* (Ball, 1977, Definition on p. 370) or (Pazy, 1983, p. 258).

An important item is the requirement $x(0) = x_0$ which appears in (Ball, 1977, Theorem, p. 371) or (Pazy, 1983, p. 259). Any result ensuring that a weak solution in Balakrishnan's sense is a weak solution in Ball's sense, possibly with an additional requirement $x(0) = x_0$, is called a *lifting theorem*.

Now we introduce the concepts of admissibility. For that we shall use the semigroups of left–shifts on $\mathrm{L}^2(0,\infty)$ which will be denoted as $\{T(t)\}_{t\geq 0}$, $(T(t)f)(\tau) := f(t+\tau)$ for almost all $t,\tau \geq 0$. Its infinitesimal generator is

$$\mathcal{L}f = f', \quad D(\mathcal{L}) = \mathrm{W}^{1,2}([0,\infty)) := \left\{ f \in \mathrm{L}^2(0,\infty) : f' \in \mathrm{L}^2(0,\infty) \right\}.$$

The adjoint of $T(t)$ is the right–shift operator on $\mathrm{L}^2(0,\infty)$ defined as

$$(T^* f)(\tau) := \left\{ \begin{array}{ccr} f(\tau - t) & \text{if} & \tau \geq t \\ 0 & \text{if} & 0 \leq \tau < t \end{array} \right\} ,$$

and it is clearly generated by $\mathcal{L}^* := \mathcal{R}$,

$$\mathcal{R}f = f', \quad D(\mathcal{R}) = \mathrm{W}^{1,2}_0([0,\infty)) := \left\{ f \in \mathrm{W}^{1,2}([0,\infty)) : f(0) = 0 \right\}.$$

Assume that $\mathcal{A}$ generates an **EXS** C_0–semigroup. Define $\mathcal{Z} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty))$ as $(\mathcal{Z}x_0)(t) := c^\# \mathcal{A}^{-1} S(t) x_0$. The operator $\Psi := \mathcal{L}\mathcal{Z}$ with natural domain $D(\Psi) = \{x \in \mathrm{H} : \mathcal{Z}x \in D(\mathcal{L})\}$ is *closed* and *densely defined*, with $\Psi|_{D(\mathcal{A})} = \mathcal{Z}\mathcal{A}$, and therefore it has *closed* and *densely defined* adjoint operator $\Psi^* = \mathcal{A}^* \mathcal{Z}^*$ with natural domain $D(\Psi^*) = \{y \in \mathrm{L}^2(0,\infty) : \mathcal{Z}^* y \in D(\mathcal{A}^*)\}$, and $\Psi^*|_{D(\mathcal{R})} = \mathcal{Z}^* \mathcal{R}, \mathcal{R} = \mathcal{L}^*$.

Definition 3.1. $c^\#$ is an admissible *observation functional* if $\Psi \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty))$; then Ψ is called the system *observability map*.

By the closed–graph theorem, several equivalent characterizations of admissibility of $c^\#$ are possible, e.g., we can require that $\mathcal{R}(\mathcal{Z}) \subset D(\mathcal{L})$ or, in the frequency–domain, that

$s \longmapsto c^{\#}(sI - \mathcal{A})^{-1}x_0 \in \mathrm{H}^2(\mathbb{C}^+)$ for every $x_0 \in \mathrm{H}$.

Lemma 3.2. If the semigroup $\{S(t)\}_{t\geq 0}$ is **EXS** and $c^{\#}$ is admissible then Ψ is also a linear densely defined and *bounded* operator from H into $\mathrm{L}^1(0,\infty)$.

Proof. Here we copy the proof of (Grabowski, 2007, Appendix C). By the semigroup property, Schwarz inequality and admissibility we have

$$\|\Psi x_0\|_{\mathrm{L}^1(0,\infty)} = \int_0^\infty \left|c^{\#}S(t)x_0\right| \mathrm{d}t = \sum_{k=0}^{\infty} \int_k^{k+1} \left|c^{\#}S(t)x_0\right| \mathrm{d}t =$$

$$= \sum_{k=0}^{\infty} \int_0^1 \left|c^{\#}S(\tau+k)x_0\right| \mathrm{d}\tau = \sum_{k=0}^{\infty} \int_0^1 \left|c^{\#}S(\tau)S(k)x_0\right| \mathrm{d}\tau \leq$$

$$\leq \sum_{k=0}^{\infty} \sqrt{\int_0^1 \left|c^{\#}S(t)S(k)x_0\right|^2 \mathrm{d}t} \leq \gamma \sum_{k=0}^{\infty} \|S(k)x_0\|_{\mathrm{H}} \quad \forall x_0 \in D(\mathcal{A}),$$

whence by **EXS**

$$\|\Psi x_0\|_{L^1(0,\infty)} \leq \gamma M \|x_0\|_{H} \sum_{k=0}^{\infty} e^{-\alpha k} = \frac{\gamma M}{1-e^{-\alpha}} \|x_0\|_{H} \quad \forall x_0 \in D(\mathcal{A}).$$

Since, by **EXS**, Ψ is well–defined on $D(\mathcal{A})$, a dense subspace of H, it extends uniquely by continuity to the closure of Ψ, $\overline{\Psi} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^1(0,\infty))$, moreover the Laplace transform of $\overline{\Psi}x_0$ clearly equals $\left(\widehat{\overline{\Psi}x_0}\right)(s) = c^{\#}(sI-\mathcal{A})^{-1}x_0 = \left(\widehat{\Psi x_0}\right)(s)$. Hence by injectivity of the Laplace transformation $\overline{\Psi}x_0 = \Psi x_0 \in \mathrm{L}^1(0,\infty)$ for any $x_0 \in \mathrm{H}$. □

Lemma 3.3. The observation functional $c^{\#}$ given by (1.2) is admissible. Moreover, by the analyticity of $\{S(t)\}_{t\geq 0}$,

$$(\Psi x_0)(t) = c^{\#}S(t)x_0 = h^*\mathcal{A}S(t)x_0, \qquad x_0 \in \mathrm{H},\ t>0\ .$$

Proof. Indeed, in virtue of the analyticity and **EXS** of $\{S(t)\}_{t\geq 0}$,

$$x_0 \in D(\mathcal{A}) \Longrightarrow (\Psi x_0)(t) = (\mathcal{Z}\mathcal{A}x_0)(t) = \langle S(t)\mathcal{A}x_0, h\rangle_{\mathrm{H}} = \\ = \langle x_0, \mathcal{A}S(t)h\rangle_{\mathrm{H}}$$

which reveals in

$$\|\Psi x_0\|^2_{\mathrm{L}^2(0,\infty)} \leq \|x_0\|^2_{\mathrm{H}} \, \|\mathcal{A}S(\cdot)h\|^2_{\mathrm{L}^2(0,\infty)}$$

and, by (2.1), Ψ is bounded. Hence Ψ uniquely extends to $\Psi = \mathcal{L}\mathcal{Z} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty))$, and $\frac{\mathrm{d}}{\mathrm{d}t}\left[h^* S(t)x_0\right] = h^*\mathcal{A}S(t)x_0$. □

An alternative proof has be provided in (Grabowski, 1990, p. 324).

Still assuming that $\mathcal{A}$ generates an **EXS** C_0–semigroup we define $\mathcal{W} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{H})$ as $\mathcal{W}f := \int_0^\infty S(t)df(t)\mathrm{d}t$. The operator $\Phi := \mathcal{A}\mathcal{W}$ with natural domain $D(\Phi) = \{u \in \mathrm{L}^2(0,\infty) : \mathcal{W}u \in D(\mathcal{A})\}$ is *closed* and *densely defined*, with $\Phi|_{D(\mathcal{R})} = \mathcal{W}\mathcal{R}$, $\mathcal{R} = \mathcal{L}^*$, and therefore it has *closed* and *densely defined* adjoint operator $\Phi^* = \mathcal{L}\mathcal{W}^*$ with natural domain $D(\Phi^*) = \{x \in \mathrm{H} : \mathcal{W}^* x \in D(\mathcal{L})\}$, with $\Phi^*|_{D(\mathcal{A}^*)} = \mathcal{W}^*\mathcal{A}^*$.

Definition 3.2. d is an admissible *control vector* if $\Phi \in \mathbf{L}(\mathrm{L}^2(0,\infty), \mathrm{H})$; then Ψ is the system *reachability map*.

By the closed–graph theorem, several equivalent characterizations of admissibility of d are possible, e.g., we can require that $R(\mathcal{W}) \subset D(\mathcal{A})$, or using duality arguments, that $d^*\mathcal{A}^*$ is admissible observation operator with respect to the semigroup $\{S^*(t)\}_{t\geq 0}$.

In what follows $\mathrm{BUC}[0,\infty;\mathrm{Z})$ will denote the Banach space of bounded, uniformly continuous functions defined on $[0,\infty)$ and taking values in a Hilbert space Z, equipped with standard norm $\|f\|_{\mathrm{BUC}[0,\infty;\mathrm{Z})} := \sup_{t\geq 0} \|f(t)\|_{\mathrm{Z}}$, $f \in \mathrm{BUC}[0,\infty;\mathrm{Z})$, whilst $\mathrm{BUC}_0[0,\infty;\mathrm{H})$ will stand for its closed subspace consisting of functions that have zero limit at infinity.

Theorem 3.1. If $\mathcal{A}$ generates an **EXS** C_0–semigroup and d is admissible then for every $x_0 \in \mathrm{H}$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$

$$x(t) = S(t)x_0 + \Phi R_t u, \quad (R_t u)(\tau) := \left\{ \begin{array}{ccc} u(t-\tau) & \text{if} & \tau \le t \\ 0 & \text{if} & \tau > t \end{array} \right\},$$

where $R_t \in \mathbf{L}(\mathrm{L}^2(0,\infty))$, $R_t = R_t^*$, $\|R_t\|_{\mathbf{L}(\mathrm{L}^2(0,\infty))} = 1$ is called the *operator of reflection* at t, is a weak solution of (1.3) in Balakrishnan's sense.

Actually $x \in \mathrm{BUC}_0([0,\infty),\mathrm{H})$, whence x is also a weak solution in Ball's sense. Moreover $x(0) = x_0$.

Proof. Recall that the admissibility of d holds iff $R(\mathcal{W}) \subset D(\mathcal{A})$

resulting in $\mathcal{A}\mathcal{W} = \Phi \in \mathbf{L}(\mathrm{L}^2(0,\infty), \mathrm{H})$. Now, if $w \in D(\mathcal{A}^*)$ then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), w\rangle_{\mathrm{H}} = & \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle S(t)x_0, w\rangle_{\mathrm{H}} + \frac{\mathrm{d}}{\mathrm{d}t}\langle \mathcal{A}\mathcal{W}R_t u, w\rangle_{\mathrm{H}} = \\
= & \quad \langle x_0, S^*(t)\mathcal{A}^* w\rangle_{\mathrm{H}} + \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}R_t u, \mathcal{A}^* w \right\rangle_{\mathrm{H}} = \\
= & \quad \langle S(t)x_0, \mathcal{A}^* w\rangle_{\mathrm{H}} + \langle \mathcal{A}\mathcal{W}R_t u + du(t), \mathcal{A}^* w\rangle_{\mathrm{H}}
\end{aligned}$$

where the last equality is met as, by $R(\mathcal{W}) \subset D(\mathcal{A})$, $\mathcal{W}R_t u$ is a *strong* solution of $\dot{x} = \mathcal{A}x + du$ with null initial condition (Pazy, 1983, Theorem 2.9/(ii), p. 109).

For any fixed $u \in \mathrm{L}^2(0,\infty)$, the function $t \longmapsto R_t u$ is in $\mathrm{BUC}[0,\infty; \mathrm{L}^2(0,\infty))$. Indeed,

$$\|R_t u - R_s u\|^2_{\mathrm{L}^2(0,\infty)} =$$

$$\int_0^\infty \left[\left\{ \begin{array}{ll} u(t-\tau), & 0 \le \tau < t \\ 0, & \tau \ge t \end{array} \right\} - \left\{ \begin{array}{ll} u(s-\tau), & 0 \le \tau < s \\ 0, & \tau \ge s \end{array} \right\} \right]^2 \mathrm{d}\tau.$$

Let $s > t$. Then

$$\|R_t u - R_s u\|^2_{\mathrm{L}^2(0,\infty)} =$$

$$= \int_0^\infty \left[\left\{ \begin{array}{lll} u(t-\tau) - u(s-\tau) & \text{if} & 0 \le \tau < t \\ -u(s-\tau) & \text{if} & t \le \tau < s \\ 0 & \text{if} & \tau \ge s \end{array} \right\} \right]^2 \mathrm{d}\tau =$$

$$= \int_0^t [u(t-\tau) - u(s-\tau)]^2 \, \mathrm{d}\tau + \int_t^s u^2(s-\tau)\mathrm{d}\tau =$$

$$= \int_0^t [u(\xi) - u(s-t+\xi)]^2 \, \mathrm{d}\xi +$$

$$+ \int_0^{s-t} u^2(\xi)\mathrm{d}\xi \le \|u - T(s-t)u\|^2_{\mathrm{L}^2(0,\infty)} + \int_0^{s-t} u^2(\xi)\mathrm{d}\xi \ .$$

Similarly, for $t > s$ we get

$$\|R_t u - R_s u\|^2_{\mathrm{L}^2(0,\infty)} \leq \|T(t-s)u - u\|^2_{\mathrm{L}^2(0,\infty)} + \int_0^{t-s} u^2(\xi)\mathrm{d}\xi \ .$$

Both these estimates together yield

$$\|R_t u - R_s u\|^2_{\mathrm{L}^2(0,\infty)} \leq \varepsilon(|t-s|), \qquad \forall t, s \geq 0 \ ,$$

$$\varepsilon(\delta) := \|T(\delta)u - u\|^2_{\mathrm{L}^2(0,\infty)} + \int_0^{\delta} u^2(\xi)\mathrm{d}\xi \ .$$

The uniform continuity and boundedness hold as the function ε is continuous, nonnegative and bounded on $[0,\infty)$ with the upper bound $5\|u\|^2_{\mathrm{L}^2(0,\infty)}$, and $\varepsilon(0) = 0$. The sharpest upper bound for the function $t \longmapsto R_t u$ directly follows from observation that the reflection operator is a contraction on $\mathrm{L}^2(0,\infty)$.

Since $\Phi \in \mathbf{L}(\mathrm{L}^2(0,\infty), \mathrm{H})$, the function $t \longmapsto \Phi R_t u$ is in $\mathrm{BUC}[0,\infty;\mathrm{H})$. Thus the linear operator given by $(\mathcal{P}u)(t) := \Phi R_t u$

belongs to $\mathbf{L}(\mathrm{L}^2(0,\infty), \mathrm{BUC}[0,\infty;\mathrm{H}))$ as for every $u \in \mathrm{L}^2(0,\infty)$:

$$\|\mathcal{P}u\|_{\mathrm{BUC}[0,\infty;\mathrm{H})} = \sup_{t\geq 0} \|\Phi R_t u\|_{\mathrm{H}} \leq \|\Phi\|_{\mathbf{L}(\mathrm{L}^2(0,\infty),\mathrm{H})} \|u\|_{\mathrm{L}^2(0,\infty)} \ .$$

Since $\overline{D(\mathcal{R})} = \mathrm{L}^2(0,\infty)$, any $u \in \mathrm{L}^2(0,\infty)$ can be represented as $\mathrm{L}^2(0,\infty)$ - limit of a sequence $\{u_n\}_{n\in\mathbb{N}} \subset D(\mathcal{R}) = \mathrm{W}_0^{1,2}[0,\infty)$. Then by (Pazy, 1983, Corollary 2.10, p. 109):

$$(\mathcal{P}u_n)(t) = \Phi R_t u_n = \mathcal{A}\int_0^t S(t-\tau)du_n(\tau)\mathrm{d}\tau =$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t S(t-\tau)du_n(\tau)\mathrm{d}\tau - du_n(t) = \int_0^t S(t-\tau)d\dot{u}_n(\tau)\mathrm{d}\tau - du_n(t). \tag{3.1}$$

Since $\mathrm{L}^2(0,\infty;\mathrm{H}) \star \mathrm{L}^2(0,\infty) \subset \mathrm{BUC}_0[0,\infty;\mathrm{H})$, $S(\cdot)d \in \mathrm{L}^2(0,\infty;\mathrm{H})$ and $\dot{u}_n \in \mathrm{L}^2(0,\infty)$, then the last convolution in (3.1) is in $\mathrm{BUC}_0[0,\infty;\mathrm{H})$, whence $\{\mathcal{P}u_n\}_{n\in\mathbb{N}} \subset \mathrm{BUC}_0[0,\infty;\mathrm{H})$, and $\mathcal{P}u_n \to \mathcal{P}u$ in $\mathrm{BUC}[0,\infty;\mathrm{H})$. By the closedness of $\mathrm{BUC}_0[0,\infty;\mathrm{H})$

we have $\mathcal{P}u \in \mathrm{BUC}_0[0,\infty;\mathrm{H})$. Actually, $(\mathcal{P}u_n)(0) = 0$, whence $(\mathcal{P}u)(0) = 0$. □

Lemma 3.4. The factor control vector d given by (1.2) is not admissible.

Proof. If d were admissible then by Definition 3.2 we would have for every $f \in \mathrm{L}^2(0,\infty) \iff \hat{f} \in \mathrm{H}^2(\mathbb{C}^+)$ (we use the Paley-Wiener theory with $\hat{f}$ standing for the Laplace transform of f)

$$\begin{aligned} \infty > \quad & \|\Phi f\|_{\mathrm{H}}^2 = \sum_{n=0}^{\infty} |\langle \Phi f, e_n \rangle_{\mathrm{H}}|^2 = \sum_{n=0}^{\infty} \left| \langle f, \Phi^* e_n \rangle_{\mathrm{L}^2(0,\infty)} \right|^2 = \\ = \quad & \sum_{n=0}^{\infty} \left| \langle f, d^* \mathcal{A} S(\cdot) e_n \rangle_{\mathrm{L}^2(0,\infty)} \right|^2 = \sum_{n=0}^{\infty} \left| \langle f, e_n'(0) e^{\lambda_n (\cdot)} \rangle_{\mathrm{L}^2(0,\infty)} \right|^2 = \\ = \quad & \sum_{n=0}^{\infty} |e_n'(0)|^2 \left| \hat{f}(-\lambda_n) \right|^2 = \frac{\pi^2}{2} \sum_{n=0}^{\infty} (2n+1)^2 \left| \hat{f}(-\lambda_n) \right|^2 \end{aligned} \tag{3.2}$$

However, for $f \in \mathrm{L}^2(0,\infty)$, $f(t) = t^{-1/4}e^{-t}$ we have, by (Bateman et al, 1954, Formula (1), p.137), $\hat{f}(s) = (s+1)^{-3/4}\Gamma(\frac{3}{4})$ and therefore

$$\left[\Gamma\left(\frac{3}{4}\right)\right]^{-2} \frac{\pi^3}{8} \sum_{n=0}^{\infty} (2n+1)^2 \left|\hat{f}(-\lambda_n)\right|^2 = \sum_{n=0}^{\infty} \frac{(2n+1)^2}{\left[\frac{4}{\pi^2} + (2n+1)^2\right]^{3/2}}$$

$$\geq \frac{\sqrt{2}}{4} \sum_{n=0}^{\infty} \frac{1}{2n+1} = \infty$$

which contradicts (3.2). □

This result is borrowed from (Grabowski and Callier, 2001b, Lemma 5.2, p. 27 and p. 33).

It follows also from our earlier result (Grabowski and Callier, 1999, pp. 97-98).

Remark 3.1. Taking $x_0 = 0$ and $u(t) = \chi_{[0,T]}(t)\frac{1}{(T-t)^\alpha}$, $T > 0$,

$\alpha \in [\frac{1}{4}, \frac{1}{2})$ we have $u \in \mathrm{L}^2(0,\infty)$ and, similarly to (3.2), we get:

$$\|x(T)\|^2_{\mathrm{H}} = \|\Phi R_T u\|^2_{\mathrm{H}} = \sum_{n=0}^{\infty} |\langle \Phi R_T u, e_n \rangle_{\mathrm{H}}|^2 =$$

$$= \sum_{n=0}^{\infty} |\langle u, R_T \Phi^* e_n \rangle_{\mathrm{L}^2(0,\infty)}|^2 = \sum_{n=0}^{\infty} \left| \int_0^T \frac{1}{(T-t)^\alpha} d^* \mathcal{A} S(T-t) e_n \mathrm{d}t \right|^2$$

$$= \sum_{n=0}^{\infty} \left| e_n'(0) \int_0^T \frac{e^{\lambda_n (T-t)}}{(T-t)^\alpha} \mathrm{d}t \right|^2 = 2 \sum_{n=0}^{\infty} (-\lambda_n) \left[\int_0^T \frac{e^{\lambda_n t}}{t^\alpha} \mathrm{d}t \right]^2 = \infty$$

because (Miller, 2006, p. 55, last line in the proof of Proposition 2.1 with $\lambda = -\alpha$ and $x = -\lambda_n$)

$$\int_0^T \frac{e^{\lambda_n t}}{t^\alpha} \mathrm{d}t = \frac{\Gamma(1-\alpha)}{(-\lambda_n)^{(1-\alpha)}} + o\left(e^{T\lambda_n}\right) \quad \text{as} \quad n \to \infty \ .$$

Remark 3.1 shows that, for the $\mathfrak{RC}$–transmission line dynamics, Theorem 3.1 does not provide any lifting of a weak solution.

Now we pass to the *construction of the system output* in operator form. By Lemma 3.3, the homogeneous part y_{h} of the system output y reads as $y_{\mathrm{h}} = (\Psi x_0)$ for every $x_0 \in \mathrm{H}$.

To construct y_{nh}–the nonhomogeneous part of the output we assume initially that $u \in D(\mathcal{R})$. Then

$$\begin{aligned} \mathcal{A} \int_0^t S(t-\tau) d u(\tau) \mathrm{d}\tau = & \quad \frac{\mathrm{d}}{\mathrm{d}t} \int_0^t S(t-\tau) d u(\tau) \mathrm{d}\tau - d u(t) = \\ = & \quad \int_0^t S(t-\tau) d \dot{u}(\tau) \mathrm{d}\tau - d u(t). \end{aligned}$$

By the maximal L^2–parabolic regularity, the last convolution term belongs to $D(\mathcal{A}) \subset D(c^{\#})$, whence

$$\begin{aligned} y_{\mathrm{nh}}(t) = & \quad h^* \mathcal{A} \int_0^t S(t-\tau) d \dot{u}(\tau) \mathrm{d}\tau - c^{\#} d u(t) = \\ = & \quad \int_0^t \Psi d(t-\tau) \dot{u}(\tau) \mathrm{d}\tau - c^{\#} d u(t) \ , \end{aligned} \tag{3.3}$$

where thanks to Lemma 3.2 and $\mathrm{L}^1(0,\infty) \star \mathrm{L}^2(0,\infty) \subset \mathrm{L}^2(0,\infty)$,

$$\begin{aligned}(\mathcal{K}u)(t) &:= \int_0^t \Psi d(t-\tau)u(\tau)\mathrm{d}\tau, \quad \mathcal{K} \in \mathbf{L}(\mathrm{L}^2(0,\infty)), \\ (\mathcal{K}^* v)(t) &= \int_t^\infty \Psi d(\tau - t)v(\tau)\mathrm{d}\tau \ .\end{aligned}$$

On this way we have determined the densely defined *input–output operator* $\mathbb{F} := -\mathcal{K}\mathcal{R} - c^\# dI$. Applying the Laplace transformation we obtain

$$\hat{y}_{\mathrm{nh}}(s) = (\widehat{\mathbb{F}u})(s) = \hat{g}(s)\hat{u}(s) \ ,$$

$$\begin{aligned}\hat{g}(s) &:= s\widehat{\Psi d}(s) - c^\# d = sc^\#(sI - \mathcal{A})^{-1}d - c^\# d = \\ &= s^2 h^*(sI - \mathcal{A})^{-1}d - sh^* d - c^\# d \ .\end{aligned} \tag{3.4}$$

If the system *transfer function* $\hat{g}$ satisfies

$$\hat{g} \in \mathrm{H}^\infty(\mathbb{C}^+) \tag{3.5}$$

then, by the Paley–Wiener theory, $\mathbb{F}$ is also bounded and therefore closable. Hence it has a bounded densely defined adjoint operator $\mathbb{F}^* = -\mathcal{L}\mathcal{K}^* - c^\# dI$, and the closure $\overline{\mathbb{F}}$ of $\mathbb{F}$, being the unique extension of $\mathbb{F}$ by continuity onto $\mathrm{L}^2(0,\infty)$, is given by $\overline{\mathbb{F}} = \mathbb{F}^{**}$. It is not difficult to see that on $D(\mathcal{L})$ the operators $\mathcal{K}^*$ and $\mathcal{L}$ commute and $\mathbb{F}^*|_{D(\mathcal{L})} = -\mathcal{K}^*\mathcal{L} - c^\# dI$, which yields $\mathbb{F}^{**} = \overline{\mathbb{F}} = -\mathcal{R}\mathcal{K} - c^\# dI \in \mathbf{L}(\mathrm{L}^2(0,\infty))$, i.e.,

$$(\mathbb{F}^{**}u)(t) = \left(\overline{\mathbb{F}}u\right)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (\Psi d)(t-\tau)u(\tau)\mathrm{d}\tau - c^\# du(t). \tag{3.6}$$

For the sake of simplicity we shall still use $\mathbb{F}$ to denote $\mathbb{F}^{**}$ or $\overline{\mathbb{F}}$ getting the output equation in operator form:

$$y = y_{\mathrm{h}} + y_{\mathrm{nh}} = \Psi x_0 + \mathbb{F}u, \qquad x_0 \in \mathrm{H},\ u \in \mathrm{L}^2(0,\infty)\ . \tag{3.7}$$

In particular, in the case of $\mathfrak{RC}$–transmission line (3.5) holds true.

Indeed, using (1.2), (1.4) and (1.6) we find

$$\hat{g}(s) = \frac{1}{\cosh\sqrt{s}}, \qquad s \notin \{\lambda_n\}_{n\in\mathbb{Z}^*} \quad .$$

By the last line of (3.4) and **EXS**, $\hat{g}$ grows trinomially on

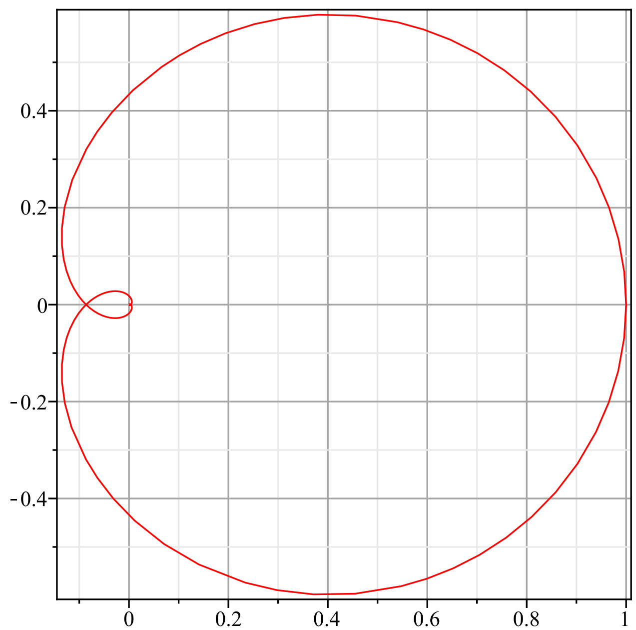

Figure 3.1: The Nyquist curve $\{\hat{g}(j\omega) : -\infty < \omega < \infty\}$.

$\overline{\mathbb{C}^+}$, and $\hat{g}$ is continuous and bounded on $j\mathbb{R}$. It follows from the *Phragmén–Lindelöf theorem* that (3.5) holds. Boundedness of $\hat{g}$ on $j\mathbb{R}$ is confirmed by the *Nyquist curve* depicted in Figure 3.1.

Actually, due to parabolic regularity, we know that $\frac{\mathrm{d}}{\mathrm{d}t}(\Psi d)(t)$ exists for $t > 0$. This suggests to transfer the time derivative in (3.6) from the front of convolution to the integrand using a version of Leibniz's rule or equivalently to integrate by parts the convolution integral in (3.3). This can be done in virtue of the following result.

Proposition 3.1. The functions Ψd and g, the inverse Laplace transform of $\hat{g}$, are continuous on $[0, \infty)$, $(\Psi d)(0) = -1 = c^{\#} d$ and

$$(1 - 3e^{-2\pi})\gamma(t) \leq g(t) \leq \gamma(t) := \min\left\{\frac{e^{-1/4t}}{t\sqrt{\pi t}}, \pi e^{-t\pi^2/4}\right\}. \quad (3.8)$$

Proof. Step 1. The function $f_{\sqcap\sqcap} \in \mathrm{L}^\infty(0, \infty)$,

$$f_{\sqcap\sqcap}(t) := 2\sum_{n=0}^{\infty} \chi_{[4n+1, 4n+3]}(t), \qquad \|f_{\sqcap\sqcap}\|_{\mathrm{L}^\infty(0,\infty)} = 2\ ,$$

having the Laplace transform:

$$\widehat{f_{\sqcap\sqcap}}(s) = \int_0^\infty f_{\sqcap\sqcap}(\tau)e^{-st}\mathrm{d}\tau = \sum_{n=0}^{\infty} 2\int_{4n+1}^{4n+3} e^{-st}\mathrm{d}t =$$

$$= \frac{2(e^{-s}-e^{-3s})}{s}\sum_{k=0}^{\infty}(e^{-4s})^k = \frac{2e^{-s}(1-e^{-2s})}{s(1-e^{-4s})} = \frac{1}{s\cosh s} \ ,$$

determines a linear and bounded functional on $\mathrm{L}^1(0,\infty)$,

$$f^*_{\sqcap\sqcap}(f) := \int_0^\infty f_{\sqcap\sqcap}(\tau)f(\tau)\mathrm{d}\tau, \qquad f \in \mathrm{L}^1(0,\infty) \ .$$

Consider the $\mathrm{L}^1(0,\infty)$–valued function

$$(0,\infty) \ni t \mapsto \varphi(t) \in \mathrm{L}^2(0,\infty), \quad \varphi(t)(\tau) := \frac{1}{\sqrt{t\pi}}e^{-\tau^2/4t}, \quad \tau \geq 0.$$

Since $\|\varphi(t)\|_{\mathrm{L}^1(0,\infty)} = \frac{1}{\sqrt{t\pi}}\int_0^\infty e^{-\tau^2/4t}\mathrm{d}\tau = 1$ (Dwight, 1961, 860.11 with $r^2 = 1/4t$), it takes values on a unit sphere; we show that φ is continuous. Indeed,

$$\sqrt{\pi}\,\|\varphi(t_1)-\varphi(t_2)\|_{\mathrm{L}^1(0,\infty} \le$$

$$\underbrace{\frac{1}{\sqrt{t_1}}\left\|e^{-(\cdot)^2/4t_1}-e^{-(\cdot)^2/4t_2}\right\|_{\mathrm{L}^1(0,\infty)}}_{✂} + \left[\frac{1}{\sqrt{t_1}}-\frac{1}{\sqrt{t_2}}\right]\left\|e^{-(\cdot)^2/4t_2}\right\|_{\mathrm{L}^1(0,\infty)}$$

If $t_1 \ge t_2$ then we extract $e^{-(\cdot)^2/4t_1}$ from ✂ getting

$$\sqrt{\pi}\,\|\varphi(t_1)-\varphi(t_2)\|_{\mathrm{L}^1(0,\infty} \le + \left[\frac{1}{\sqrt{t_1}}-\frac{1}{\sqrt{t_2}}\right]\left\|e^{-(\cdot)^2/4t_2}\right\|_{\mathrm{L}^1(0,\infty)} +$$

$$\frac{1}{\sqrt{t_1}}\left\|e^{-(\cdot)^2/4t_1}\right\|_{\mathrm{L}^1(0,\infty)} \underbrace{\left\|1-e^{-\frac{(\cdot)^2}{4}\left[\frac{1}{t_2}-\frac{1}{t_1}\right]}\right\|_{\mathrm{L}^\infty(0,\infty)}}_{\le 2 \text{ as then } \frac{\tau^2}{4}\left[\frac{1}{t_2}-\frac{1}{t_1}\right]>0}$$

and continuity follows by employing the Lebesgue dominated convergence theorem. If $t_1 \le t_2$ then we extract $e^{-(\cdot)^2/4t_2}$ and proceed similarly.

A consequence of the above facts is that the composite scalar function $h_{\text{step}} : (0, \infty) \ni t \mapsto h_{\text{step}}(t) := f^*_{\sqcap\sqcup\sqcap}[\varphi(t)]$ is continuous positive and bounded by 2; its Laplace transform can be computed as follows ($\hat{\varphi}$ is taken from (Bateman et al, 1954, p. 135, Formula (27) with $\alpha = \tau^2$)):

$$\widehat{h_{\text{step}}} = f^*_{\sqcap\sqcup\sqcap}(\hat{\varphi}), \qquad \hat{\varphi}(s) = \frac{1}{\sqrt{s}} e^{-\tau\sqrt{s}} \ ,$$

which reveals in

$$\widehat{h_{\text{step}}}(s) = \frac{1}{\sqrt{s}} \int_0^\infty f_{\sqcap\sqcup\sqcap}(\tau) e^{-\tau\sqrt{s}} \mathrm{d}\tau = \frac{1}{\sqrt{s}} \hat{f}(\sqrt{s}) = \frac{1}{s \cosh \sqrt{s}} \ .$$

On the other side

$$\widehat{\Psi d}(s) = c^{\#}(sI - \mathcal{A})^{-1} d = \frac{1}{s \cosh \sqrt{s}} - \frac{1}{s} = \widehat{h_{\text{step}}}(s) - \frac{1}{s} \ ,$$

whence

$$(\Psi d)(t) = h_{\text{step}}(t) - 1, \qquad t > 0 \ ,$$

and

$$h_{\text{step}}(t) = \tfrac{1}{\sqrt{t\pi}} \int_0^\infty f_{\sqcap\sqcup\sqcap}(\tau) e^{-\tau^2/4t} \mathrm{d}\tau =$$

$$= 2 \sum_{n=0}^{\infty} \left[\operatorname{erf}\left(\frac{4k+3}{2\sqrt{t}} \right) - \operatorname{erf}\left(\frac{4k+1}{2\sqrt{t}} \right) \right], \ \operatorname{erf}(\sigma) := \frac{2}{\sqrt{\pi}} \int_0^\sigma e^{-\xi^2} \mathrm{d}\xi.$$

Moreover,

$$h_{\text{step}}(t) = \tfrac{1}{\sqrt{t\pi}} \int_0^\infty f_{\sqcap\sqcup\sqcap}(\tau) e^{-\tau^2/4t} \mathrm{d}\tau = \int_0^\infty f_{\sqcap\sqcup\sqcap}(2\xi\sqrt{t}) \frac{2}{\sqrt{\pi}} e^{-\xi^2} \mathrm{d}\xi$$

$$= \int_{\frac{1}{2\sqrt{t}}}^\infty f_{\sqcap\sqcup\sqcap}(2\xi\sqrt{t}) \frac{2}{\sqrt{\pi}} e^{-\xi^2} \mathrm{d}\xi \longrightarrow 0 \quad \text{as} \quad t \to 0 \ .$$

This jointly with $(\Psi d)(0) = c^{\#} d = -1$ shows that Ψd is continuous on $[0, \infty)$, whilst h_{step} can be continuously prolongated to $[0, \infty)$ by taking $h_{\text{step}}(0) = 0$.

Step 2. We shall demonstrate that

$$\frac{\mathrm{d}\,(\Psi d)\,(t)}{\mathrm{d}t} = \frac{\mathrm{d}h_{\text{step}}(t)}{\mathrm{d}t} = \int_0^\infty f_{\sqcap\sqcup\sqcap}(\tau)\,\underbrace{\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{t\pi}}e^{-\frac{\tau^2}{4t}}\right]}_{=\dot{\varphi}(t)}\mathrm{d}\tau. \tag{3.9}$$

Indeed,

$$\frac{h_{\text{step}}(t+\delta) - h_{\text{step}}(t)}{\delta} = \int_0^\infty f_{\sqcap\sqcup\sqcap}(\tau)\frac{\varphi(t+\delta)(\tau) - \varphi(t)(\tau)}{\delta\sqrt{\pi}}\mathrm{d}\tau\ ,$$

where

$$\frac{\varphi(t+\delta)(\tau) - \varphi(t)(\tau)}{\delta} = \frac{1}{\delta}\left[\frac{e^{-\frac{\tau^2}{4(t+\delta)}}}{\sqrt{t+\delta}} - \frac{e^{-\frac{\tau^2}{4t}}}{\sqrt{t}}\right] \to \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{t}}e^{-\tau^2/4t}\right)$$

as δ tends to 0; t and τ are fixed positive.

Next

$$\frac{\varphi(t+\delta)(\tau)-\varphi(t)(\tau)}{\delta} = \underbrace{\frac{1}{\sqrt{t+\delta}}\frac{e^{-\frac{\tau^2}{4(t+\delta)}}-e^{-\frac{\tau^2}{4t}}}{\delta}}_{①} + \underbrace{e^{-\frac{\tau^2}{4t}}\frac{\frac{1}{\sqrt{t+\delta}}-\frac{1}{\sqrt{t}}}{\delta}}_{②}$$

$$\|②\|_{L^1(0,\infty)} \le \left\|e^{-\frac{\tau^2}{4t}}\right\|_{L^1(0,\infty)} \left|\frac{\sqrt{t}-\sqrt{t+\delta}}{\delta\sqrt{t}\sqrt{t+\delta}}\right| \le \left\|e^{-\frac{\tau^2}{4t}}\right\|_{L^1(0,\infty)} \frac{1}{2t\sqrt{t}} \ .$$

If $1 \ge \delta > 0$ then we extract $\tau^2 e^{-\frac{\tau^2}{4(t+\delta)}}$ from ①; with $\beta(\delta) := \frac{\tau^2}{4(t+\delta)}$ one has

$$\|①\|_{L^1(0,\infty)} \le \frac{1}{\sqrt{t}}\left\|\tau^2 e^{-\frac{\tau^2}{4(t+1)}}\right\|_{L^1(0,\infty)} \left\|\frac{1-e^{-[\beta(0)-\beta(\delta)]}}{\beta(0)-\beta(\delta)}\,\frac{\beta(0)-\beta(\delta)}{\delta\tau^2}\right\|_{L^\infty(0,\infty)} \le$$

$$\frac{1}{\sqrt{t}}\left\|\tau^2 e^{-\frac{\tau^2}{4(t+1)}}\right\|_{L^1(0,\infty)} \left\|\frac{\beta(0)-\beta(\delta)}{\delta\tau^2}\right\|_{L^\infty(0,\infty)} = \frac{1}{4t^2\sqrt{t}}\left\|\tau^2 e^{-\frac{\tau^2}{4(t+1)}}\right\|_{L^1(0,\infty)} .$$

If $\delta < 0$, $t+\delta > 0$ then we extract $\tau^2 e^{-\frac{\tau^2}{4t}}$ from ① and still with

$\beta(\delta) := \frac{\tau^2}{4(t+\delta)}$ one obtains

$$\|①\|_{L^1(0,\infty)} \le \frac{1}{\sqrt{t}} \left\| \tau^2 e^{-\frac{\tau^2}{4t}} \right\|_{L^1(0,\infty)} \left\| \frac{1-e^{-[\beta(\delta)-\beta(0)]}}{\beta(\delta)-\beta(0)} \; \frac{\beta(\delta)-\beta(0)}{\delta\tau^2} \right\|_{L^\infty(0,\infty)} \le$$

$$\le \frac{1}{\sqrt{t}} \left\| \tau^2 e^{-\frac{\tau^2}{4t}} \right\|_{L^1(0,\infty)} \left\| \frac{\beta(\delta)-\beta(0)}{\delta\tau^2} \right\|_{L^\infty(0,\infty)} = \frac{1}{4t^2\sqrt{t}} \left\| \tau^2 e^{-\frac{\tau^2}{4t}} \right\|_{L^1(0,\infty)}.$$

Taking into account that $\|f_{\sqcap\sqcup\sqcap}\|_{L^\infty(0,\infty)} = 2$ and applying the Lebesgue dominated convergence theorem we get (3.9).

Since, the Laplace transform of $\dot{h}_{\text{step}}$ is

$$s\widehat{h_{\text{step}}}(s) - h_{\text{step}}(0) = s\widehat{h_{\text{step}}}(s) = \frac{1}{\cosh\sqrt{s}} = \hat{g}(s)$$

then, by (3.9),

$$g(t) = h^*\mathcal{A}^2 S(t)d = \int_0^\infty f_{\sqcap\sqcup\sqcap}(\tau)\frac{\partial}{\partial t}\left[\frac{1}{\sqrt{t\pi}}e^{-\frac{\tau^2}{4t}}\right]\mathrm{d}\tau =$$

$$= \frac{1}{t^2\sqrt{t\pi}} \int_0^\infty f_{\sqcap\sqcup\sqcap}(\tau) e^{-\frac{\tau^2}{4t}} \left[\frac{\tau^2}{4} - \frac{t}{2}\right] \mathrm{d}\tau =$$

$$\frac{2}{t^2\sqrt{t\pi}} \sum_{n=0}^{\infty} \int_{4n+1}^{4n+3} e^{-\frac{\tau^2}{4t}} \left[\frac{\tau^2}{4} - \frac{t}{2}\right] \mathrm{d}\tau = \frac{2}{t\sqrt{\pi}} \sum_{n=0}^{\infty} \int_{\frac{4n+1}{2\sqrt{t}}}^{\frac{4n+3}{2\sqrt{t}}} e^{-\xi^2} [2\xi^2 - 1] \mathrm{d}\xi$$

$$= \frac{2}{t\sqrt{\pi}} \sum_{n=0}^{\infty} \left[-\xi e^{-\xi^2}\right]_{\frac{4n+1}{2\sqrt{t}}}^{\frac{4n+3}{2\sqrt{t}}} \, ,$$

whence the system *impulse response* satisfies the estimates:

$$\frac{1}{t\sqrt{\pi t}} e^{-1/4t} = \frac{2}{t\sqrt{\pi}} \left[-\xi e^{-\xi^2}\right]_{\frac{1}{2\sqrt{t}}}^{\infty} \geq g(t) \geq$$

$$\geq \max \left\{ \underbrace{\frac{2}{t\sqrt{\pi}} \left[-\xi e^{-\xi^2}\right]_{\frac{1}{2\sqrt{t}}}^{\frac{3}{2\sqrt{t}}}}_{=\frac{1}{t\sqrt{\pi t}} e^{-1/4t}\left(1-3e^{-2/t}\right)} , \frac{1}{t\sqrt{\pi t}} e^{-1/4t} \left(1 - 3e^{-2\pi}\right) \right\} \tag{3.10}$$

from which we conclude that $g(0) = 0$, g is continuous and g is positive on $(0, \frac{1}{\pi}]$. Furthermore g has the series representation

$$g(t) = \sum_{n=0}^{\infty} \left[\frac{4n+1}{t\sqrt{\pi t}} e^{-(4n+1)^2/4t} - \frac{4n+3}{t\sqrt{\pi t}} e^{-(4n+3)^2/4t} \right], \quad t > 0. \quad (3.11)$$

Step 3. The function g satisfies on $[0, \infty)$ the following identity:

$$g(t) \equiv (t\pi)^{-3/2} g\left(\frac{1}{t\pi^2}\right) \quad , \qquad (3.12)$$

as its both sides have the same Laplace transform $\hat{g}$, $\hat{g}(s) = \frac{1}{\cosh\sqrt{s}}$. To justify this assertion we firstly define and auxiliary function $g_{\mathrm{a}}(t) := \frac{1}{t\sqrt{t}} g\left(\frac{1}{t}\right)$ and we compute its Laplace transform $\widehat{g_{\mathrm{a}}}$ using (Bateman et al, 1954, p. 122, Formula (25) with $\nu = -\frac{1}{2}$):

$$\widehat{g_{\mathrm{a}}}(s) = s^{1/4} \int_0^{\infty} u^{-1/4} J_{-1/2}(2\sqrt{us}) \hat{g}(u) \mathrm{d}u \quad ,$$

where $J_{-1/2}$ stands for the Bessel function of the first kind and of order $-\frac{1}{2}$. Since (Dwight, 1961, 804.21) (this identity holds also for complex variable z)

$$J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z$$

we get

$$\widehat{g_{\mathrm{a}}}(s) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{u}} \frac{\cos(2\sqrt{us})}{\cosh\sqrt{u}} \mathrm{d}u = \frac{\sqrt{\pi}}{\cosh(\pi\sqrt{s})} \ ,$$

where the last equality follows from (Dwight, 1961, 861.62 with $m = 2\sqrt{s}$ and $a = 1$).

Secondly, rescaling the time $t \rightsquigarrow t\pi^2$ we shall find with the aid of (Bateman et al, 1954, p. 120, Formula (4) with $a = \pi^2$, $b = 0$) the

Laplace transform of rescaled function

$$g_{\mathrm{r}}(t) = g_{\mathrm{a}}(t\pi^2), \qquad \widehat{g_{\mathrm{r}}}(s) = \frac{1}{\pi^2}\widehat{g_{\mathrm{a}}}\left(\frac{s}{\pi^2}\right) = \pi^{-3/2}\frac{1}{\cosh\sqrt{s}} \ .$$

But

$$g_{\mathrm{a}}(t\pi^2) = t^{-3/2}\pi^{-3} g\left(\frac{1}{t\pi^2}\right) \ ,$$

whence the Laplace transform of $\pi^{3/2}g_{\mathrm{a}}(t\pi^2) = (t\pi)^{-3/2}g\left(\frac{1}{t\pi^2}\right)$ is $\frac{1}{\cosh\sqrt{s}} = \hat{g}(s)$.

Step 4. From (3.11) and (3.12) we obtain an equivalent series representation of g:

$$g(t) = 2\sum_{n=0}^{\infty}\left\{\left(\tfrac{\pi}{2} + 2n\pi\right)e^{-t\left(\frac{\pi}{2}+2n\pi\right)^2} - \left(\tfrac{3\pi}{2} + 2n\pi\right)e^{-t\left(\frac{3\pi}{2}+2n\pi\right)^2}\right\},$$

whilst from (3.10) and (3.12) we get purely exponential estimates for g

$$\begin{aligned}&\pi e^{-t\pi^2/4} \geq g(t) \geq \\ &\geq \max\left\{(1-3e^{-2t\pi^2})\pi e^{-t\pi^2/4}, (1-3e^{-2\pi})\pi e^{-t\pi^2/4}\right\}\end{aligned} \tag{3.13}$$

On $[0, \frac{1}{\pi}]$, (3.10) implies the worse but simpler estimate

$$\frac{1}{t\sqrt{\pi t}}e^{-1/4t} \geq g(t) \geq \frac{1}{t\sqrt{\pi t}}e^{-1/4t}\left(1-3e^{-2\pi}\right)$$

which is however better than (3.13) on the same interval (max in (3.13) is achieved on the second element and $\frac{1}{t\sqrt{\pi t}}e^{-1/4t} \leq \pi e^{-t\pi^2/4}$).

On $[\frac{1}{\pi}, \infty)$, (3.13) implies the worse but simpler estimate

$$\pi e^{-t\pi^2/4} \geq g(t) \geq (1-3e^{-2\pi})\pi e^{-t\pi^2/4}$$

which is however better than (3.10) on the same interval (max in

(3.10) is achieved on the second element and $\frac{1}{t\sqrt{\pi t}}e^{-1/4t} \geq \pi e^{-t\pi^2/4}$). Thus we come to the estimate (3.8) which is very precise as depicted in Figure 3.2. □

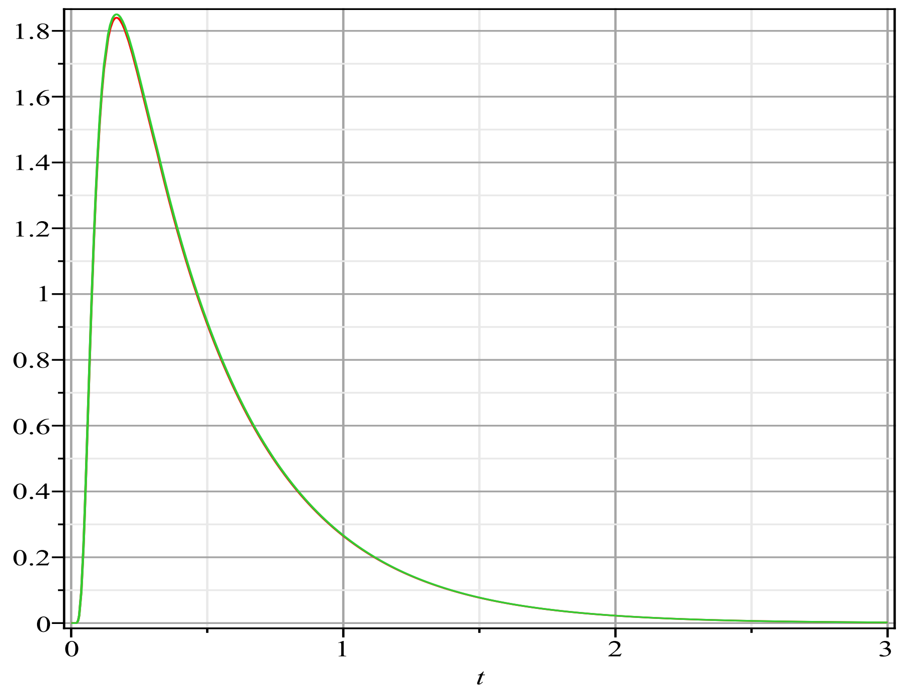

Figure 3.2: Lower and upper estimates of g.

It follows from Proposition 3.1 that Ψd also decays exponentially. Indeed,

$$|(\Psi d)(t)| = -(\Psi d)(t) = \int_t^\infty g(\tau)\mathrm{d}\tau \le \int_t^\infty M e^{-\alpha\tau}\mathrm{d}\tau = \frac{M e^{-\alpha t}}{\alpha} .$$

By Proposition 3.1, we can integrate the last convolution in (3.3) by parts getting an equivalent form of the input–output operator

$$(\mathbb{F}u)(t) = \int_0^t g(t-\tau)u(\tau)\mathrm{d}\tau, \qquad g \in \mathrm{L}^1(0,\infty)$$

and

$$1 = \hat{g}(0) \le |\hat{g}(s)| \le \int_0^\infty g(t)\mathrm{d}t = \|g\|_{\mathrm{L}^1(0,\infty)}$$

whence $\hat{g}(0) = \|\hat{g}\|_{\mathrm{H}^\infty(\mathbb{C}^+)} = \|g\|_{\mathrm{L}^1(0,\infty)} = 1 = \|\mathbb{F}\|_{\mathbf{L}(\mathrm{L}^2(0,\infty))}$ (the Nyquist curve determines spectrum $\sigma(\mathbb{F}) = \overline{\hat{g}(\mathbb{C}^+)}$ of $\mathbb{F}$).

4 Lur'e problem

Consider the *Lur'e system* of automatic feedback control depicted in Figure 4.1.

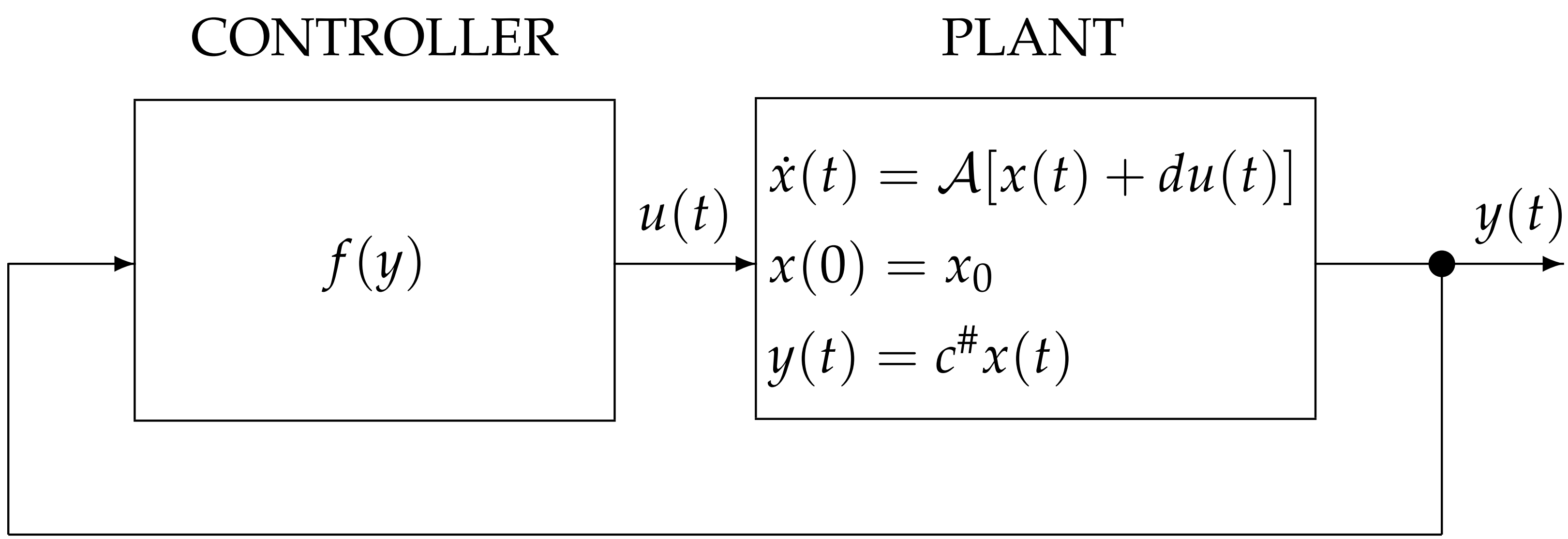

Figure 4.1: The Lur'e control system

Let $f \in W^{1,\infty}(\mathbb{R})$ and $f(0) = 0$. f may represent the static characteristic of an operational amplifier with gain k and level of saturation M,

$$f(y) = \left\{ \begin{array}{cccc} M, & \text{if} & y & \geq & M/k \\ ky, & \text{if} & |y| & \leq & M/k \\ -M, & \text{if} & y & \leq & -M/k \end{array} \right\} .$$

The closed–loop system dynamics reads as

$$\left\{ \begin{array}{l} \dot{x}(t) = \{\mathcal{A}\left[x(t) - f[c^{\#}x(t)]\right]\} \\ x(0) = x_0 \end{array} \right\} . \qquad (4.1)$$

If the Lipschitz constant of f is m then f induces the nonlinear *Nemytskii operator of superposition*

$$(\mathcal{N}y)(t) := f[y(t)] \qquad \text{for almost all } t \geq 0 \ ,$$

satisfying in $L^2(0, \infty)$ the Lipschitz condition with the same

Lipschitz constant m,

$$\|\mathcal{N}y_1 - \mathcal{N}y_2\|^2_{\mathrm{L}^2(0,\infty)} = \int_0^\infty \{f[y_1(t)] - f[y_2(t)]\}^2 \,\mathrm{d}t \leq$$

$$m^2 \int_0^\infty [y_1(t) - y_2(t)]^2 \, dt = m^2 \|y_1 - y_2\|^2_{\mathrm{L}^2(0,\infty)} \quad \forall y_1, y_2 \in \mathrm{L}^2(0,\infty).$$

Inserting the controller equation $u = -\mathcal{N}y$ into (3.7) we get

$$y = \Psi x_0 - \mathbb{F}\mathcal{N}y \ . \tag{4.2}$$

The RHS of (4.2) is clearly a Lipschitz operator with Lipschitz constant $m \|\mathbb{F}\|_{\mathbf{L}(\mathrm{L}^2(0,\infty))} = m\hat{g}(0) = m$. If $m < 1$ then, by *Banach's fixed point theorem*, (4.2) has a unique solution $y^c \in \mathrm{L}^2(0,\infty)$ and, consequently, $u^c \in \mathrm{L}^2(0,\infty)$.

The complete operator–theoretic description of the closed–loop system is depicted in Figure 4.2.

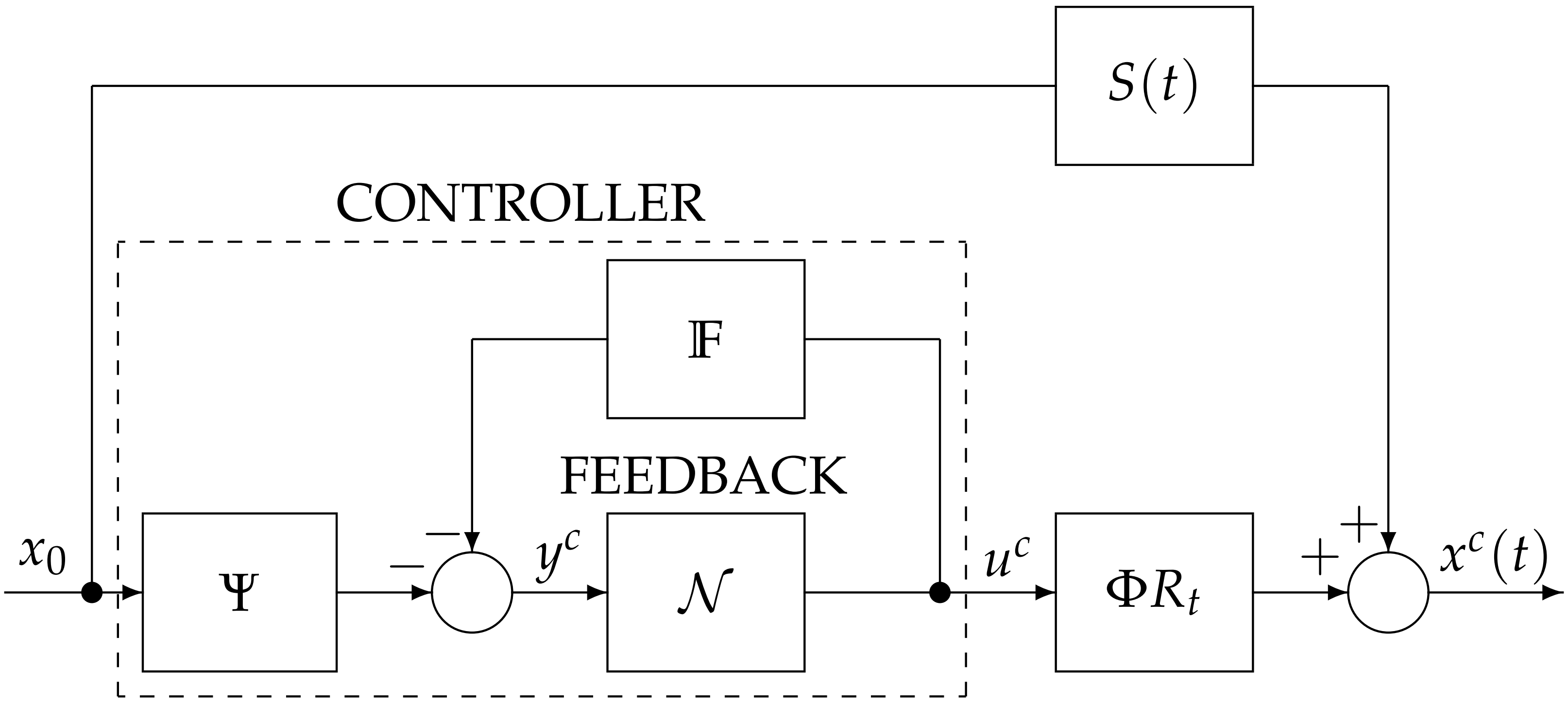

Figure 4.2: The operator–theoretic diagram of the Lur'e control system

In accordance with Figure 4.2 we have

$$x^c(t) = S(t)x_0 - \int_0^t \mathcal{A}S(t-\tau)d\,\underbrace{f[y^c(\tau)]}_{=-u^c}\,\mathrm{d}\tau \quad , \tag{4.3}$$

however at this stage of generality we know only that x is a weak solution in Balakrishnan's sense.

Theorem 4.1. Actually x^c is a weak solution in Ball's sense satisfying $x^c(0) = x_0$. The null equilibrium is globally strongly asymptotically stable **AS**.

Proof. We shall use a more detailed description of the RHS of (4.3) following from Figure 4.2 and the conicity of f, i.e., $|f(y)| \leq m|y|$,

$$\begin{aligned} \|x^c(t)\|_{\mathrm{H}} \leq & \quad \|S(t)x_0\|_{\mathrm{H}} + m\int_0^t \|\mathcal{A}S(t-\tau)d\|_{\mathrm{H}}|y^c(\tau)|\mathrm{d}\tau \leq \\ \leq & \quad \|S(t)x_0\|_{\mathrm{H}} + m\underbrace{\int_0^t \|\mathcal{A}S(t-\tau)d\|_{\mathrm{H}}|\left(\Psi x_0\right)(\tau)|\mathrm{d}\tau}_{❶} + \\ & +m\underbrace{\int_0^t \|\mathcal{A}S(t-\tau)d\|_{\mathrm{H}}|\left(\mathbb{F}u^c\right)(\tau)|\mathrm{d}\tau}_{❷} \leq \end{aligned}$$

Since $\mathrm{L}^2(0,\infty) \star \mathrm{L}^2(0,\infty) \subset \mathrm{BUC}_0[0,\infty)$, and, by Proposition 3.1, $g \in \mathrm{L}^2(0,\infty)$ then, $\mathbb{F}u^c = g \star u^c \in \mathrm{BUC}_0[0,\infty)$ with

$$\|\mathbb{F}u^c\|_{\mathrm{BUC}[0,\infty)} \leq \|g\|_{\mathrm{L}^2(0,\infty)} \|u^c\|_{\mathrm{L}^2(0,\infty)} \leq \|g\|_{\mathrm{L}^2(0,\infty)} m \|y^c\|_{\mathrm{L}^2(0,\infty)} \ ,$$

where, by (4.2),

$$\|y^c\|_{\mathrm{L}^2(0,\infty)} \leq \frac{\|\Psi\|_{\mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty))} \|x_0\|_{\mathrm{H}}}{1 - m\hat{g}(0)}$$

Next, since $\mathrm{L}^1(0,\infty) \star \mathrm{BUC}_0[0,\infty) \subset \mathrm{BUC}_{00}[0,\infty)$ then, ❷ $\in \mathrm{BUC}_{00}[0,\infty)$, where $\mathrm{BUC}_{00}[0,\infty)$ denotes closed subspace of $\mathrm{BUC}_0[0,\infty)$ of functions vanishing at 0, and ❷ is estimated as follows:

$$
\begin{aligned}
❷ \leq \quad & \|\mathcal{A}S(\cdot)d\|_{\mathrm{L}^1(0,\infty)} \|\mathbb{F}u^c\|_{\mathrm{BUC}[0,\infty)} \leq \\
\leq \quad & \|\mathcal{A}S(\cdot)d\|_{\mathrm{L}^1(0,\infty)} \|g\|_{\mathrm{L}^2(0,\infty)} m \frac{\|\Psi\|_{\mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty))} \|x_0\|_{\mathrm{H}}}{1 - m\hat{g}(0)} \ .
\end{aligned}
\tag{4.4}
$$

❶ is estimated by

$$\int_0^t \|\mathcal{A}S(t-\tau)d\|_{\mathrm{H}} \left| \left(\Psi x_0\right)(\tau)\right| \mathrm{d}\tau \leq$$
$$\leq \|x_0\|_{\mathrm{H}} \underbrace{\int_0^t \|\mathcal{A}S(t-\tau)d\|_{\mathrm{H}} \|\mathcal{A}S(\tau)h\|_{\mathrm{H}} \mathrm{d}\tau}_{❸} \quad . \tag{4.5}$$

For ❸ one has by (2.1) and (2.2) with

$$\eta(t) := \sqrt{2}\sqrt{\sqrt{t}+1} \text{ and } \zeta(t) := \frac{\pi}{\sqrt{2}}\sqrt{t\sqrt{t}+1} \ ,$$

$$❸ \leq \int_0^t \zeta(t-\tau)\frac{e^{\lambda_0(t-\tau)}}{(t-\tau)^{3/4}}\eta(\tau)\frac{e^{\lambda_0 \tau}}{\tau^{1/4}}\mathrm{d}\tau \leq$$
$$\leq \zeta(t)\eta(t)e^{\lambda_0 t}\int_0^t \frac{\mathrm{d}\tau}{(t-\tau)^{3/4}\tau^{1/4}} = \zeta(t)\eta(t)e^{\lambda_0 t}\int_0^1 \frac{\mathrm{d}\xi}{(1-\xi)^{3/4}\xi^{1/4}} =$$
$$= \pi^2\sqrt{(t\sqrt{t}+1)(\sqrt{t}+1)}e^{\lambda_0 t}\sqrt{2} \leq \pi^2\sqrt{3}(1+t)e^{-\pi^2 t/4} \ ,$$

where the last integral is the *Beta–function*

$$B(\tfrac{1}{4}, \tfrac{3}{4}) = \Gamma(\tfrac{1}{4})\Gamma(\tfrac{3}{4}) = \pi\sqrt{2} \ .$$

In particular, this implies that the mapping

$$\mathrm{H} \ni x_0 \longmapsto \Psi x_0 \longmapsto \mathcal{A}S(\cdot)d \star \Psi x_0 \in \mathrm{L}^\infty(0,\infty;\mathrm{H})$$

is a bounded everywhere defined linear operator. But for $x_0 \in D(\mathcal{A})$ one has by **EXS**: $\Psi x_0 \in \mathrm{BUC}_0[0,\infty)$, whence in this case the value of this operator is in $\mathrm{BUC}_0[0,\infty)$ as $\mathrm{L}^1(0,\infty) \star \mathrm{BUC}_0[0,\infty) \subset \mathrm{BUC}_{00}([0,\infty);\mathrm{H})$. But $\mathrm{BUC}_{00}([0,\infty);\mathrm{H})$, the subspace of $\mathrm{BUC}_0([0,\infty);\mathrm{H})$ vanishing at 0 is a closed subspace of $\mathrm{L}^\infty(0,\infty;\mathrm{H})$ (partially confirmed by (Reed and Simon, 1980, pp. 67 - 68)) and $\overline{D(\mathcal{A})} = \mathrm{H}$.

Finally $\mathrm{H} \ni x_0 \longmapsto x^c \in \mathrm{BUC}_0([0,\infty);\mathrm{H})$ and the null equilibrium point is globally strongly asymptotically stable (stability follows from estimates (4.4) and (4.5)). Moreover $x^c(0) = x_0$, and x^c is being lifted to weak solution in Ball's sense. □

5 Discussion and conclusions

- It is possible to use Theorem 3.1, to get global asymptotic stability with admissible d but this requires changing the state space to be $\mathrm{H}^{-1/4}$ - the completion of H with respect to the norm induced by the scalar product

$$\langle x_1, x_2 \rangle_{\mathrm{H}_{-\alpha}} := \langle (-A)^{-\alpha} x_1, (-A)^{-\alpha} x_2 \rangle_{\mathrm{H}}$$

  (Grabowski and Callier, 2001b, Appendix C), however then, final stability results loose its power in comparison with $\mathrm{L}^2(0,1)$–topology.

- We have proved that despite the fact that the factor control vector d for $\mathfrak{RC}$–electric transmission line is not admissible we can get asymptotic stability employing parabolic regularity of the problem. The same method (used to prove Theorem 4.1) is applicable while constructing the lq–controller problem for this line. The results are in progress and will be presented elsewhere. They offer an alternative for the *bootstrapping arguments* proposed in (Lasiecka and Triggiani, 2000), used to show that the closed–loop system with lq–controller is well–defined and stable. (Lasiecka and Triggiani, 2000) use dual system, which is not required here.

- An open problem is to examine whether the output equation (4.2) has a solution under weaker assumptions imposed on f.
- In (Grabowski and Callier, 2011) a nonlinear semigroup approach jointly with certain lq–problem and Lyapunov's method has been applied to get the circle criterion for the Lur'e system. This method relies on subsequent application of

the following results.

Theorem 5.1. Assume that there exist finite $k_1, k_2 \in \mathbb{R}$ such that:

(i) $f : \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the incremental sector condition

$$k_1 < \frac{f(y_1) - f(y_2)}{y_1 - y_2} < k_2 \quad \forall y_1, y_2 \in \mathbb{R},\ f(0) = 0, \quad (5.1)$$

(ii) with

$$q := k_1 k_2, \quad e := -\frac{k_1 + k_2}{2} + k_1 k_2 c^\# d \ ,$$
$$\delta := (1 - k_1 c^\# d)(1 - k_2 c^\# d) = 1 + 2ec^\# d - q\left(c^\# d\right)^2 \geq 0$$

the *linear operator inequality*

$$\mathfrak{M} := \begin{bmatrix} \mathcal{A}^{-*}\mathcal{H} + \mathcal{H}\mathcal{A}^{-1} - qhh^* & \mathcal{H}d - eh \\ d^*\mathcal{H} - eh^* & -\delta \end{bmatrix} \leq 0 \quad (5.2)$$

holds for some $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* \geq \eta I > 0$,

(iii) the transfer function $\hat{g}$ is *regular*, i.e., $\lim_{s \to \infty, s \in \mathbb{R}} \hat{g}(s) = 0$.

Then, the (nonlinear closed–loop) operator

$$\begin{aligned} \mathcal{A}^c x := &\quad \mathcal{A}\left[x - df(c^{\#}x)\right], \\ D(\mathcal{A}^c) = &\quad \left\{x \in D(c^{\#}) \subset \mathrm{H} :\ x - df(c^{\#}x) \in D(\mathcal{A})\right\}, \end{aligned}$$

is *dissipative* with respect to an *equivalent* scalar product $\langle x_1, x_2 \rangle_{\mathcal{H}} := \langle x_1, \mathcal{H}x_2 \rangle_{\mathrm{H}}$ and it satisfies the range condition

$$R(\lambda I - \mathcal{A}^c) = \mathrm{H} \qquad \forall \lambda > 0 \ .$$

Furthermore, $\mathcal{A}$ is *demiclosed* and *densely defined*.

Finally, for $x_0 \in \mathcal{D}(\mathcal{A})$, (4.1) has a unique strong solution $x \in \mathrm{W}^{1,\infty}([0,\infty), \mathrm{H})$ (the Sobolev space of absolutely continuous functions $x(t) \in \mathrm{H}$ with both x and $\dot{x}$ in $\mathrm{L}^{\infty}((0,\infty), \mathrm{H})$) and the output y of the Lur'e feedback system of Figure 4.1 is in $L^\infty(0,\infty)$.

Remark 5.1. Weak inequality can be taken in (5.1). $\hat{g}$ is clearly a regular transfer function. $\mathcal{H} > 0$ still defines a scalar product which however induces a weaker norm (topology) in H. But the statement of Theorem 5.1 remains true in H equipped with this scalar product.

Lemma 5.1. Assume that the observation functional $c^\#$ is admissible and exactly observable, $\hat{g} \in H^\infty(\mathbb{C}^+)$ and there exist $k_1, k_2 > k_1$ such that the *coercive frequency–domain inequality of the circle–type* holds,

$$1 + (k_1 + k_2)\,\mathrm{Re}\,[\hat{g}(j\omega)] + k_1 k_2\,|\hat{g}(j\omega)|^2 \geq \eta > 0 \quad \forall \omega \in \mathbb{R}\ .$$

Assume that $q \leq 0$. Then, there exists $\mathcal{H} \in \mathbf{L}(H)$, $\mathcal{H} = \mathcal{H}^* \geq \eta I > 0$, satisfying the *Riccati operator equation*

$$\mathcal{A}^{-*}\mathcal{H} + \mathcal{H}\mathcal{A}^{-1} - qhh^* + \frac{1}{\delta}(-eh + \mathcal{H}d)(-eh + \mathcal{H}d)^* = 0 \quad (5.3)$$

and therefore inequality (5.2).

Remark 5.2. If $c^{\#}$ is merely *approximate observable*, i.e., $\ker \Psi = \{0\}$ then a weak statement holds true, namely, there exists $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} = \mathcal{H}^* > 0$, satisfying (5.3). It can be shown that here the kernel of Ψ is trivial.

Theorem 5.2. Let the assumptions of Lemma 5.1 hold and let for the given k_1 and $k_2 \in \mathbb{R}$ the incremental sector condition (5.1) be satisfied. Assume that the transfer function $\hat{g}$ is regular. Moreover, let d be an admissible factor control vector. Then the null equilibrium of (4.1) is globally strongly asymptotically stable (**GAS**).

Remark 5.3. The proof given in (Grabowski and Callier, 2011, pp. 3078-3081) relies on employing the quadratic form dictated by $\mathcal{H}$ as a Lyapunov functional which enables us to get y^c, $u^c \in \mathrm{L}^2(0, \infty)$. Then the statement follows from the latter and Theorem 3.1, provided that d is admissible. If d is not admissible we use Theorem 4.1 of the present presentation.

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