ADMISSIBILITY of OBSERVATION OPERATORS.
Pt. II: Time–delay systems. Semigroup criteria.

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AFA Seminar, 11^{th} **and** 25^{th} **January,** 29^{th} **November 2023.**

Last opening or modification: November 25, 2023 at 12:15

Abstract

This presentation contains a survey of selected papers on admissibility of observation operators as well as some new criteria ensuring admissibility.

The results are illustrated by some comparative examples.

1 Basic theory

The contents of this section has been presented previously, see also (Grabowski, 2021, Sections 2 and the beginning of Section 3) or (Grabowski, 2022, pp. 350-370).

Consider a class of control systems with observation governed by the model in factor form

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \mathcal{A}\left[x(t)+\mathcal{D}u(t)\right] \\ x(0) & = & x_0 \\ y(t) & = & \mathcal{C}x(t) \end{array}\right\}, \tag{1.1}$$

where the linear *state operator* $\mathcal{A} : (D(\mathcal{A}) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ acts on a Hilbert *state space* H with scalar product $\langle \cdot, \cdot \rangle_{\mathrm{H}}$ and is invertible with $\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$.

$\mathcal{C} : (D(\mathcal{C}) \subset \mathrm{H}) \longrightarrow \mathrm{Y}$ is an *observation (output) operator*, such that $D(\mathcal{A}) \subset D(\mathcal{C})$ and $H := \mathcal{C}\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H}, \mathrm{Y})$[a]. Here Y denotes an *output space* which is a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_{\mathrm{Y}}$.

$\mathcal{D} \in \mathbf{L}(\mathrm{U}, \mathrm{H})$ with range $R(\mathcal{D}) \subset D(\mathcal{C}), \mathcal{C}\mathcal{D} \in \mathbf{L}(\mathrm{U}, \mathrm{Y})$ is a *factor control operator* and U stands for a space of controls which is also a

[a]Since $\mathcal{A}$ is boundedly invertible the norms $\|\mathcal{A}x\|_{\mathrm{H}}$, $\|x\|_{\mathrm{H}} + \|\mathcal{A}x\|_{\mathrm{H}}$ are equivalent, whence without loss of generality: $\mathcal{C} = H\mathcal{A}$, $H \in \mathbf{L}(\mathrm{H}, \mathrm{Y})$.

Hilbert space with scalar product $\langle\cdot,\cdot\rangle_{\mathrm{U}}$.

The function

$$G(s) = \mathcal{C}\mathcal{A}(sI-\mathcal{A})^{-1}\mathcal{D} = s\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{D} - \mathcal{C}\mathcal{D}, \quad s \in \rho(\mathcal{A})$$

is call the *transfer function* of the system (1.1).

1.1 Semigroups and state operators

Definition 1.1. A family $\{S(t)\}_{t\geq 0} \subset \mathbf{L}(\mathrm{H})$ is a C_0–*semigroup* on H if **(i)** $S(0) = I$, $S(t+\tau) = S(t)S(\tau)$ for $t, \tau \geq 0$ and **(ii)** $S(t)x_0 \to x_0$ as $t \to 0$ for every $x_0 \in \mathrm{H}$.

$\{S(t)\}_{t\geq 0}$ is *uniformly bounded* if there exist $M \geq 1$ such that

$$\|S(t)x_0\|_{\mathrm{H}} \leq M \qquad \forall t \geq 0. \tag{1.2}$$

$\{S(t)\}_{t\geq 0}$ is *asymptotically stable* (**AS**) if $\|S(t)x_0\|_{\mathrm{H}} \longrightarrow 0$ as $t \to \infty$,

$x_0 \in \mathrm{H}$.

$\{S(t)\}_{t\geq 0}$ is *exponentially stable* (**EXS**) if there exist $M \geq 1$, $\alpha > 0$ such that

$$\|S(t)\|_{\mathbf{L}(\mathrm{H})} \leq Me^{-\alpha t} \qquad \forall t \geq 0. \tag{1.3}$$

The *generator* of a C_0 semigroup $\{S(t)\}_{t\geq 0}$ is defined by

$$\mathcal{A}x_0 = \lim_{h\to 0} \frac{1}{h}[S(h)x_0 - x_0],$$
$$D(\mathcal{A}) = \{x_0 \in \mathrm{H} : \exists \lim_{h\to 0} \frac{1}{h}[S(h)x_0 - x_0]\}.$$

Theorem 1.1 (Hille–Phillips–Yosida)**.** A linear operator $\mathcal{A} : (D(\mathcal{A}) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ generates C_0–semigroup $\{S(t)\}_{t\geq 0}$ satisfying the growth estimate $\|S(t)\|_{\mathbf{L}(\mathrm{H})} \leq Me^{\omega t}$ for $t \geq 0$ and some $M \geq 1$, $\omega \in \mathbb{R}$ (by the principle of boundedness every C_0–semigroups satisfies this estimate) iff $\mathcal{A}$ is closed densely defined and its resolvent $(sI - \mathcal{A})^{-1}$ satisfies the estimate

$$\|(sI - \mathcal{A})^{-n}\|_{\mathbf{L}(\mathrm{H})} \leq \frac{M}{(s-\omega)^n} \qquad \forall s > \omega, \quad \forall n \in \mathbb{N}$$

For a good sufficient generation criterion – see (Walker, 1980).

Theorem 1.2 (Walker)**.** Let H be a Hilbert space with scalar product $\langle \cdot, \cdot \rangle_{\mathrm{H}}$. Assume that $\mathcal{A} : \ (D(\mathcal{A}) \subset \mathrm{H}) \longrightarrow \mathrm{H}$ is a linear operator for which the following conditions holds:

(i) there exists $\lambda_0 > 0$ such that $\mathcal{R}(\lambda I - \mathcal{A}) = \mathrm{H}$ for $\lambda > \lambda_0$,

(ii) there exist $\omega \in \mathbb{R}$ and an *equivalent* scalar product $\langle \cdot, \cdot \rangle_e$ in H such that $\mathcal{A}$ is *ω-dissipative with respect to* $\langle \cdot, \cdot \rangle_e$, i.e.,

$$\langle \mathcal{A}x, x \rangle_e + \langle x, \mathcal{A}x \rangle_e \leq 2\omega \|x\|_e^2 \qquad \forall x \in D(\mathcal{A}).$$

Then $\mathcal{A}$ generates C_0-semigroup $\{S(t)\}_{t \geq 0}$ on H satisfying the estimate

$$\|S(t)x\|_e \leq e^{\omega t}\|x\|_e \qquad \forall t \geq 0 \quad \forall x \in \mathrm{H}. \tag{1.4}$$

Theorem 1.3 (Prüss-Huang-Weiss). A C_0 – semigroup generated by $\mathcal{A}$ is **EXS** iff $s \longmapsto (sI - \mathcal{A})^{-1}$ is in the Hardy class $\mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{H}))$, $\mathbb{C}^+ = \{s \in \mathbb{C} : \operatorname{Re} s > 0\}$.

In a Hilbert space H: $\mathcal{A}$ generates a semigroup $\{S(t)\}_{t\geq 0}$/**EXS** iff $\mathcal{A}^*$ generates semigroup $\{S^*(t)\}_{t\geq 0}$/**EXS**. Then H and $\mathcal{A}$ are called the *state space* and *state operators*, respectively.

The resolvent $s \longmapsto (sI - \mathcal{A})^{-1}x_0$ is the *Laplace transform* of $t \mapsto S(t)x_0$. In particular, if $\{S(t)\}_{t\geq 0}$ is **EXS** then, by (1.3), the half–plane $\{s \in \mathbb{C} : \operatorname{Re} s > -\alpha\}$ is contained in $\rho(\mathcal{A})$ – the resolvent set of $\mathcal{A}$ which, in particular, implies that $\mathcal{A}$ is invertible with $\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H})$.

Definition 1.2. Let $x_0 \in \mathrm{H}$ and $u \in \mathrm{L}^2(0, \infty; \mathrm{U})$. A continuous vector valued function $t \mapsto x(t) \in \mathrm{H}$ is called a *weak solution* of (1.1) if $x(0) = x_0$ and x satisfies (1.1) in a *weak sense*, i.e., the function $t \mapsto \langle x(t), w\rangle_{\mathrm{H}}$ is absolutely continuous and for almost all $t \geq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), w\rangle_{\mathrm{H}} = \langle x(t), \mathcal{A}^* w\rangle_{\mathrm{H}} + \langle \mathcal{D}u(t), \mathcal{A}^* w\rangle_{\mathrm{H}}, \quad w \in D(\mathcal{A}^*).$$

Theorem 1.4 (Ball)**.** A linear operator $\mathcal{A}$ generates a C_0–semigroup $\{S(t)\}_{t\geq 0}$ on H iff $\mathcal{A}$ is closed densely defined and for each $x_0 \in \mathrm{H}$ there exists a unique weak solution of (1.1) with $\mathcal{D} = 0$ and $\mathcal{C} = 0$.

It is known that if X is a Hilbert space then

$$\mathcal{L}_{\mathrm{X}} f = f', \quad D(\mathcal{L}_{\mathrm{X}}) = \mathrm{W}^{1,2}([0,\infty);\mathrm{X}) := \\ \left\{ f \in \mathrm{L}^2(0,\infty;\mathrm{X}) : f' \in \mathrm{L}^2(0,\infty;\mathrm{X}) \right\} \subset \mathrm{C}([0,\infty);\mathrm{X})$$

generates the C_0–semigroup $\{T_{\mathrm{X}}(t)\}_{t\geq 0}$ of *left–shifts* on $\mathrm{L}^2(0,\infty;\mathrm{X})$,

$$(T_{\mathrm{X}}(t) f)(\tau) := f(t+\tau), \qquad t \geq 0,$$

whilst its adjoint $\mathcal{L}_{\mathrm{X}}^{*} := \mathcal{R}_{\mathrm{X}}$,

$$\mathcal{R}_{\mathrm{X}} f = -f', \quad D(\mathcal{R}_{\mathrm{X}}) = \mathrm{W}_0^{1,2}([0,\infty);\mathrm{X})$$
$$\mathrm{W}_0^{1,2}([0,\infty);\mathrm{X}) := \left\{ f \in \mathrm{W}^{1,2}([0,\infty);\mathrm{X}) : f(0) = 0 \right\}.$$

generates adjoint C_0–semigroup of *right–shifts* on $\mathrm{L}^2(0,\infty;\mathrm{X})$,

$$\left(T_{\mathrm{X}}^{*}(t) f\right)(\tau) := \left\{ \begin{array}{ccc} f(\tau - t) & \text{if} & \tau \geq t \\ 0 & \text{if} & 0 \leq \tau < t \end{array} \right\}, \ t \geq 0. \tag{1.5}$$

1.2 Admissible observation operators

Define $\mathcal{Z} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{Y}))$,

$$\left(\mathcal{Z} x_0\right)(t) := H S(t) x_0 \quad \left[\Leftrightarrow \mathcal{Z}^* f = \int_0^\infty S^*(t) H^* f(t) \mathrm{d}t \right].$$

The operator, called the *observability map*,

$$\mathbf{\Psi} := \mathcal{L}_{\mathrm{Y}}\mathcal{Z}, \qquad D(\mathbf{\Psi}) = \{x \in \mathrm{H} : \mathcal{Z}x \in D(\mathcal{L}_{\mathrm{Y}})\}$$

is closed and densely defined, with $\mathbf{\Psi}|_{D(\mathcal{A})} = \mathcal{Z}\mathcal{A}$, and therefore it has closed and densely defined adjoint operator

$$\mathbf{\Psi}^* = \mathcal{A}^*\mathcal{Z}^*, \quad D(\mathbf{\Psi}^*) = \{y \in \mathrm{L}^2(0,\infty;\mathrm{Y}) : \mathcal{Z}^*y \in D(\mathcal{A}^*)\},$$

and $\mathbf{\Psi}^*|_{D(\mathcal{R}_{\mathrm{Y}})} = \mathcal{Z}^*\mathcal{R}_{\mathrm{Y}}$.

Definition 1.3. $\mathcal{C}$ is an admissible *observation (output) operator*[b] if $\mathbf{\Psi} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{Y}))$.

Here $\mathbf{\Psi} \in \mathbf{L}(\mathrm{H}, \mathrm{L}^2(0,\infty;\mathrm{Y}))$ can be replaced by $\mathbf{\Psi}$ is bounded or, by the closed graph theorem, $R(\mathcal{Z}) \subset D(\mathcal{L}_{\mathrm{Y}})$).

Lemma 1.1. If $\mathcal{C}$ is admissible then $\mathbf{\Psi}$ is also a linear densely

[b] To be more precise *infinite–time admissible*. It is shown in (Grabowski, 1995, Lemma 1.1) that *finite–time admissibility* is an equivalent concept under the assumption of **EXS**.

defined and *bounded* operator from H into $\mathrm{L}^1(0,\infty;\mathrm{Y})$.

This result is proved in (Grabowski, 2007, Lemma 2.1).

1.3 Admissible control operators

Define $\mathcal{W} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}),\mathrm{H})$

$$\mathcal{W}f := \int_0^\infty S(t)\mathcal{D}f(t)\mathrm{d}t \quad \left[\Leftrightarrow (\mathcal{W}^* x_0)(t) = \mathcal{D}^* S^*(t) x_0\right].$$

The operator, called the *reachability map*,

$$\mathbf{\Phi} := \mathcal{A}\mathcal{W},\ D(\mathbf{\Phi}) = \{u \in \mathrm{L}^2(0,\infty;\mathrm{U}) : \mathcal{W}u \in D(\mathcal{A})\}$$

is closed and densely defined, with $\mathbf{\Phi}|_{D(\mathcal{R}_\mathrm{U})} = \mathcal{W}\mathcal{R}_\mathrm{U}$, and therefore it has closed and densely defined adjoint operator

$$\mathbf{\Phi}^* = \mathcal{L}_\mathrm{Y}\mathcal{W}^*,\ D(\mathbf{\Phi}) = \{x \in \mathrm{H} :\ \mathcal{W}^* x \in D(\mathcal{L}_\mathrm{U})\},$$

with $\mathbf{\Phi}^*|_{D(\mathcal{A}^*)} = \mathcal{W}^* \mathcal{A}^*$.

Definition 1.4. $\mathcal{D}$ is an admissible *factor control operator* if $\mathbf{\Phi} \in \mathbf{L}(\mathrm{L}^2(0, \infty; \mathrm{U}), \mathrm{H})$.

Here $\mathbf{\Phi} \in \mathbf{L}(\mathrm{L}^2(0, \infty; \mathrm{U}), \mathrm{H})$ can be replaced by $\mathbf{\Phi}$ is bounded or, by the closed graph theorem, $R(\mathcal{W}) \subset D(\mathcal{A})$.

Using duality arguments, we can state the following result (Grabowski and Callier, 1999).

Lemma 1.2. $\mathcal{D}$ is an admissible factor control operator iff $\mathcal{D}^* \mathcal{A}^*$ is an admissible observation operator with respect to the semigroup $\{S^*(t)\}_{t \geq 0}$.

The admissibilty notion is widely surveyed in the paper by (Jacob and Partington, 2004).

2 Lyapunov criterion of admissibility

This criterion has been proposed and developed in (Grabowski, 1983a,b, 1990, 1991, 1997, 1999, 2022) especially for a class of time–delay systems of neutral type.

Theorem 2.1. $\mathcal{C}$ is admissible iff there exists $\mathcal{H} = \mathcal{H}^* \in \mathbf{L}(\mathrm{H})$, $\mathcal{H} \geq 0$, and $\mathcal{H}$ satisfies the *Lyapunov operator equation*[c]

$$\langle \mathcal{A}x, \mathcal{H}z \rangle_{\mathrm{H}} + \langle x, \mathcal{H}\mathcal{A}z \rangle_{\mathrm{H}} = -\langle \mathcal{C}x, \mathcal{C}z \rangle_{\mathrm{Y}} \qquad \forall x, z \in D(\mathcal{A}) \qquad (2.1)$$

If the solution to (2.1) is unique then it is called the *Gramian of observability* with infinite horizon of observation.

Theorem 2.2. Let $\mathcal{C}$ be admissible. If the semigroup $\{S(t)\}_{t \geq 0}$ is **AS** then (2.1) has the unique solution.

Definition 2.1. Let H_1 and H_2 be Hilbert spaces with scalar

[c] If, in addition, $\mathcal{C}$ is *closed* then $\mathcal{C}$ has closed densely defined adjoint operator $\mathcal{C}^*$ and we have $\mathcal{H}D(\mathcal{A}) \subset D(\mathcal{A}^*)$ while (2.1) reduces to $\mathcal{A}^*\mathcal{H}z + \mathcal{H}\mathcal{A}z = -\mathcal{C}^*\mathcal{C}z$.

products $\langle \cdot, \cdot \rangle_{\mathrm{H}_1}$, $\langle \cdot, \cdot \rangle_{\mathrm{H}_2}$, respectively. An operator $\mathcal{T} \in \mathbf{L}(\mathrm{H}_1, \mathrm{H}_2)$ is called a *compact operator* if $\langle f_n, g \rangle_{\mathrm{H}_1} \longrightarrow \langle f, g \rangle_{\mathrm{H}_1}$ for all $g \in \mathrm{H}_1$ implies $\|\mathcal{T} f_n - \mathcal{T} f\|_{\mathrm{H}_2} \longrightarrow 0$ as $n \to \infty$. $\mathcal{T}$ is called a *Hilbert–Schmidt* operator (**HS**) if there exists (equivalently, for every) ONB $\{e_k\}_{k \in \mathbb{N}}$ in H_1 such that $\sum_{k=1}^{\infty} \|\mathcal{T} e_k\|_{\mathrm{H}_2}^2 < \infty$ (the sum does not depend on a choice of ONB). $\mathcal{T}$ is called a *nuclear* operator if for any ONBs $\{e_k\}_{k \in \mathbb{N}}$ in H_1 and $\{f_k\}_{k \in \mathbb{N}}$ in H_2 there holds $\sum_{k=1}^{\infty} \left| \langle \mathcal{T} e_k, f_k \rangle_{\mathrm{H}_2} \right| < \infty$.

It can be shown with an aid of the Cauchy–Schwarz inequality that any nuclear operator is a **HS** operator and that a composition of any two **HS** operators is a nuclear operator (it is less obvious that conversely, every nuclear operator is a composition of two **HS** operators).

Theorem 2.3. Suppose that $\mathrm{Y} = \mathbb{R}$ and $\mathcal{C}$ is such that the estimate

$$|\mathcal{C}S(t)x_0| \leq k(t)\,\|x_0\|_{\mathrm{H}} \qquad \forall x_0 \in D(\mathcal{A}) \text{ and almost all } t \geq 0 \tag{2.2}$$

holds for some $k \in \mathrm{L}^2(0,\infty)$. Then (2.1) has a solution $\mathcal{H} \in \mathbf{L}(\mathrm{H})$, with $\mathcal{H} = \mathcal{H}^*$ and $\mathcal{H} \geq 0$, and $\mathcal{H}$ is a nuclear operator.

3 What was presented in Pt. I.

1. Translation semigroups and admissibilty
2. Introduction to a class of time–delay systems

4 To be presented

1. Corrections concerning translation semigroups on a finite

space interval

2. Admissible observations for time–delay systems
3. Two general spectral criteria of admissibility

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