THE LQ–CONTROLLER SYNTHESIS PROBLEM FOR INFINITE–DIMENSIONAL SYSTEMS IN FACTOR FORM. PART I ABSTRACT

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1. Introduction

Consider a control system governed by the model in factor form

$$\left\{\begin{array}{rcl} \dot{x}(t) & = & \mathcal{A}\left[x(t)+\mathcal{D}u(t)\right] \\ y(t) & = & \mathcal{C}x(t) \end{array}\right\} \tag{1.1}$$

where the *state operator* $\mathcal{A}$ generates an **EXS** semigroup $\{S(t)\}_{t\geq 0}$ on a Hilbert space H with scalar product $\langle\cdot,\cdot\rangle_{\mathrm{H}}$, , i.e., there exit $M \geq 1$ and $\alpha > 0$ such that

$$\|S(t)x_0\|_{\mathrm{H}} \leq Me^{-\alpha t}\|x_0\|_{\mathrm{H}} \qquad \forall t \geq 0, \quad \forall x_0 \in \mathrm{H} \ ; \tag{1.2}$$

$\mathcal{C} : (D(\mathcal{C}) \subset \mathrm{H}) \longrightarrow \mathrm{Y}$, $\mathcal{C}\mathcal{A}^{-1} \in \mathbf{L}(\mathrm{H},\mathrm{Y})$, $\mathcal{D} \in \mathbf{L}(\mathrm{U},\mathrm{H})$ with $R(\mathcal{D}) \subset D(\mathcal{C})$, $\mathcal{C}\mathcal{D} \in \mathbf{L}(\mathrm{U},\mathrm{Y})$ and $u \in \mathrm{L}^2(0,\infty;\mathrm{U})$. Here Y and U are Hilbert spaces with scalar products $\langle\cdot,\cdot\rangle_{\mathrm{Y}}$ and $\langle\cdot,\cdot\rangle_{\mathrm{U}}$, respectively.

The LQ–optimal control problem with infinite time horizon is to minimize the quadratic integral performance index

$$J(x_0,u) = \int_0^\infty \begin{bmatrix} y(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & N \\ N^* & R \end{bmatrix} \begin{bmatrix} y(t) \\ u(t) \end{bmatrix} \mathrm{d}t, \tag{1.3}$$

where $Q = Q^* \in \mathbf{L}(\mathrm{Y})$, $N \in \mathbf{L}(\mathrm{U},\mathrm{Y})$ and $R = R^* \in \mathbf{L}(\mathrm{U})$, on trajectories of (1.1).

To solve this problem we shall assume that:

(**A1**) $\mathcal{C}$ is an admissible *observation operator*, i.e., $\mathcal{R}(\mathcal{Z}) \subset D(\mathcal{L}_{\mathrm{Y}})$, where

$\mathcal{Z} \in \mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty;\mathrm{Y}))$, $(\mathcal{Z}x_0)(t) := \mathcal{C}\mathcal{A}^{-1}S(t)x_0$; $\mathcal{L}_{\mathrm{Y}}f = f'$, $D(\mathcal{L}_{\mathrm{Y}}) = \mathrm{W}^{1,2}([0,\infty);\mathrm{Y})$.

Since $\mathcal{L}_{\mathrm{Y}}$ generates the *semigroup of left–shifts* on $\mathrm{L}^2(0,\infty;\mathrm{Y})$ then, by the closed–graph theorem, the admissibility of $\mathcal{C}$ holds iff

$$\Psi = \mathcal{L}_{\mathrm{Y}}\mathcal{Z} \in \mathbf{L}(\mathrm{H},\mathrm{L}^2(0,\infty;\mathrm{Y})) \ ,$$

and Ψ is called the system *observability map*.

(**A2**) $\mathcal{D}$ is an admissible *factor control operator*, i.e., $\mathcal{R}(\mathcal{W}) \subset D(\mathcal{A})$, where

$$\mathcal{W} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}),\mathrm{H}), \qquad \mathcal{W}f := \int_0^\infty S(t)\mathcal{D}f(t)\mathrm{d}t \ .$$

By the closed–graph theorem, the admissibility of $\mathcal{D}$ holds iff

$$\Phi = \mathcal{A}\mathcal{W} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}),\mathrm{H}) \ ,$$

and Φ is the system *reachability map*.

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(**A3**) The system *transfer function* $\hat{G}(s) := s\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{D} - \mathcal{C}\mathcal{D}$ satisfies[1]

$$\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+, \mathbf{L}(\mathrm{U},\mathrm{Y})) \ .$$

If the latter is met then the *input–output operator*, given by

$$(\mathbb{F}u)(t) := \frac{\mathrm{d}}{\mathrm{d}t}\int_0^t (\Psi\mathcal{D})(t-\tau)u(\tau)\mathrm{d}\tau - (\mathcal{C}\mathcal{D})u(t) \ .$$

satisfies $\mathbb{F} \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}), \mathrm{L}^2(0,\infty;\mathrm{Y}))$. This follows from the Paley–Wiener theorem upon taking the Laplace transforms: $(\widehat{\mathbb{F}u})(s) = \hat{G}(s)\hat{u}(s)$, $s \in \mathbb{C}^+$.

Let us remark that since $\hat{G}(s) = s^2\left(\mathcal{C}\mathcal{A}^{-1}\right)(sI-\mathcal{A})^{-1}\mathcal{D} - s\left(\mathcal{C}\mathcal{A}^{-1}\right)\mathcal{D} - \mathcal{C}\mathcal{D}$ then, by **EXS**, $\hat{G}$ is analytic on a set containing $\overline{\mathbb{C}^+}$, which jointly with (**A3**) yields $\|\hat{G}(j\omega)\|_{\mathbf{L}(\mathrm{U},\mathrm{Y})} \leq \|\hat{G}\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathbf{L}(\mathrm{U},\mathrm{Y}))}$ for every $\omega \in \mathbb{R}$.

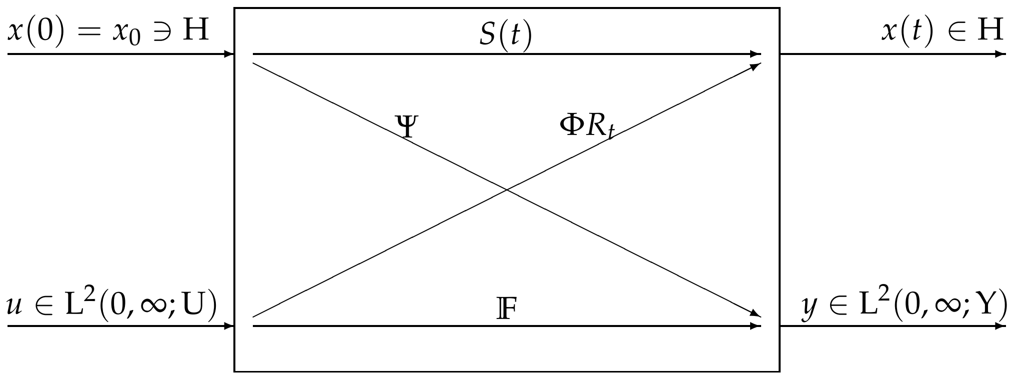

FIGURE 1.1. Basic control–theoretic operators and their action.

Our aim is to prove the following theorem.

Theorem 1.1. Let $\mathcal{A}$ generates an **EXS** semigroup on H. Assume that the assumptions (**A1**), (**A2**) and (**A3**) hold. If the operator

$$\mathcal{R} := R + N^*\mathbb{F} + \mathbb{F}^*Q\mathbb{F} + \mathbb{F}^*N = \mathcal{R}^* \in \mathbf{L}(\mathrm{L}^2(0,\infty;\mathrm{U}))$$

is coercive then there exists a unique optimal control, given by

$$u = -\mathcal{R}^{-1}(\mathbb{F}^*Q + N^*)\Psi x_0 \in \mathrm{L}^2(0,\infty;\mathrm{U}) \ . \tag{1.4}$$

on which the performance index J achieves its minimum. The minimal value is

$$J(x_0) = x_0^*\left[\Psi^*Q\Psi - \Psi^*(Q\mathbb{F}+N)\mathcal{R}^{-1}(\mathbb{F}^*Q+N^*)\Psi\right]x_0 \ . \tag{1.5}$$

Let

$$N_- := N - Q(\mathcal{C}\mathcal{D}), \qquad R_- := R - (\mathcal{C}\mathcal{D})^*N - N^*(\mathcal{C}\mathcal{D}) + (\mathcal{C}\mathcal{D})^*Q(\mathcal{C}\mathcal{D}) = R_-^* \ .$$

[1] Recall that $\hat{G} \in \mathrm{H}^\infty(\mathbb{C}^+,\mathrm{Z})$, for some Banach space Z, if $\hat{G} : \mathbb{C}^+ \ni s \longmapsto \hat{G}(s) \in \mathrm{Z}$ is holomorphic and $\|\hat{G}\|_{\mathrm{H}^\infty(\mathbb{C}^+,\mathrm{Z})} = \sup_{s\in\mathbb{C}^+}\|\hat{G}(s)\|_{\mathrm{Z}} < \infty$. This definition applies as $\mathrm{Z} = \mathbf{L}(\mathrm{U},\mathrm{Y})$ is a Banach space.

Assume, in addition, R_- is coercive and $\mathcal{H} = \mathcal{H}^* \in \mathbf{L}(\mathrm{H})$ solves the *Riccati operator equation*

$$\begin{aligned}&\langle \mathcal{A}z, \mathcal{H}z\rangle_{\mathrm{H}} + \langle z, \mathcal{H}\mathcal{A}z\rangle_{\mathrm{H}} + \langle Q\mathcal{C}z, \mathcal{C}z\rangle_{\mathrm{Y}} = \\ &= \left\langle -\mathcal{D}^*\mathcal{H}\mathcal{A}z + N_-^*\mathcal{C}z, R_-^{-1}\left(-\mathcal{D}^*\mathcal{H}\mathcal{A}z + N_-^*\mathcal{C}z\right)\right\rangle_{\mathrm{U}}, \quad z \in D(\mathcal{A}) \ .\end{aligned} \tag{1.6}$$

Let us define

$$\mathcal{G}z := -\mathcal{D}^*\mathcal{H}\mathcal{A}z + N_-^*\mathcal{C}z, \qquad z \in D(\mathcal{A}) \ . \tag{1.7}$$

If the *implicitly* defined pointwise control in feedback form

$$u(t) = -R_-^{-1}\mathcal{G}\left[x(t) + \mathcal{D}u(t)\right], \qquad t \geq 0 \ , \tag{1.8}$$

is in $\mathrm{L}^2(0,\infty;\mathrm{U})$, then at this control J achieves its minimal value $J(x_0) = \mathcal{V}(x_0) = \langle x_0, \mathcal{H}x_0\rangle_{\mathrm{H}}$, whence, by the uniqueness of optimal control and (1.5) one necessarily has

$$\mathcal{H} := \Psi^*Q\Psi - \Psi^*(Q\mathbb{F} + N)\mathcal{R}^{-1}(\mathbb{F}^*Q + N^*)\Psi = \mathcal{H}^* \in \mathbf{L}(\mathrm{U}) \ . \tag{1.9}$$

Finally, if $\mathcal{G}$ extends to an operator $\mathcal{G}_\Lambda$ with the domain $D(\mathcal{G}_\Lambda)$ such that $D(\mathcal{A}) \subset D(\mathcal{G}_\Lambda)$, $R(\mathcal{D}) \subset D(\mathcal{G}_\Lambda)$, $\mathcal{G}_\Lambda\mathcal{D} \in \mathbf{L}(\mathrm{U},\mathrm{Y})$ and $\mathcal{G}_\Lambda z := \lim_{s\to\infty, s\in\mathbb{R}} s\mathcal{G}(sI - \mathcal{A})^{-1}z$ for $z \in D(\mathcal{G}_\Lambda)$ ($\mathcal{G}_\Lambda$ is a restriction of the *Yosida approximation* of $\mathcal{G}$ being defined for those $z \in \mathrm{H}$ for which this limit exists), then the optimal controller can be written in the *explicit* feedback form

$$u = -\left(R_- + \mathcal{G}_\Lambda\mathcal{D}\right)^{-1}\mathcal{G}_\Lambda x, \qquad x \in D(\mathcal{G}_\Lambda) \ , \tag{1.10}$$

provided that the operator $(R_- + \mathcal{G}_\Lambda\mathcal{D})$ is boundedly invertible and this control, as a function of t, is in $\mathrm{L}^2(0,\infty;\mathrm{U})$.

An example illustrating the results will also be presented.

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