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GLOBAL EXISTENCE AND UNIQUENESS OF A CLASSICAL SOLUTION TO SOME DIFFERENTIAL EVOLUTIONARY SYSTEM

LUCJAN SAPA

ABSTRACT. Theorems of global existence and uniqueness of a classical solution to a nonlinear differential evolutionary system with initial conditions are proved. This system is composed of one partial hyperbolic second-order equation and an ordinary subsystem with a parameter. In the proof of the theorems we use the Picard iteration method, the monotone method of lower and upper solutions, the integral form of the differential problem, weak differential inequalities and the Arzeli-Ascola lemma.

1. Introduction. Let the functions $f : \mathbb{R}_0^+ \times \mathbb{R}^{2+k} \to \mathbb{R}$, $g = (g_1, \dots, g_k) : \mathbb{R}_0^+ \times \mathbb{R}^{2+k} \to \mathbb{R}^k$ of variables $(t, x, p, r) \in \mathbb{R}_0^+ \times \mathbb{R}^{2+k}$, $\varphi_0, \varphi_1 : \mathbb{R} \to \mathbb{R}$, $\psi_0 = (\psi_{01}, \dots, \psi_{0k}) : \mathbb{R} \to \mathbb{R}^k$ and a constant $c \in \mathbb{R}$ be given, where $\mathbb{R}_0^+ = [0, \infty)$. Consider a nonlinear second-order partial differential system of $(1 + k)$ equations of the form

$$\begin{cases} u_{tt} - u_{xx} + cu_t = f(t, x, u, v) & (t, x) \in \mathbb{R}_0^+ \times \mathbb{R}, \\ v_t = g(t, x, u, v) & (t, x) \in \mathbb{R}_0^+ \times \mathbb{R}, \end{cases} \tag{1.1}$$

with the initial conditions

$$\begin{cases} u(0, x) = \varphi_0(x) & x \in \mathbb{R}, \\ v(0, x) = \psi_0(x) & x \in \mathbb{R}, \\ u_t(0, x) = \varphi_1(x) & x \in \mathbb{R}, \end{cases} \tag{1.2}$$

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where $v = (v_1, \ldots, v_k)$. The equations are weakly coupled. If $c > 0$, then the first hyperbolic wave equation in (1.1) is called the *telegraph equation*. The other equations in (1.1) are of first-order with a space parameter x.

The existence, uniqueness and estimates of local solutions to problem (1.1), (1.2) in Hölder spaces were studied in [**21**]. Some information, especially regarding maximum principles and the existence of time-periodic bounded weak solutions for the wave or telegraph equations, is given in [**13, 17**].

Physical motivation of a particular system of the form (1.1) with $c = 1/\tau$, $\tau \geq 0$ the time of relaxation, and f, g of a special form, together with a construction of the solitary wave solutions and their stability, are given in [**14**], see Example 4.12. This system is a generalization of the Hodgkin-Huxley, FitzHugh-Nagumo and McKean models, taking into account effects of memory connected with internal media structure.

The existence, uniqueness and continuous dependence on initial values of global classical solutions to a similar system, but with the parabolic leading equation instead of our telegraph or wave equations, were studied by Evans and Shenk [**5, 9**]. Moreover, Evans [**5**]–[**8**] considered stability in the suitable sense of stationary and traveling wave classical solutions to such systems, usually called *partly parabolic*. Those systems describe, for example, the dynamics of a nerve impulse in axons, and they cover, in particular, the Hodgkin-Huxley system. In [**12, 20**], a connection is described between fast and slow waves in the FitzHugh-Nagumo system and in some systems with non continuous right-hand side. A review of recent results on the stability of traveling wave solutions in partly parabolic reaction-diffusion systems is given in [**10**]. Similar evolutionary systems also appear in [**16**]. Stability of traveling wave solutions of hyperbolic, like the first equation in (1.1), and parabolic convection-reaction-diffusion equations is studied in [**23**]. A realistic view of wave mechanics was first proposed by de Broglie [**4**]. In his inspiring work, Madelung related the linear Schrödinger equation to the hydrodynamic type system [**15**]. The various aspects of hydrodynamic [**2**] and mechanistic [**3**] formulations of the nonlinear Schrödinger equation are still at the center of interest. These formulations lead to systems comprised of second or third order partial differential evolution equations together with first order subsystems.

The general consensus from the authors cited above is that adding first order subsystems (kinetic equations) to the second order equations has a physical motivation and improves the stability of stationary and traveling wave solutions. In order to study the stability of such special solutions we usually need theorems on the existence and uniqueness of global solutions in time of initial problems, with the initial data from a suitable large class of functions.

In this paper, we study global in time bounded or unbounded classical solutions of the initial differential problem (1.1), (1.2). By a *classical solution*, we mean a function

$$(u, v) \in C^2(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}) \times C^1(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^k),$$

which fulfills differential system (1.1) and initial conditions (1.2). We give two theorems on existence and uniqueness. In the first theorem, we assume, in particular, the global Lipschitz condition on f, g (and some of their first order derivatives) with respect to p, r, and we prove it with the use of the Picard iteration method. In the second, we assume the local Lipschitz condition on f, g (and some of their first order derivatives) with respect to p, r, together with suitable monotonicity for f, g in some interval, and we prove it with the use of the monotone method of upper and lower solutions. This theorem also implies the localization of the unique solution. The proof of the theorems is based upon the equivalent integral form of the differential problem. Moreover, we give a theorem on weak linear hyperbolic differential inequalities which is a tool in the proof of convergence based on the monotone method. Monotone methods for parabolic finite and infinite systems are studied in **[1, 18, 19]**.

The paper is organized in the following way. In Section 2, the properties of suitable linear hyperbolic equations and operators related to the first equation in (1.1) are discussed. In particular, weak linear hyperbolic differential inequalities, which are sometimes called the maximum principle for inequalities, are given. The integral system, equivalent under some assumptions to the initial differential problem (1.1), (1.2), is given in Section 3. In Section 4, theorems on global existence and uniqueness are formulated and proven. Moreover, the construction of upper and lower solutions in the case of bounded $f, g, \varphi_0, \varphi_1, \psi_0$ and examples of differential problems are presented.

2. Fundamental solution and the maximum principle in a linear case. In this section, we discuss the properties of suitable linear equations and operators related to the first equation in (1.1).

The next simple lemma will be useful in our future considerations.

Lemma 2.1. *The* ansatz $u = we^{-(ct)/2}$ *transforms the equation*

$$u_{tt} - u_{xx} + cu_t + a(t,x)u = 0, \tag{2.1}$$

where $c = \text{const} \in \mathbb{R}$, $a : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$, *to the equivalent equation*

$$w_{tt} - w_{xx} = \left(\frac{c^2}{4} - a(t,x)\right)w. \tag{2.2}$$

Observe that, for $a(t,x) \equiv \lambda > c^2/4$, $\lambda = \text{const}$, equation (2.2) is the Klein-Gordon equation, and this equation together with the initial conditions does not fulfill the weak maximum principle, see Example 2.3.

Define the differential operator

$$\mathcal{L}u = u_{tt} - u_{xx} + cu_t + \frac{c^2}{4}u \quad \text{in } \mathbb{R}^2, \tag{2.3}$$

acting on scalar functions on $\mathbb{R}^2$. Let $\mathcal{D}'(\mathbb{R}^2)$ be the space of distributions on $\mathbb{R}^2$, and let δ be the Dirac distribution in $\mathcal{D}'(\mathbb{R}^2)$. It is well known that the function

$$U_0(t,x) = \begin{cases} 1/2 & |x| < t, \\ 0 & |x| \geq t, \end{cases} \tag{2.4}$$

is the *fundamental solution* of the wave operator $\Box u = u_{tt} - u_{xx}$ in $\mathcal{D}'(\mathbb{R}^2)$, namely,

$$(U_0)_{tt} - (U_0)_{xx} = \delta \quad \text{in } \mathcal{D}'(\mathbb{R}^2).$$

After setting $a(t,x) \equiv c^2/4$ in Lemma 2.1, we easily see that the fundamental solution of the operator $\mathcal{L}$ in $\mathcal{D}'(\mathbb{R}^2)$ is given by

$$U(t,x) = e^{-(ct)/2}U_0(t,x), \tag{2.5}$$

that is,

$$U_{tt} - U_{xx} + cU_t + \frac{c^2}{4}U = \delta \quad \text{in } \mathcal{D}'(\mathbb{R}^2),$$

see [**13**].

Now, we give a theorem concerning weak differential inequalities, sometimes called the *weak maximum principle* for inequalities. Let

$$\mathcal{L}_a u = u_{tt} - u_{xx} + cu_t + a(t,x)u \quad \text{in } \mathbb{R}^+ \times \mathbb{R} \tag{2.6}$$

$$\mathcal{L}_0 u = u_t + \frac{c}{2}u \quad \text{on } \{0\} \times \mathbb{R} \tag{2.7}$$

act on scalar functions on $\mathbb{R}_0^+ \times \mathbb{R}$; $\mathbb{R}_0^+ = [0,\infty)$, $\mathbb{R}^+ = (0,\infty)$.

Theorem 2.2. *If $c \geq 0$, $a(t,x) \leq c^2/4$ in $\mathbb{R}_0^+ \times \mathbb{R}$ is continuous and $u \in C^1(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}) \cap C^2(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$ satisfies*

$$\begin{aligned} \mathcal{L}_a u \leq 0 &\quad \textit{in } \mathbb{R}^+ \times \mathbb{R}, \\ \mathcal{L}_0 u \leq 0 &\quad \textit{on } \{0\} \times \mathbb{R}, \\ u \leq 0 &\quad \textit{on } \{0\} \times \mathbb{R}, \end{aligned}$$

then

$$u \leq 0 \quad \textit{on } \mathbb{R}_0^+ \times \mathbb{R}.$$

Proof. These are a direct consequence of [**17,** Chapter 4, Theorem 3], if we multiply the operator $\mathcal{L}_a$ by the positive function $\widetilde{u}(t,x) = e^{(ct)/2}$. □

The example below shows that Theorem 2.2 is invalid for $a(t,x) \equiv \lambda > c^2/4$, $\lambda = \text{const}$.

Example 2.3. Let $c = 0$, $\lambda = 1$. The function $u(t,x) = -\cos t$ satisfies the inequalities in the assumptions of Theorem 2.2; however, it is not a non positive function in $\mathbb{R}_0^+ \times \mathbb{R}$.

3. Integral system.

In this section, we give a lemma which will be crucial for our future studies.

Consider a nonlinear integral system of $(1+k)$ equations of the form

(3.1)
$$\begin{cases} u(t,x) = \frac{1}{2}\int\limits_0^t \int\limits_{x-(t-s)}^{x+(t-s)} e^{-[c(t-s)]/2} \\ \quad \times [f(s,y,u(s,y),v(s,y)) + \frac{c^2}{4}u(s,y)]\,dy\,ds \\ \qquad + \frac{1}{2}e^{-(ct)/2}\int\limits_{x-t}^{x+t}\varphi_1(y)\,dy + \frac{c}{4}e^{-(ct)/2}\int\limits_{x-t}^{x+t}\varphi_0(y)\,dy \\ \qquad + \frac{1}{2}e^{-(ct)/2}[\varphi_0(x+t)+\varphi_0(x-t)] & (t,x)\in\mathbb{R}_0^+\times\mathbb{R}, \\ v(t,x) = \psi_0(x) + \int\limits_0^t g(s,x,u(s,x),v(s,x))\,ds & (t,x)\in\mathbb{R}_0^+\times\mathbb{R}. \end{cases}$$

Lemma 3.1. *If*

(i) $f \in C^1(\mathbb{R}_0^+ \times \mathbb{R}^{2+k}, \mathbb{R})$, $g \in C^1(\mathbb{R}_0^+ \times \mathbb{R}^{2+k}, \mathbb{R}^k)$,
(ii) $\varphi_0 \in C^2(\mathbb{R},\mathbb{R})$, $\varphi_1 \in C^1(\mathbb{R},\mathbb{R})$, $\psi_0 \in C^1(\mathbb{R},\mathbb{R}^k)$,

then the differential initial problem (1.1), (1.2) *and the integral system* (3.1) *are equivalent in the sense that any solution* $(u,v) \in C^2(\mathbb{R}_0^+ \times \mathbb{R},\mathbb{R}) \times C^1(\mathbb{R}_0^+ \times \mathbb{R},\mathbb{R}^k)$ *of* (1.1), (1.2) *is a solution of* (3.1), *and any solution* $(u,v) \in C(\mathbb{R}_0^+ \times \mathbb{R},\mathbb{R}^{1+k})$, $v_x \in C(\mathbb{R}_0^+ \times \mathbb{R},\mathbb{R}^k)$, *of* (3.1) *belongs to* $C^2(\mathbb{R}_0^+ \times \mathbb{R},\mathbb{R}) \times C^1(\mathbb{R}_0^+ \times \mathbb{R},\mathbb{R}^k)$ *and fulfills* (1.1), (1.2).

Proof. The first equation in (1.1) is equivalent to:

$$u_{tt} - u_{xx} + cu_t + \frac{c^2}{4}u = f(t,x,u,v) + \frac{c^2}{4}u, \quad (t,x)\in\mathbb{R}_0^+\times\mathbb{R}.$$

Hence, the ansatz $u = we^{-(ct)/2}$ transforms problem (1.1), (1.2) to the equivalent differential system

(3.2)
$$\begin{cases} w_{tt} - w_{xx} = e^{(ct)/2}f(t,x,we^{-(ct)/2},v) + \frac{c^2}{4}w & (t,x)\in\mathbb{R}_0^+\times\mathbb{R}, \\ v_t = g(t,x,we^{-(ct)/2},v) & (t,x)\in\mathbb{R}_0^+\times\mathbb{R}, \end{cases}$$

with the equivalent initial conditions

$$\begin{cases} w(0,x) = \varphi_0(x) & x\in\mathbb{R}, \\ v(0,x) = \psi_0(x) & x\in\mathbb{R}, \\ w_t(0,x) = \frac{c}{2}\varphi_0(x) + \varphi_1(x) & x\in\mathbb{R}. \end{cases} \tag{3.3}$$

The continuity of $f, g, \varphi_0, \psi_0, \varphi_1$, the use of Riemann's method, see [**17,** page 196], for the first equation in (3.2) and integration of the second equation imply that any solution $(w, v) \in C^2(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}) \times C^1(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^k)$ of (3.2), (3.3) is a solution of a nonlinear integral system of $(1+k)$ equations of the form

(3.4)
$$\begin{cases} w(t,x) = \frac{1}{2}\int\limits_0^t \int\limits_{x-(t-s)}^{x+(t-s)} \\ \quad \times [e^{(cs)/2} f(s,y,w(s,y)e^{-(cs)/2}, v(s,y)) + \frac{c^2}{4} w(s,y)]\, dy\, ds \\ \qquad + \frac{1}{2}\int\limits_{x-t}^{x+t} \left[\frac{c}{2}\varphi_0(y) + \varphi_1(y)\right] dy \\ \qquad + \frac{1}{2}[\varphi_0(x+t) + \varphi_0(x-t)] & (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}, \\ v(t,x) = \psi_0(x) + \int\limits_0^t g(s,x,w(s,x)e^{-(cs)/2}, v(s,x))\, ds & (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}. \end{cases}$$

Multiplying the first equation in (3.4) by $e^{-(ct)/2}$, we have that any solution $(u, v) \in C^2(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}) \times C^1(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^k)$ of the differential initial problem (1.1), (1.2) is a solution of the integral system (3.1).

On the other hand, let $(u, v) \in C(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^{1+k})$, $v_x \in C(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^k)$, be a solution of (3.1). For simplicity, put

$$A(t,x,s) = (s, x+(t-s), u(s, x+(t-s)), v(s, x+(t-s))),$$
$$B(t,x,s) = (s, x-(t-s), u(s, x-(t-s)), v(s, x-(t-s))).$$

From the regularity of $f, g, \varphi_0, \psi_0, \varphi_1$ and differentiation of the integrals in (3.1), we have

$$\begin{aligned} u_t(t,x) = &-\frac{c}{4}\int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-[c(t-s)]/2}\left[f(s,y,u(s,y), v(s,y)) + \frac{c^2}{4}u(s,y)\right] dy ds \\ &+ \frac{1}{2}\int_0^t e^{-[c(t-s)]/2}\left[f(A(t,x,s)) + \frac{c^2}{4}u(s, x+(t-s))\right] ds \\ &+ \frac{1}{2}\int_0^t e^{-[c(t-s)]/2}\left[f(B(t,x,s)) + \frac{c^2}{4}u(s, x-(t-s))\right] ds \\ &- \frac{c}{4}e^{-(ct)/2}\int_{x-t}^{x+t} \varphi_1(y)\, dy \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}e^{-(ct)/2}[\varphi_1(x+t)+\varphi_1(x-t)]-\frac{c^2}{8}e^{-(ct)/2}\int_{x-t}^{x+t}\varphi_0(y)dy\\
&+\frac{1}{2}e^{-(ct)/2}[(\varphi_0)_x(x+t)-(\varphi_0)_x(x-t)]
\end{aligned}
$$

$$
\begin{aligned}
u_x(t,x)=&\frac{1}{2}\int_0^t e^{-[c(t-s)]/2}\Big[f(A(t,x,s))+\frac{c^2}{4}u(s,x+(t-s))\Big]ds\\
&-\frac{1}{2}\int_0^t e^{-[c(t-s)]/2}\Big[f(B(t,x,s))+\frac{c^2}{4}u(s,x-(t-s))\Big]ds\\
&+\frac{1}{2}e^{-(ct)/2}[\varphi_1(x+t)-\varphi_1(x-t)]\\
&+\frac{c}{4}e^{-(ct)/2}[\varphi_0(x+t)-\varphi_0(x-t)]\\
&+\frac{1}{2}e^{-(ct)/2}[(\varphi_0)_x(x+t)+(\varphi_0)_x(x-t)],
\end{aligned}
$$

$$
\begin{aligned}
u_{tt}(t,x)=&\frac{c^2}{8}\int_0^t\int_{x-(t-s)}^{x+(t-s)} e^{-[c(t-s)]/2}\Big[f(s,y,u(s,y),v(s,y))+\frac{c^2}{4}u(s,y)\Big]dyds\\
&-\frac{c}{2}\int_0^t e^{-[c(t-s)]/2}\Big[f(A(t,x,s))+\frac{c^2}{4}u(s,x+(t-s))\Big]ds\\
&-\frac{c}{2}\int_0^t e^{-[c(t-s)]/2}\Big[f(B(t,x,s))+\frac{c^2}{4}u(s,x-(t-s))\Big]ds\\
&+\frac{1}{2}\int_0^t e^{-[c(t-s)]/2}\Big[f_x(A(t,x,s))+f_p(A(t,x,s))u_x(s,x+(t-s))\\
&\qquad+\sum_{i=1}^k f_{r_i}(A(t,x,s))(v_i)_x(s,x+(t-s))\\
&\qquad\qquad+\frac{c^2}{4}u_x(s,x+(t-s))\Big]ds\\
&-\frac{1}{2}\int_0^t e^{-[c(t-s)]/2}\Big[f_x(B(t,x,s))+f_p(B(t,x,s))u_x(s,x-(t-s))\\
&\qquad+\sum_{i=1}^k f_{r_i}(B(t,x,s))(v_i)_x(s,x-(t-s))\\
&\qquad\qquad+\frac{c^2}{4}u_x(s,x-(t-s))\Big]ds
\end{aligned}
$$

$$
\begin{aligned}
&+ f(t,x,u(t,x),v(t,x)) + \frac{c^2}{4}u(t,x)\\
&+ \frac{c^2}{8}e^{-(ct)/2}\int_{x-t}^{x+t}\varphi_1(y)\,dy - \frac{c}{2}e^{-(ct)/2}[\varphi_1(x+t)+\varphi_1(x-t)]\\
&+ \frac{1}{2}e^{-(ct)/2}[(\varphi_1)_x(x+t)-(\varphi_1)_x(x-t)]\\
&+ \frac{c^3}{16}e^{-(ct)/2}\int_{x-t}^{x+t}\varphi_0(y)\,dy - \frac{c^2}{8}e^{-(ct)/2}[\varphi_0(x+t)+\varphi_0(x-t)]\\
&- \frac{c}{4}e^{-(ct)/2}[(\varphi_0)_x(x+t)-(\varphi_0)_x(x-t)]\\
&+ \frac{1}{2}e^{-(ct)/2}[(\varphi_0)_{xx}(x+t)+(\varphi_0)_{xx}(x-t)],
\end{aligned}
$$

$$
\begin{aligned}
u_{xx}(t,x) = \frac{1}{2}\int_0^t e^{-[c(t-s)]/2}\Bigg[&f_x(A(t,x,s))+f_p(A(t,x,s))u_x(s,x+(t-s))\\
&+\sum_{i=1}^{k} f_{r_i}(A(t,x,s))(v_i)_x(s,x+(t-s))\\
&+\frac{c^2}{4}u_x(s,x+(t-s))\Bigg]ds\\
-\frac{1}{2}\int_0^t e^{-[c(t-s)]/2}\Bigg[&f_x(B(t,x,s))+f_p(B(t,x,s))u_x(s,x-(t-s))\\
&+\sum_{i=1}^{k} f_{r_i}(B(t,x,s))(v_i)_x(s,x-(t-s))\\
&+\frac{c^2}{4}u_x(s,x-(t-s))\Bigg]ds\\
&+ \frac{1}{2}e^{-(ct)/2}[(\varphi_1)_x(x+t)-(\varphi_1)_x(x-t)]\\
&+ \frac{c}{4}e^{-(ct)/2}[(\varphi_0)_x(x+t)-(\varphi_0)_x(x-t)]\\
&+ \frac{1}{2}e^{-(ct)/2}[(\varphi_0)_{xx}(x+t)+(\varphi_0)_{xx}(x-t)],
\end{aligned}
$$

$$
v_t(t,x) = g(t,x,u(t,x),v(t,x)).
$$

It is clear that $(u,v) \in C^2(\mathbb{R}_0^+ \times \mathbb{R},\mathbb{R}) \times C^1(\mathbb{R}_0^+ \times \mathbb{R},\mathbb{R}^k)$ and fulfills (1.1), (1.2). $\square$

4. Global existence and uniqueness of a solution. We show the global in time existence and uniqueness of a solution $(u, v) \in C^2(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}) \times C^1(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^k)$ of (1.1), (1.2). Firstly, we prove it by using the Picard iteration method, assuming the global Lipschitz condition on f, g with respect to p, r, and secondly, by using the monotone method of upper and lower solutions, assuming the local Lipschitz condition on f, g with respect to p, r.

4.1. The Picard iteration method. Let $\| \ \|$ be the maximum norm in $\mathbf{R}^d$, i.e.,

$$\|y\| = \max_{i=1,\dots,d} |y_i|, \tag{4.1}$$

where $y \in \mathbf{R}^d$. In the space of continuous functions $C(\Omega, \mathbf{R}^d)$, we define the *maximum norm*

$$\|z\|_\Omega = \max\{\|z(\omega)\| : \omega \in \Omega\}, \tag{4.2}$$

where $z \in C(\Omega, \mathbf{R}^d)$, $\Omega \subset \mathbf{R}^m$ is a compact set. Moreover, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Now, we formulate two well-known lemmas which will be useful further on in the paper, and then we prove a theorem on global existence and uniqueness under the global Lipschitz condition on f, g with respect to p, r.

Lemma 4.1. *Let (y_n) be a sequence of functions $y_n : \mathbb{R}^m \supset D \to \mathbb{R}^d$, $n \in \mathbb{N}_0$. If, for all $x \in D$ and for all $n \in \mathbb{N}_0$,*

$$\|y_{n+1}(x) - y_n(x)\| \le \alpha_n,$$

and the number series $\sum_{n=0}^{\infty} \alpha_n$ is convergent, then (y_n) is uniformly convergent in D. Moreover, if all y_n are continuous, then the limit is also continuous.

Lemma 4.2. *Let $h : [\alpha, \beta] \times \mathbb{R}^m \to \mathbb{R}^d$ be continuous, and let (y_n), $y_n : [\alpha, \beta] \to \mathbb{R}^m$, $n \in \mathbb{N}_0$, be a sequence of continuous functions uniformly convergent in $[\alpha, \beta]$ to y. Then,*

$$\lim_{n\to\infty} \int_\alpha^s h(t, y_n(t))\, dt = \int_\alpha^s h(t, y(t))\, dt, \quad s \in [\alpha, \beta].$$

Theorem 4.3. *Assume that*

(i) *$f \in C^1(\mathbb{R}_0^+ \times \mathbb{R}^{2+k}, \mathbb{R})$, $g \in C^1(\mathbb{R}_0^+ \times \mathbb{R}^{2+k}, \mathbb{R}^k)$;*
(ii) *f, g are Lipschitz continuous, with a constant L, with respect to p, r in $\mathbb{R}_0^+ \times \mathbb{R}^{2+k}$;*
(iii) *g_x, g_p, g_{r_i}, $i = 1, \dots, k$, are Lipschitz continuous, with a constant L_1, with respect to x, p, r in $\mathbb{R}_0^+ \times \mathbb{R}^{2+k}$;*
(iv) *$\varphi_0 \in C^2(\mathbb{R}, \mathbb{R})$, $\varphi_1 \in C^1(\mathbb{R}, \mathbb{R})$, $\psi_0 \in C^1(\mathbb{R}, \mathbb{R}^k)$;*
(v) *$(\psi_0)_x$ is Lipschitz continuous with a constant L_0 in $\mathbb{R}$.*

Then there exists a unique solution $(u, v) \in C^2(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}) \times C^1(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^k)$ of (1.1), (1.2).

Proof. It follows from Lemma 3.1 that the differential initial problem (1.1), (1.2) is equivalent to the integral system (3.1) in a suitable sense. From this equivalence, it is sufficient to find a unique continuous solution (u, v) of (3.1) such that v_x exists and is continuous.

Let $T \in \mathbb{R}^+$ be fixed, and let $X \in \mathbb{R}$ be such that the lines passing through the points $(X-T, 0)$, (X, T) and $(X+T, 0)$, (X, T) create angles $\pi/4$, $-\pi/4$ with the x-axis, respectively. Denote by $\Delta(X, T) \subset \mathbb{R} \times \mathbb{R}_0^+$ an isosceles triangle with the vertices $(X - T, 0)$, (X, T), $(X + T, 0)$.

We will construct a continuous solution (u, v) in $\Delta(X, T)$, with v_x continuous in $\Delta(X, T)$, of (3.1), and then, we will prove its uniqueness. Define a sequence (u_n, v_n) of functions

$$u_n : \Delta(X, T) \longrightarrow \mathbb{R}, \qquad v_n : \Delta(X, T) \longrightarrow \mathbb{R}^k,$$
$$v_n = (v_{1n}, \dots, v_{kn}), \quad n \in N_0,$$

by the Picard recurrence formula

$$\begin{aligned} u_0(t, x) &= \frac{1}{2} e^{-(ct)/2} \int_{x-t}^{x+t} \varphi_1(y)\, dy + \frac{c}{4} e^{-(ct)/2} \int_{x-t}^{x+t} \varphi_0(y)\, dy \\ &\quad + \frac{1}{2} e^{-(ct)/2} [\varphi_0(x + t) + \varphi_0(x - t)], \\ v_0(t, x) &= \psi_0(x), \end{aligned} \tag{4.3}$$

$$\begin{aligned} u_{n+1}(t, x) = \frac{1}{2} \int_0^t \int_{x-(t-s)}^{x+(t-s)} & e^{-[c(t-s)]/2} \\ & \times \left[f(s, y, u_n(s, y), v_n(s, y)) + \frac{c^2}{4} u_n(s, y) \right] dy\, ds \end{aligned}$$

$$+\frac{1}{2}e^{-(ct)/2}\int_{x-t}^{x+t}\varphi_1(y)\,dy+\frac{c}{4}e^{-(ct)/2}\int_{x-t}^{x+t}\varphi_0(y)\,dy$$
$$+\frac{1}{2}e^{-(ct)/2}[\varphi_0(x+t)+\varphi_0(x-t)],$$
$$v_{n+1}(t,x)=\psi_0(x)+\int_0^t g(s,x,u_n(s,x),v_n(s,x))\,ds.$$

By induction and assumptions (i) and (iv), this sequence is well defined. Using induction and assumptions (i), (ii) and (iv), we obtain the estimates

$$|u_{n+1}(t,x)-u_n(t,x)|\le\frac{\gamma^{n+1}M(L+c^2/4)^nT(1+T)^nt^{n+1}}{(n+1)!},\tag{4.4}$$
$$\|v_{n+1}(t,x)-v_n(t,x)\|\le\frac{\gamma^{n+1}M(L+c^2/4)^n(1+T)^nt^{n+1}}{(n+1)!},$$

where

$$M=\max\Big\{\|f(\cdot,\cdot,u_0(\cdot,\cdot),v_0(\cdot,\cdot))\|_{\Delta(X,T)}+c^2/4\|u_0(\cdot,\cdot)\|_{\Delta(X,T)},$$
$$\|g(\cdot,\cdot,u_0(\cdot,\cdot),v_0(\cdot,\cdot))\|_{\Delta(X,T)}\Big\},$$

$\gamma=1$ if $c\ge 0$ and $\gamma=e^{-(cT)/2}$ if $c<0$, for $(t,x)\in\Delta(X,T)$, $n\in\mathbb{N}_0$. Note that $\gamma\ge 1$. It follows from the d'Alembert criterion that the number series

$$\sum_{n=0}^{\infty}\frac{\gamma^{n+1}M(L+(c^2/4))^nT(1+T)^nT^{n+1}}{(n+1)!},$$
$$\sum_{n=0}^{\infty}\frac{\gamma^{n+1}M(L+(c^2/4))^n(1+T)^nT^{n+1}}{(n+1)!}$$

are convergent. Lemma 4.1 implies that

$$\lim_{n\to\infty}u_n(t,x)=u(t,x),\qquad\lim_{n\to\infty}v_n(t,x)=v(t,x)\tag{4.5}$$

uniformly in $\Delta(X,T)$, and u, v are continuous. From Lemma 4.2, regarding the limit transition under the sign of an integral, we obtain that (u,v) is a solution of (3.1) in $\Delta(X,T)$.

We must show that v_x exists and it is continuous in $\Delta(X,T)$. In order to do so, we will prove the uniform boundedness of the sequence $((v_n)_x)$ in $\Delta(X,T)$ and the Lipschitz continuity of the functions $(v_n)_x$

in $\Delta(X,T)$ with the same constant for all $n \in \mathbb{N}_0$, and then we will use the Arzeli-Ascola lemma. For simplicity, set

$$\begin{aligned}
A_n(t,x,s) &= (s, x+(t-s), u_n(s,x+(t-s)), v_n(s,x+(t-s))),\\
B_n(t,x,s) &= (s, x-(t-s), u_n(s,x-(t-s)), v_n(s,x-(t-s))),\\
C_n(x,s) &= (s,x,u_n(s,x),v_n(s,x)).
\end{aligned}$$

Induction and assumptions (i) and (iv) imply the existence of $(u_n)_x$, $(v_n)_x$, $n \in \mathbb{N}_0$, and moreover,

(4.6)

$$\begin{aligned}
(u_0)_x(t,x) &= \frac{1}{2}e^{-(ct)/2}[\varphi_1(x+t)-\varphi_1(x-t)]\\
&\quad+\frac{c}{4}e^{-(ct)/2}[\varphi_0(x+t)-\varphi_0(x-t)]\\
&\quad+\frac{1}{2}e^{-(ct)/2}[(\varphi_0)_x(x+t)+(\varphi_0)_x(x-t)],\\
(v_0)_x(t,x) &= (\psi_0)_x(x),\\
(u_{n+1})_x(t,x) &= \frac{1}{2}\int_0^t e^{-[c(t-s)]/2}\left[f(A_n(t,x,s))+\frac{c^2}{4}u_n(s,x+(t-s))\right]ds\\
&\quad-\frac{1}{2}\int_0^t e^{-[c(t-s)]/2}\left[f(B_n(t,x,s))+\frac{c^2}{4}u_n(s,x-(t-s))\right]ds\\
&\quad+\frac{1}{2}e^{-(ct)/2}[\varphi_1(x+t)-\varphi_1(x-t)]\\
&\quad+\frac{c}{4}e^{-(ct)/2}[\varphi_0(x+t)-\varphi_0(x-t)]\\
&\quad+\frac{1}{2}e^{-(ct)/2}[(\varphi_0)_x(x+t)+(\varphi_0)_x(x-t)],\\
(v_{n+1})_x(t,x) &= (\psi_0)_x(x)\\
&\quad+\int_0^t \bigg[g_x(C_n(x,s))+g_p(C_n(x,s))(u_n)_x(s,x)\\
&\qquad\qquad+\sum_{i=1}^k g_{r_i}(C_n(x,s))(v_{in})_x(s,x)\bigg]ds.
\end{aligned}$$

The sequences (u_n) and (v_n) are uniformly bounded in $\Delta(X,T)$ since they are uniformly convergent in $\Delta(X,T)$. Moreover, it follows from (4.6) that the sequence $((u_n)_x)$ is also uniformly bounded in $\Delta(X,T)$, i.e.,

$$\|(u_n)_x\|_{\Delta(X,T)} \le c, \quad n \in \mathbb{N}_0, \tag{4.7}$$

where c is a positive constant. Hence, by continuity of $(\psi_0)_x$, g_x, g_p, g_{r_i}, there are positive constants a and b such that

$$\|(v_{n+1})_x(t,x)\| \le a + b\int_0^t \|(v_n)_x(s,x)\|\, ds, \quad n \in \mathbb{N}_0,$$

and $\|(v_0)_x(t,x)\| \le a$ for $(t,x) \in \Delta(X,T)$. Elementary calculations imply the estimates

$$\|(v_n)_x(t,x)\| \le a\sum_{i=0}^{n} \frac{(bt)^i}{i!} \le a\exp(bt), \quad n \in \mathbb{N}_0$$

for $(t,x) \in \Delta(X,T)$, and consequently,

$$\|(v_n)_x\|_{\Delta(X,T)} \le a\exp(bT), \quad n \in \mathbb{N}_0. \tag{4.8}$$

Thus, the sequence $((v_n)_x)$ is uniformly bounded in $\Delta(X,T)$. From assumptions (i), (ii), (iv), boundedness of the sequences (u_n), (v_n), $((u_n)_x)$, $((v_n)_x)$, the mean value theorem for u_n, the theorem on the estimate of an increment of v_n and additivity of an integral, we get that the functions $(u_n)_x$ fulfill the Lipschitz condition in $\Delta(X,T)$ with the same constant L_2 for all $n \in \mathbb{N}_0$. The technical details of finding L_2 are omitted. The same arguments, together with assumption (iii) instead of (ii) and assumption (v), give

$$\begin{aligned}
\|(v_{n+1})_x(t,x) - (v_{n+1})_x(\bar{t},\overline{x})\| &\le \|(\psi_0)_x(x) - (\psi_0)_x(\overline{x})\| \\
&+ \left\| \int_0^t g_x(C_n(x,s))\, ds - \int_0^{\bar{t}} g_x(C_n(\overline{x},s))\, ds \right\| \\
&+ \Bigg\| \int_0^t g_p(C_n(x,s))(u_n)_x(s,x)\, ds \\
&\qquad - \int_0^{\bar{t}} g_p(C_n(\overline{x},s))(u_n)_x(s,\overline{x})\, ds \Bigg\| \\
&+ \sum_{i=1}^{k} \Bigg\| \int_0^t g_{r_i}(C_n(x,s))(v_{in})_x(s,x)\, ds \\
&\qquad - \int_0^{\bar{t}} g_{r_i}(C_n(\overline{x},s))(v_{in})_x(s,\overline{x})\, ds \Bigg\|,
\end{aligned} \tag{4.9}$$

$$
\begin{aligned}
&\left\| \int_0^t g_x(C_n(x,s))\,ds - \int_0^{\bar t} g_x(C_n(\overline{x},s))\,ds \right\| \\
&\quad = \left\| \int_0^t g_x(C_n(x,s))\,ds - \int_0^t g_x(C_n(\overline{x},s))\,ds - \int_t^{\bar t} g_x(C_n(\overline{x},s))\,ds \right\| \\
&\quad \le \int_0^t \|g_x(C_n(x,s)) - g_x(C_n(\overline{x},s))\|\,ds + d_1|t-\bar t| \\
&\quad \le L_1 \int_0^t [|x-\overline{x}| + |u_n(s,x) - u_n(s,\overline{x})| \\
&\qquad\qquad + \|v_n(s,x) - v_n(s,\overline{x})\|]\,ds + d_1|t-\bar t| \\
&\quad \le L_1 \int_0^t [|x-\overline{x}| + |(u_n)_x(s,x_1)||x-\overline{x}| \\
&\qquad\qquad + \|(v_n)_x(s,x_2)\||x-\overline{x}|]\,ds + d_1|t-\bar t| \\
&\quad \le L_1 T(1+c+a\exp{(bT)})|x-\overline{x}| + d_1|t-\bar t|,
\end{aligned}
$$

$$
\begin{aligned}
&\left\| \int_0^t g_p(C_n(x,s))(u_n)_x(s,x)\,ds - \int_0^{\bar t} g_p(C_n(\overline{x},s))(u_n)_x(s,\overline{x})\,ds \right\| \\
&\quad = \Bigg\| \int_0^t g_p(C_n(x,s))(u_n)_x(s,x)\,ds - \int_0^t g_p(C_n(\overline{x},s))(u_n)_x(s,\overline{x})\,ds \\
&\qquad\qquad - \int_t^{\bar t} g_p(C_n(\overline{x},s))(u_n)_x(s,\overline{x})\,ds \Bigg\| \\
&\quad \le \int_0^t \|g_p(C_n(x,s))(u_n)_x(s,x) - g_p(C_n(\overline{x},s))(u_n)_x(s,\overline{x})\|\,ds + d_2c|t-\bar t| \\
&\quad \le \int_0^t \|g_p(C_n(x,s))(u_n)_x(s,x) - g_p(C_n(x,s))(u_n)_x(s,\overline{x})\|\,ds \\
&\qquad + \int_0^t \|g_p(C_n(x,s))(u_n)_x(s,\overline{x}) - g_p(C_n(\overline{x},s))(u_n)_x(s,\overline{x})\|\,ds \\
&\qquad\quad + d_2c|t-\bar t| \\
&\quad \le \int_0^t \|g_p(C_n(x,s))\||(u_n)_x(s,x) - (u_n)_x(s,\overline{x})|\,ds \\
&\qquad + cL_1 \int_0^t [|x-\overline{x}| + |u_n(s,x) - u_n(s,\overline{x})| + \|v_n(s,x) - v_n(s,\overline{x})\|]\,ds \\
&\qquad\quad + d_2c|t-\bar t|
\end{aligned}
$$

$$\leq d_2 L_2 \int_0^t |x-\overline{x}|\, ds$$
$$+ cL_1 \int_0^t [|x-\overline{x}|+|(u_n)_x(s,x_1)||x-\overline{x}|+\|(v_n)_x(s,x_2)\||x-\overline{x}|]\, ds$$
$$+ d_2 c|t-\overline{t}|$$
$$\leq [d_2 L_2 T + cL_1 T(1+c+a\exp{(bT)})]|x-\overline{x}| + d_2 c|t-\overline{t}|,$$

$$\left\| \int_0^t g_{r_i}(C_n(x,s))(v_{in})_x(s,x)\, ds - \int_0^{\overline{t}} g_{r_i}(C_n(\overline{x},s))(v_{in})_x(s,\overline{x})\, ds \right\|$$
$$= \left\| \int_0^t g_{r_i}(C_n(x,s))(v_{in})_x(s,x)\, ds - \int_0^t g_{r_i}(C_n(\overline{x},s))(v_{in})_x(s,\overline{x})\, ds \right.$$
$$\left. - \int_t^{\overline{t}} g_{r_i}(C_n(\overline{x},s))(v_{in})_x(s,\overline{x})\, ds \right\|$$
$$\leq \int_0^t \|g_{r_i}(C_n(x,s))(v_{in})_x(s,x) - g_{r_i}(C_n(\overline{x},s))(v_{in})_x(s,\overline{x})\|\, ds$$
$$+ d_3 a\exp{(bT)}|t-\overline{t}|$$
$$\leq \int_0^t \|g_{r_i}(C_n(x,s))(v_{in})_x(s,x) - g_{r_i}(C_n(x,s))(v_{in})_x(s,\overline{x})\|\, ds$$
$$+ \int_0^t \|g_{r_i}(C_n(x,s))(v_{in})_x(s,\overline{x}) - g_{r_i}(C_n(\overline{x},s))(v_{in})_x(s,\overline{x})\|\, ds$$
$$+ d_3 a\exp{(bT)}|t-\overline{t}|$$
$$\leq \int_0^t \|g_{r_i}(C_n(x,s))\||(v_{in})_x(s,x) - (v_{in})_x(s,\overline{x})|\, ds$$
$$+ a\exp{(bT)}L_1 \int_0^t [|x-\overline{x}| + |u_n(s,x) - u_n(s,\overline{x})|$$
$$+ \|v_n(s,x) - v_n(s,\overline{x})\|]\, ds + d_3 a\exp{(bT)}|t-\overline{t}|$$
$$\leq d_3 \int_0^t \|(v_n)_x(s,x) - (v_n)_x(s,\overline{x})\|\, ds$$
$$+ a\exp{(bT)}L_1 \int_0^t [|x-\overline{x}| + |(u_n)_x(s,x_1)||x-\overline{x}|$$
$$+ \|(v_n)_x(s,x_2)\||x-\overline{x}|]\, ds + d_3 a\exp{(bT)}|t-\overline{t}|$$
$$\leq [a\exp{(bT)}L_1 T(1+c+a\exp{(bT)})]|x-\overline{x}| + d_3 a\exp{(bT)}|t-\overline{t}|$$

$$+ d_3 \int_0^t \|(v_n)_x(s,x) - (v_n)_x(s,\overline{x})\| \, ds,$$

where $x_1, x_2 \in [X-T, X+T]$ are some intermediate points and d_1, d_2, d_3 are positive constants. Set

(4.10)

$$\begin{aligned} a_1(t,\overline{t},x,\overline{x}) = \{&L_0 + L_1 T(1+c+a\exp(bT)) \\ &+ [d_2 L_2 T + c L_1 T(1+c+a\exp(bT))] \\ &+ ka\exp(bT) L_1 T(1+c+a\exp(bT))\}|x-\overline{x}| \\ &+ \{d_1 + d_2 c + k d_3 a \exp(bT)\}|t-\overline{t}|, \quad b_1 = k d_3. \end{aligned}$$

Hence, by (4.9) and (4.10),

$$\begin{aligned} \|(v_{n+1})_x(t,x) - (v_{n+1})_x(\overline{t},\overline{x})\| &\le a_1(t,\overline{t},x,\overline{x}) \\ &+ b_1 \int_0^t \|(v_n)_x(s,x) - (v_n)_x(s,\overline{x})\| \, ds, \end{aligned}$$

$n \in \mathbb{N}_0$, and $\|(v_0)_x(t,x) - (v_0)_x(\overline{t},\overline{x})\| \le a_1(t,\overline{t},x,\overline{x})$ for $(t,x) \in \Delta(X,T)$. Elementary calculations, together with the relation $a_1(s,s,x,\overline{x}) \le a_1(t,\overline{t},x,\overline{x})$, imply the estimates

$$\begin{aligned} \|(v_n)_x(t,x) - (v_n)_x(\overline{t},\overline{x})\| &\le a_1(t,\overline{t},x,\overline{x}) \sum_{i=0} n \frac{(b_1 t)^i}{i!} \\ &\le \exp(b_1 T) a_1(t,\overline{t},x,\overline{x}), \quad n \in \mathbb{N}_0, \end{aligned}$$

for $(t,x) \in \Delta(X,T)$; thus, the functions $(v_n)_x$ fulfill the Lipschitz condition in $\Delta(X,T)$ with the same constant for all $n \in \mathbb{N}_0$. It follows from (4.8), (4.11) and the Arzeli-Ascola lemma that there exists a subsequence $((v_{n_l})_x)$, uniformly convergent in $\Delta(X,T)$, and consequently, v_x equals the limit of this subsequence.

Let $(\widetilde{u}, \widetilde{v})$ be any continuous solution of (3.1) in $\Delta(X,T)$. Induction and assumptions (i), (ii), (iv) lead to the estimates

$$(4.11) \qquad |u_n(t,x) - \widetilde{u}(t,x)| \le \frac{\gamma^{n+1} \widetilde{M} (L + c^2/4)^n T (1+T)^n t^{n+1}}{(n+1)!},$$

$$\|v_n(t,x) - \widetilde{v}(t,x)\| \le \frac{\gamma^{n+1} \widetilde{M} (L + c^2/4)^n (1+T)^n t^{n+1}}{(n+1)!},$$

where

$$\widetilde{M} = \max\Big\{\|f(\cdot,\cdot,\widetilde{u}(\cdot,\cdot),\widetilde{v}(\cdot,\cdot))\|_{\Delta(X,T)} + \frac{c^2}{4}\|\widetilde{u}(\cdot,\cdot)\|_{\Delta(X,T)}, \\ \|g(\cdot,\cdot,\widetilde{u}(\cdot,\cdot),\widetilde{v}(\cdot,\cdot))\|_{\Delta(X,T)}\Big\},$$

for $(t,x) \in \Delta(X,T)$, $n \in \mathbb{N}_0$. From (4.5) and (4.11), we have $|u(t,x) - \widetilde{u}(t,x)| \le 0$, $\|v(t,x) - \widetilde{v}(t,x)\| \le 0$ in $\Delta(X,T)$, and consequently, $u = \widetilde{u}$, $v = \widetilde{v}$ in $\Delta(X,T)$.

A global solution is constructed by piecing together the solutions in all of the triangles $\Delta(X,T)$ and, by Lemma 3.1, the proof is complete. □

Remark 4.4. Due to $u = \widetilde{u}$, $v = \widetilde{v}$ in $\Delta(X,T)$ in the proof of Theorem 4.3, inequalities (4.11) give the rate of convergence of the analytical method

$$|u_n(t,x) - u(t,x)| \le \frac{\gamma^{n+1}M_1(L + c^2/4)^n T(1+T)^n t^{n+1}}{(n+1)!},$$
$$\|v_n(t,x) - v(t,x)\| \le \frac{\gamma^{n+1}M_1(L + c^2/4)^n (1+T)^n t^{n+1}}{(n+1)!},$$

$$M_1 = \max\Big\{\|f(\cdot,\cdot,u(\cdot,\cdot),v(\cdot,\cdot))\|_{\Delta(X,T)} + \frac{c^2}{4}\|u(\cdot,\cdot)\|_{\Delta(X,T)}, \\ \|g(\cdot,\cdot,u(\cdot,\cdot),v(\cdot,\cdot))\|_{\Delta(X,T)}\Big\},$$

for $(t,x) \in \Delta(X,T)$, $n \in \mathbb{N}_0$.

Remark 4.5. From (4.4), (4.6), and after simple calculations, we have the estimate

$$|(u_{n+1})_x(t,x) - (u_n)_x(t,x)| \le \frac{\gamma^{n+1}M(L + c^2/4)^n (1+T)^n t^{n+1}}{(n+1)!} \tag{4.12}$$

for $(t,x) \in \Delta(X,T)$, $n \in \mathbb{N}_0$. It follows from Lemma 4.1 that the sequence $((u_n)_x)$ is uniformly convergent in $\Delta(X,T)$.

4.2. The monotone method of upper and lower solutions. In this section, we use a monotone method of upper and lower solutions to prove a theorem on global existence and uniqueness under the local, in a some sector, Lipschitz condition on f, g with respect to p, r. This theorem also gives the localization of the unique solution.

In $\mathbb{R}^k$, the following order is introduced: for $y = (y_1, \dots, y_k), \widetilde{y} = (\widetilde{y}_1, \dots, \widetilde{y}_k) \in \mathbb{R}^k$, the inequality $y \le \widetilde{y}$ means that $y_i \le \widetilde{y}_i$, $i = 1, \dots, k$. Moreover, we define the order in the space $C^2(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R})$ as follows: for $u, \widetilde{u} \in C^2(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R})$, the inequality $u \le \widetilde{u}$ means that $u(t,x) \le \widetilde{u}(t,x)$, $(t,x) \in \mathbb{R}_0^+ \times \mathbb{R}$. Similarly, in the space $C^1(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^k)$: for $v = (v_1, \dots, v_k), \widetilde{v} = (\widetilde{v}_1, \dots, \widetilde{v}_k) \in C^1(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^k)$, the inequality $v \le \widetilde{v}$ means that $v_i(t,x) \le \widetilde{v}_i(t,x)$, $(t,x) \in \mathbb{R}_0^+ \times \mathbb{R}$, $i = 1, \dots, k$.

A function $(u,v) \in C^2(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}) \times C^1(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^k)$ satisfying the system of inequalities

$$(4.13) \qquad \begin{cases} u_{tt} - u_{xx} + cu_t \le f(t,x,u,v) & (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}, \\ v_t \le g(t,x,u,v) & (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}, \\ \varphi_0(x) \ge u(0,x) & x \in \mathbb{R}, \\ \psi_0(x) \ge v(0,x) & x \in \mathbb{R}, \\ \varphi_1(x) \ge u_t(0,x) & x \in \mathbb{R}, \end{cases}$$

is called a *lower solution* of (1.1), (1.2) in $\mathbb{R}_0^+ \times \mathbb{R}$. If the inequalities are inverse, we call it an *upper solution* of (1.1), (1.2) in $\mathbb{R}_0^+ \times \mathbb{R}$.

Assumption A. There exists at least one pair of lower and upper solutions $(\underline{u}_0, \underline{v}_0)$, $(\overline{u}_0, \overline{v}_0)$, respectively, of (1.1), (1.2) such that

$$(4.14) \qquad \underline{u}_0 \le \overline{u}_0, \ \underline{v}_0 \le \overline{v}_0 \quad \text{in } \mathbb{R}_0^+ \times \mathbb{R}.$$

For a given pair of lower and upper solutions $(\underline{u}_0, \underline{v}_0)$, $(\overline{u}_0, \overline{v}_0)$, respectively, of (1.1), (1.2) satisfying (4.14), we define a sector

$$\begin{aligned} \langle (\underline{u}_0, \underline{v}_0), (\overline{u}_0, \overline{v}_0) \rangle = \{ (u,v) \in C^2(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}) \times C^1(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^k) : \\ \underline{u}_0(t,x) \le u(t,x) \le \overline{u}_0(t,x), \\ \underline{v}_0(t,x) \le v(t,x) \le \overline{v}_0(t,x), \\ (t,x) \in \mathbb{R}_0^+ \times \mathbb{R} \} \end{aligned}$$

and an interval

$$\langle m, M \rangle = \{ (u,v) \in \mathbb{R}^{1+k} : m_0 \le u \le M_0, \ m \le v \le M \},$$

where $m = (m_1, \dots, m_k)$, $M = (M_1, \dots, M_k)$,

$$m_0 = \inf\{ \underline{u}_0(t,x) : (t,x) \in \mathbb{R}_0^+ \times \mathbb{R} \},$$

$$m_i = \inf\{\underline{v}_{0i}(t,x) : (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}\},$$
$$M_0 = \sup\{\overline{u}_0(t,x) : (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}\},$$
$$M_i = \sup\{\overline{v}_{0i}(t,x) : (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}\},$$

$\underline{v} = (\underline{v}_{01}, \dots, \underline{v}_{0k})$, $\overline{v} = (\overline{v}_{01}, \dots, \overline{v}_{0k})$, $i = 1, \dots, k$.

Define two sequences $(\underline{u}_n, \underline{v}_n)$, $(\overline{u}_n, \overline{v}_n)$ of functions $\underline{u}_n, \overline{u}_n : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}$, $\underline{v}_n, \overline{v}_n : \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}^k$, $n \in N_0$, by the linear recurrence formulae

(4.15)
$$\begin{cases} \mathcal{L}\underline{u}_{n+1} = f(t,x,\underline{u}_n(t,x),\underline{v}_n(t,x)) + (c^2/4)\underline{u}_n(t,x) & (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}, \\ (\underline{v}_{n+1})_t = g(t,x,\underline{u}_n(t,x),\underline{v}_n(t,x)) & (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}, \\ \underline{u}_{n+1}(0,x) = \varphi_0(x) & x \in \mathbb{R}, \\ \underline{v}_{n+1}(0,x) = \psi_0(x) & x \in \mathbb{R}, \\ (\underline{u}_{n+1})_t(0,x) = \varphi_1(x) & x \in \mathbb{R}, \end{cases}$$

(4.16)
$$\begin{cases} \mathcal{L}\overline{u}_{n+1} = f(t,x,\overline{u}_n(t,x)\overline{v}_n(t,x)) + (c^2/4)\overline{u}_n(t,x) & (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}, \\ (\overline{v}_{n+1})_t = g(t,x,\overline{u}_n(t,x),\overline{v}_n(t,x)) & (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}, \\ \overline{u}_{n+1}(0,x) = \varphi_0(x) & x \in \mathbb{R}, \\ \overline{v}_{n+1}(0,x) = \psi_0(x) & x \in \mathbb{R}, \\ (\overline{u}_{n+1})_t(0,x) = \varphi_1(x) & x \in \mathbb{R}, \end{cases}$$

where $\mathcal{L}$ is given in (2.3).

Theorem 4.6. *Suppose that Assumption* A *is satisfied and*

(i) $f \in C^1(\mathbb{R}_0^+ \times \mathbb{R} \times \langle m, M \rangle, \mathbb{R})$, $g \in C^1(\mathbb{R}_0^+ \times \mathbb{R} \times \langle m, M \rangle, \mathbb{R}^k)$;
(ii) f, g *are Lipschitz continuous, with a constant* L*, with respect to* p, r *in* $\mathbb{R}_0^+ \times \mathbb{R} \times \langle m, M \rangle$;
(iii) g_x, g_p, g_{r_i}, $i = 1, \dots, k$, *are Lipschitz continuous, with a constant* L_1*, with respect to* x, p, r *in* $\mathbb{R}_0^+ \times \mathbb{R} \times \langle m, M \rangle$;
(iv) $f(t,x,p,r) + (c^2/4)p$ *and* g *are nondecreasing with respect to* p, r *in* $\mathbb{R}_0^+ \times \mathbb{R} \times \langle m, M \rangle$;
(v) $c \geq 0$;
(vi) $\varphi_0 \in C^2(\mathbb{R}, \mathbb{R})$, $\varphi_1 \in C^1(\mathbb{R}, \mathbb{R})$, $\psi_0 \in C^1(\mathbb{R}, \mathbb{R}^k)$;
(vii) $(\psi_0)_x$ *is Lipschitz continuous with a constant* L_0 *in* $\mathbb{R}$.

Then,

(a) *there exist unique solutions* $(\underline{u}_n, \underline{v}_n), (\overline{u}_n, \overline{v}_n) \in C^2(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}) \times C^1(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^k)$, $n \in \mathbb{N}_0$, *of* (4.15), (4.16), *respectively*;

(b) *the inequalities*

$$\underline{u}_0 \le \underline{u}_1 \le \underline{u}_2 \le \cdots \le \overline{u}_2 \le \overline{u}_1 \le \overline{u}_0,$$
$$\underline{v}_0 \le \underline{v}_1 \le \underline{v}_2 \le \cdots \le \overline{v}_2 \le \overline{v}_1 \le \overline{v}_0$$

hold in $\mathbb{R}_0^+ \times \mathbb{R}$;

(c) $(\underline{u}_n, \underline{v}_n), (\overline{u}_n, \overline{v}_n)$, $n \in \mathbb{N}_0$, *are lower and upper solutions of* (1.1), (1.2) *in* $\mathbb{R}_0^+ \times \mathbb{R}$, *respectively*;

(d) $\lim_{n\to\infty}(\overline{u}_n(t,x) - \underline{u}_n(t,x)) = 0$, $\lim_{n\to\infty}(\overline{v}_n(t,x) - \underline{v}_n(t,x)) = 0$ *almost uniformly in* $\mathbb{R}_0^+ \times \mathbb{R}$;

(e) *the function*

$$\begin{aligned}(u(t,x), v(t,x)) &= \lim_{n\to\infty}(\underline{u}_n(t,x), \underline{v}_n(t,x)) \\ &= \lim_{n\to\infty}(\overline{u}_n(t,x), \overline{v}_n(t,x)) \in C^2(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}) \\ &\qquad \times C^1(\mathbb{R}_0^+ \times \mathbb{R}, \mathbb{R}^k)\end{aligned}$$

is a unique solution of (1.1), (1.2) *in the sector* $\langle(\underline{u}_0, \underline{v}_0), (\overline{u}_0, \overline{v}_0)\rangle$.

Proof. Observe that, for a fixed $n \in \mathbb{N}_0$, the right-hand sides of (4.15) and (4.16) are known and depend only upon t, x. Hence, Theorem 4.3 implies (a).

Statements (b) and (c) are an intermediate consequence of induction, Assumption A and the following implication. If $(\underline{u}_n, \underline{v}_n), (\overline{u}_n, \overline{v}_n) \in \langle(\underline{u}_0, \underline{v}_0), (\overline{u}_0, \overline{v}_0)\rangle$, $n \in \mathbb{N}_0$, are lower and upper solutions of (1.1) and (1.2) in $\mathbb{R}_0^+ \times \mathbb{R}$, respectively, $\underline{u}_n \le \overline{u}_n$, $\underline{v}_n \le \overline{v}_n$, then

$$\underline{u}_n \le \underline{u}_{n+1} \le \overline{u}_n, \qquad \underline{v}_n \le \underline{v}_{n+1} \le \overline{v}_n, \tag{4.17}$$
$$\underline{u}_n \le \overline{u}_{n+1} \le \overline{u}_n, \qquad \underline{v}_n \le \overline{v}_{n+1} \le \overline{v}_n,$$
$$\underline{u}_{n+1} \le \overline{u}_{n+1}, \qquad \underline{v}_{n+1} \le \overline{v}_{n+1}, \tag{4.18}$$

and $(\underline{u}_{n+1}, \underline{v}_{n+1}), (\overline{u}_{n+1}, \overline{v}_{n+1}) \in \langle(\underline{u}_0, \underline{v}_0), (\overline{u}_0, \overline{v}_0)\rangle$, $n \in \mathbb{N}_0$, are lower and upper solutions of (1.1) and (1.2) in $\mathbb{R}_0^+ \times \mathbb{R}$, respectively. Let a predecessor of this implication hold. Now, we are able to show the first inequality in (4.17). Due to the fact that $\underline{u}_n$ is a lower solution of (1.1), (1.2) and $\underline{u}_{n+1}$ is defined in (4.15), we have

$$\mathcal{L}\underline{u}_n \le f(t, x, \underline{u}_n(t,x), \underline{v}_n(t,x)) + \frac{c^2}{4}\underline{u}_n(t,x), \quad (t,x) \in \mathbb{R}_0^+ \times \mathbb{R},$$

$$\mathcal{L}\underline{u}_{n+1} = f(t,x,\underline{u}_n(t,x),\underline{v}_n(t,x)) + \frac{c^2}{4}\underline{u}_n(t,x), \quad (t,x) \in \mathbb{R}_0^+ \times \mathbb{R},$$

and hence,

$$\mathcal{L}(\underline{u}_n - \underline{u}_{n+1}) \le 0 \quad \text{in } \mathbb{R}_0^+ \times \mathbb{R}.$$

Moreover,

$$\underline{u}_n - \underline{u}_{n+1} \le 0 \quad \text{on } \{0\} \times \mathbb{R}$$

and by assumption (v),

$$\mathcal{L}_0(\underline{u}_n - \underline{u}_{n+1}) \le 0 \quad \text{on } \{0\} \times \mathbb{R}.$$

Then, using Theorem 2.2 for $\mathcal{L}_a = \mathcal{L}$ gives

$$\underline{u}_n - \underline{u}_{n+1} \le 0 \quad \text{in } \mathbb{R}_0^+ \times \mathbb{R}.$$

Reasoning in a similar way as above and using assumption (iv), we obtain the relations

$$\begin{aligned} \mathcal{L}\underline{u}_{n+1} &= f(t,x,\underline{u}_n(t,x),\underline{v}_n(t,x)) + \frac{c^2}{4}\underline{u}_n(t,x), \quad (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}, \\ \mathcal{L}\overline{u}_n &\ge f(t,x,\overline{u}_n(t,x),\overline{v}_n(t,x)) + \frac{c^2}{4}\overline{u}_n(t,x), \quad (t,x) \in \mathbb{R}_0^+ \times \mathbb{R}, \\ \mathcal{L}(\underline{u}_{n+1} - \overline{u}_n) &\le f(t,x,\underline{u}_n(t,x),\underline{v}_n(t,x)) + \frac{c^2}{4}\underline{u}_n(t,x) \\ &\quad - f(t,x,\overline{u}_n(t,x),\overline{v}_n(t,x)) - \frac{c^2}{4}\overline{u}_n(t,x) \le 0 \quad \text{in } \mathbb{R}_0^+ \times \mathbb{R}, \end{aligned}$$

$$\begin{aligned} \underline{u}_{n+1} - \overline{u}_n &\le 0 \quad \text{on } \{0\} \times \mathbb{R}, \\ \mathcal{L}_0(\underline{u}_{n+1} - \overline{u}_n) &\le 0 \quad \text{on } \{0\} \times \mathbb{R}, \end{aligned}$$

and, from Theorem 2.2,

$$\underline{u}_{n+1} - \overline{u}_n \le 0 \quad \text{in } \mathbb{R}_0^+ \times \mathbb{R}.$$

Therefore, the second inequality in (4.17) is also true. The other inequalities in (4.17), (4.18) may be proved, in a similar way, for $\underline{v}_n$ and $\overline{v}_n$ by using the simple weak maximum principle for ordinary differential inequalities as in Theorem 2.2.

Proof of (d). Fix $T \in \mathbb{R}^+$, and consider the same isosceles triangle $\Delta(X,T) \subset \mathbb{R} \times \mathbb{R}_0^+$ as in the proof of Theorem 4.3.

Set

$$N_0 = \|\overline{u}_0 - \underline{u}_0\|_{\Delta(X,T)} + \|\overline{v}_0 - \underline{v}_0\|_{\Delta(X,T)}. \tag{4.19}$$

It follows from definitions (4.15), (4.16) and the integral formula (3.1) that

(4.20)

$$\begin{aligned}
\underline{u}_{n+1}(t,x) &= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-[c(t-s)]/2} \\
&\quad \times \left[f(s,y,\underline{u}_n(s,y),\underline{v}_n(s,y)) + \frac{c^2}{4}\underline{u}_n(s,y)\right] dy\, ds \\
&\quad + \frac{1}{2}e^{-(ct)/2}\int_{x-t}^{x+t} \varphi_1(y)\, dy + \frac{c}{4}e^{-(ct)/2}\int_{x-t}^{x+t}\varphi_0(y)\, dy \\
&\quad + \frac{1}{2}e^{-(ct)/2}[\varphi_0(x+t)+\varphi_0(x-t)], \\
\underline{v}_{n+1}(t,x) &= \psi_0(x) + \int_0^t g(s,x,\underline{u}_n(s,x),\underline{v}_n(s,x))\, ds, \\
\overline{u}_{n+1}(t,x) &= \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} e^{-[c(t-s)]/2} \\
&\quad \times \left[f(s,y,\overline{u}_n(s,y),\overline{v}_n(s,y)) + \frac{c^2}{4}\overline{u}_n(s,y)\right] dy\, ds \\
&\quad + \frac{1}{2}e^{-(ct)/2}\int_{x-t}^{x+t} \varphi_1(y)\, dy + \frac{c}{4}e^{-(ct)/2}\int_{x-t}^{x+t}\varphi_0(y)\, dy \\
&\quad + \frac{1}{2}e^{-(ct)/2}[\varphi_0(x+t)+\varphi_0(x-t)], \\
\overline{v}_{n+1}(t,x) &= \psi_0(x) + \int_0^t g(s,x,\overline{u}_n(s,x),\overline{v}_n(s,x))\, ds,
\end{aligned}$$

for $(t,x) \in \Delta(X,T)$, $n \in \mathbb{N}_0$. By induction and assumptions (i), (ii) and (vi), we obtain the following estimates

$$\begin{aligned}
\overline{u}_n(t,x) - \underline{u}_n(t,x) &\le \frac{N_0(L+c^2/4)^n T(1+T)^{n-1}t^n}{n!}, \\
\overline{v}_{ni}(t,x) - \underline{v}_{ni}(t,x) &\le \frac{N_0(L+c^2/4)^n (1+T)^{n-1}t^n}{n!},
\end{aligned} \tag{4.21}$$

for $(t,x) \in \Delta(X,T)$, $i = 1,\dots,k$, $n \in \mathbb{N}$. As a direct conclusion of (4.21) and (b), we obtain

$$\lim_{n\to\infty}(\overline{u}_n(t,x)-\underline{u}_n(t,x))=0,\quad \lim_{n\to\infty}(\overline{v}_n(t,x)-\underline{v}_n(t,x))=0 \tag{4.22}$$

uniformly in $\Delta(X,T)$, thus almost uniformly in $\mathbb{R}_0^+\times\mathbb{R}$.

The sequences of continuous functions $(\underline{u}_n,\underline{v}_n)$, $(\overline{u}_n,\overline{v}_n)$ are from (b) monotone and bounded, and (4.22) holds, so a continuous function $(u,v):\Delta(X,T)\to\mathbb{R}^{1+k}$ exists such that

$$\lim_{n\to\infty}(\underline{u}_n(t,x),\underline{v}_n(t,x))=\lim_{n\to\infty}(\overline{u}_n(t,x),\overline{v}_n(t,x))=(u(t,x),v(t,x)) \tag{4.23}$$

uniformly in $\Delta(X,T)$. From Lemma 4.2 regarding the limit transition under the sign of an integral used in (4.20), we get that (u,v) is a solution in $\Delta(X,T)$ of (3.1). The existence of the continuous v_x in $\Delta(X,T)$ is analogously proved as in the proof of Theorem 4.3. This derivative equals the limit of the suitable subsequence $((\underline{v}_{n_l})_x)$, and obviously $((\overline{v}_{n_l})_x)$. Moreover, it follows from the construction that (u,v) belongs to the restriction of the sector $\langle(\underline{u}_0,\underline{v}_0),(\overline{u}_0,\overline{v}_0)\rangle$ to $\Delta(X,T)$.

Let $(\widetilde{u},\widetilde{v})$ belonging to the restriction of the sector $\langle(\underline{u}_0,\underline{v}_0),(\overline{u}_0,\overline{v}_0)\rangle$ to $\Delta(X,T)$ be any continuous solution of (3.1) in $\Delta(X,T)$. The use of induction and assumptions (i), (ii), (vi) implies the estimates

$$|\overline{u}_n(t,x)-\widetilde{u}(t,x)|\le\frac{N_0(L+c^2/4)^nT(1+T)^{n-1}t^n}{n!},\quad \|\overline{v}_n(t,x)-\widetilde{v}(t,x)\|\le\frac{N_0(L+c^2/4)^n(1+T)^{n-1}t^n}{n!}, \tag{4.24}$$

where N_0 is defined in (4.19), for $(t,x)\in\Delta(X,T)$, $n\in\mathbb{N}$. From (4.23) and (4.24), we have $|u(t,x)-\widetilde{u}(t,x)|\le 0$, $\|v(t,x)-\widetilde{v}(t,x)\|\le 0$ in $\Delta(X,T)$, and consequently, $u=\widetilde{u}$, $v=\widetilde{v}$ in $\Delta(X,T)$.

We construct a global solution by piecing together the solutions in all of the triangles $\Delta(X,T)$ and, by Lemma 3.1, (e) follows, and the proof is complete. □

Remark 4.7. Due to the fact that $u=\widetilde{u}$, $v=\widetilde{v}$ in $\Delta(X,T)$ in the proof of Theorem 4.6, inequalities (4.24) give the effective estimate of the error of the analytical method

$$|\overline{u}_n(t,x)-u(t,x)| \le \frac{N_0(L+c^2/4)^n T(1+T)^{n-1}t^n}{n!},$$
$$\|\overline{v}_n(t,x)-v(t,x)\| \le \frac{N_0(L+c^2/4)^n (1+T)^{n-1}t^n}{n!},$$

for $(t,x)\in\Delta(X,T)$, $n\in\mathbb{N}$. Analogously, the same estimate is true for $|\underline{u}_n(t,x)-u(t,x)|$, $\|\underline{v}_n(t,x)-v(t,x)\|$.

Remark 4.8. Due to the fact that the assumptions of Theorem 4.6 only hold in the set $\mathbb{R}_0^+\times\mathbb{R}\times\langle m,M\rangle$, the uniqueness of a solution of (1.1), (1.2) is ensured only with respect to the given upper and lower solutions; therefore, this does not rule out the existence of other solutions outside the sector $\langle(\underline{u}_0,\underline{v}_0),(\overline{u}_0,\overline{v}_0)\rangle$. Thus, if the assumptions of this theorem hold in $\mathbb{R}_0^+\times\mathbb{R}^{2+k}$, then the solution of (1.1), (1.2) is unconditionally unique.

Remark 4.9. It follows from the proof of Theorems 4.3 and 4.6 that analogous proofs are true for only one equation, instead of system (1.1),

$$u_{tt}-u_{xx}+cu_t=f(t,x,u),\quad (t,x)\in\mathbb{R}_0^+\times\mathbb{R}, \tag{4.25}$$

with the initial conditions

$$\begin{cases} u(0,x)=\varphi_0(x) & x\in\mathbb{R},\\ u_t(0,x)=\varphi_1(x) & x\in\mathbb{R},\end{cases} \tag{4.26}$$

where $f:\mathbb{R}_0^+\times\mathbb{R}^2\to\mathbb{R}$, $\varphi_0,\varphi_1:\mathbb{R}\to\mathbb{R}$ and a constant $c\in\mathbb{R}$ are given.

4.3. The construction of upper and lower solutions. In the ample literature on monotone methods there is no general method for building upper and lower solutions. However, if f,g are bounded in $\mathbb{R}_0^+\times\mathbb{R}$ and $\varphi_0,\varphi_1,\psi_0$ are bounded in $\mathbb{R}$, then such a construction is possible by solving associated suitable initial ordinary differential problems.

Let

$$m(f)=\inf\{f(t,x,u,v):(t,x,u,v)\in\mathbb{R}_0^+\times\mathbb{R}^{2+k}\},$$
$$m(g)=\inf\{g(t,x,u,v):(t,x,u,v)\in\mathbb{R}_0^+\times\mathbb{R}^{2+k}\},$$

$$
\begin{aligned}
m(\varphi_0) &= \inf\{\varphi_0(x) : x \in \mathbb{R}\},\\
m(\varphi_1) &= \inf\{\varphi_1(x) : x \in \mathbb{R}\},\\
m(\psi_0) &= \inf\{\psi_0(x) : x \in \mathbb{R}\},\\
M(f) &= \sup\{f(t,x,u,v) : (t,x,u,v) \in \mathbb{R}_0^+ \times \mathbb{R}^{2+k}\},\\
M(g) &= \sup\{g(t,x,u,v) : (t,x,u,v) \in \mathbb{R}_0^+ \times \mathbb{R}^{2+k}\},\\
M(\varphi_0) &= \sup\{\varphi_0(x) : x \in \mathbb{R}\},\\
M(\varphi_1) &= \sup\{\varphi_1(x) : x \in \mathbb{R}\},\\
M(\psi_0) &= \sup\{\psi_0(x) : x \in \mathbb{R}\}.
\end{aligned}
$$

Consider the case $c > 0$. The functions

$$
\begin{aligned}
\underline{u}_0(t,x) &= \frac{1}{c^2}[m(f) - cm(\varphi_1)]e^{-ct} + \frac{1}{c}m(f)t\\
&\quad + \frac{1}{c^2}[c^2 m(\varphi_0) + cm(\varphi_1) - m(f)],\\
\underline{v}_0(t,x) &= m(g)t + m(\psi_0),\\
\overline{u}_0(t,x) &= \frac{1}{c^2}[M(f) - cM(\varphi_1)]e^{-ct} + \frac{1}{c}M(f)t\\
&\quad + \frac{1}{c^2}[c^2 M(\varphi_0) + cM(\varphi_1) - M(f)],\\
\overline{v}_0(t,x) &= M(g)t + M(\psi_0)
\end{aligned}
\tag{4.27}
$$

are lower and upper solutions of (1.1) and (1.2), respectively. For example, the lower solution $\underline{u}_0$ is a solution of the initial second-order ordinary differential problem

$$
\begin{cases}
y'' + cy' = m(f),\\
y(0) = m(\varphi_0),\\
y'(0) = m(\varphi_1).
\end{cases}
\tag{4.28}
$$

In the case $c = 0$, we have

$$
\begin{aligned}
\underline{u}_0(t,x) &= \frac{1}{2}m(f)t^2 + m(\varphi_1)t + m(\varphi_0),\\
\overline{u}_0(t,x) &= \frac{1}{2}M(f)t^2 + M(\varphi_1)t + M(\varphi_0).
\end{aligned}
\tag{4.29}
$$

The monotone methods and a construction of upper and lower solutions with the use of Green's function for parabolic finite and infinite systems are studied in [**1, 18, 19**].

It is sometimes possible to find upper and lower solutions without the use of any mathematical tools, as in Example 4.10 or for the Hodgkin-Huxley system studied in [**9**].

Example 4.10. Consider the nonlinear telegraph equation

$$u_{tt} - u_{xx} + cu_t = u(1-u), \tag{4.30}$$

with the initial conditions

$$\begin{cases} u(0,x) = \varphi_0(x) & x \in \mathbb{R}, \\ u_t(0,x) = 0 & x \in \mathbb{R}, \end{cases} \tag{4.31}$$

where $c \geq 2$ and $0 \leq \varphi_0(x) \leq 1$ in $\mathbb{R}$ is of the C^2 class, see Remark 4.9. It is clear that $\underline{u}_0(t,x) \equiv 0$ is a lower solution and $\overline{u}_0(t,x) \equiv 1$ is an upper solution of (4.30), (4.31). It follows from Theorem 4.6 that problem (4.30), (4.31) has a unique global solution u of the C^2 class in the sector $\langle \underline{u}_0, \overline{u}_0 \rangle = \langle 0, 1 \rangle$. Note that $f(t,x,p) = p(1-p)$ is not nondecreasing for $p \in \langle m, M \rangle = \langle 0, 1 \rangle$, but

$$f(t,x,p) + \frac{c^2}{4}p = p\bigg(\Big(1 + \frac{c^2}{4}\Big) - p\bigg)$$

is nondecreasing in this interval.

Example 4.11. Consider the nonlinear system

$$\begin{cases} u_{tt} - u_{xx} + cu_t = 1/(1+u^2) + \operatorname{arctg} v, \\ v_t = \operatorname{arctg}(u+v), \end{cases} \tag{4.32}$$

with the initial conditions (1.2), where $c \geq 2$, $|\varphi_0(x)| \leq 1$, $|\psi_0(x)| \leq 1$, $|\varphi_1(x)| \leq 1$ in $\mathbb{R}$ and φ_0 is of the C^2 class, ψ_0, φ_1 are of the C^1 class, and $(\psi_0)_x$ is Lipschitz continuous. We calculate the following from (4.27):

$$\underline{u}_0 = \frac{1}{c^2}\bigg(-\frac{\pi}{2} + c\bigg)e^{-ct} - \frac{\pi}{2c}t + \frac{1}{c^2}\bigg(-c^2 - c + \frac{\pi}{2}\bigg),$$

$$\underline{v}_0 = -\frac{\pi}{2}t - 1,$$

$$\overline{u}_0 = \frac{1}{c^2}\left(1 + \frac{\pi}{2} - c\right)e^{-ct} + \frac{1}{c}\left(1 + \frac{\pi}{2}\right)t + \frac{1}{c^2}\left(c^2 + c - 1 - \frac{\pi}{2}\right),$$
$$\overline{v}_0 = \frac{\pi}{2}t + 1.$$

Observe that Assumption A holds and $\underline{u}_0, \underline{v}_0, \overline{u}_0, \overline{v}_0$ are unbounded. It follows from Theorem 4.6 that problem (4.32), (1.2) has a unique global solution (u, v) of the $C^2 \times C^1$ class in the sector $\langle(\underline{u}_0, \underline{v}_0), (\overline{u}_0, \overline{v}_0)\rangle$. Note that $f(t, x, p, r) = 1/(1 + p^2) + \mathrm{arctg} r$ is not nondecreasing for $(p, r) \in \langle m, M\rangle = \mathbb{R}^2$, but

$$f(t, x, p, r) + \frac{c^2}{4}p = \frac{1}{1 + p^2} + \frac{c^2}{4}p + \mathrm{arctg} r$$

is nondecreasing in this interval.

Example 4.12. Consider the nonlinear system

$$\begin{cases} \tau u_{tt} - u_{xx} + u_t = H(u - a) - u - v, \\ v_t = bu - dv, \end{cases} \tag{4.33}$$

where $\tau \geq 0$ is the so-called time of relaxation, a, b, d are real constants and H is the Heaviside function. The term τu_{tt}, taking into account effects of memory connected with media internal structure, is generated by the Cattaneo law

$$\tau\frac{\partial J}{\partial t} + J = -k\nabla u,$$

which is a generalization of the conventional Fick law

$$J = -k\nabla u,$$

where J is a flux of u and k is a real constant, see **[11, 14]**. Physical motivation of system (4.33) together with construction of the smooth solitary wave solutions and their stability are given in **[14]**. To the present day, the author unfortunately has not proved any existence results on a classical solution to the Cauchy problem concerned (4.33) due to non-continuity of H. However, it follows from **[11, 14, 20, 22]** that, from a physical point of view, the piecewise linear term

$$H(u - a) - u$$

in (4.33) can be changed by smooth functions, for example, by some polynomials.

5. Conclusions. This manuscript deals with the Cauchy problem for the one-dimensional system of partial differential equations (1.1). This system is composed of one partial hyperbolic second-order equation and an ordinary subsystem with a space parameter x. It appears, for example, as a model for the propagation of nerve impulses along axons. As was mentioned in Example 4.12, it can be obtained from the Cattaneo law for fluxes, and it is a generalization of the Hodgkin-Huxley, FitzHugh-Nagumo and McKean models, taking into account effects of memory connected with media internal structure. Example 4.10 is concerned with only the first equation in (1.1) (the telegraph equation with the Fischer reaction term), and the abstract Example 4.11 shows that applications in other physical models are possible.

The main results of this manuscript are summarized as theorems on the global existence and uniqueness of a classical solution to the Cauchy problem (1.1), (1.2). In order to prove these, we use two different approaches: the Picard iteration method and the monotone method of upper and lower solutions. The equivalence between (1.1), (1.2) and its integral version under suitable regularity assumptions on the solutions (u, v) is crucial in the proof of the existence and uniqueness results. Moreover, in the proof of a convergence of the monotone method, an important tool is a weak maximum principle for inequalities generated by a linear homogenous version of the telegraph equation. The most important assumption for the first approach is the global Lipschitz condition on the right hand sides f, g and their first order derivatives, but for the second approach, the most important assumption is the local Lipschitz condition together with suitable monotonicity for f, g.

REFERENCES

1. S. Brzychczy and R.R. Poznański, *Mathematical neuroscience*, Academic Press, Amsterdam, 2014.

2. R. Carles, R. Danchin and J.C. Saut, *Madelung, Gross-Pitaevskii and Korteveg*, Nonlinearity **25** (2012), 2843–2873.

3. M. Danielewski, *The Planck-Kleinert crystal*, Z. Natur. **62** (2007), 564–568.

4. L. de Broglie, *La mécanique ondulatoire et la structure atomique de la matière et du rayonnement*, J. Phys. Rad. **8** (1927), 225–241.

5. J.W. Evans, *Nerve axon equations*: I, *Linear approximations*, Indiana Univ. Math. J. **21** (1972), 877–885.

6. ______, *Nerve axon equations*: II, *Stability at rest*, Indiana Univ. Math. J. **22** (1972), 75–90.

7. J.W. Evans, *Nerve axon equations*: III, *Stability of the nerve impulse*, Indiana Univ. Math. J. **22** (1972), 577–593.

8. ______, *Nerve axon equations*: IV, *The stable and the unstable impulse*, Indiana Univ. Math. J. **24** (1975), 1169–1190.

9. J. Evans and N. Shenk, *Solutions to axon equations*, Biophys. J. **10** (1970), 1090–1101.

10. A. Ghazaryan, Y. Latushkin and S. Schecter, *Stability of traveling waves in partly parabolic systems*, Math. Model. Nat. Phenom. **8** (2013), 32–48.

11. D.D. Joseph and L. Preziosi, *Heat waves*, Rev. Mod. Phys. **61** (1989), 41–73.

12. M. Krupa, B. Sanstede and P. Szmolyan, *Fast and slow waves in the FitzHugh-Nagumo equation*, J. Differ. Eq. **133** (1997), 49–97.

13. Y. Li, *Maximum principles and the method of upper and lower solutions for time periodic problems of the telegraph equations*, J. Math. Anal. Appl. **327** (2007), 997–1009.

14. W. Likus and V.A. Vladimirov, *Solitary waves in the model of active media, taking into account relaxing effects*, Rep. Math. Phys. **75** (2015), 213–230.

15. E. Madelung, *Quantentheorie in hydrodynamischer form*, Z. Phys. Hadron. Nucl. **40** (1927), 322–326.

16. C.V. Pao, *Nonlinear parabolic and elliptic equations*, Plenum, New York, 1992.

17. M.H. Protter and H.F. Weinberger, *Maximum principles in differential equations*, Springer-Verlag, New York, 1984.

18. A. Pudełko, *Monotone iteration for infinite systems of parabolic equations with functional dependence*, Ann. Polon. Math. **90** (2007), 1–19.

19. ______, *Monotone iterative method for infinite systems of parabolic functional-differential equations with nonlocal initial conditions*, Topol. Meth. Nonlin. Anal. **36** (2010), 101–117.

20. J. Rinzel and J.B. Keller, *Traveling waves solutions of a nerve conduction equation*, Biophys. J. **13** (1974), 1313–1337.

21. L. Sapa, *Existence, uniqueness and estimates of classical solutions to some evolutionary system*, Opuscula Math. **35** (2015), 935–956.

22. A.C. Scott, *Neuroscience*: *Mathematical primer*, Springer-Verlag, New York, 2002.

23. V.A. Vladimirov and Cz. Mączka, *On the stability of kink-like and soliton-like solutions of the generalized convection-reaction-diffusion equation*, Rep. Math. Phys. **70** (2012), 313–329.

AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Kraków, Poland

Email address: **sapa@agh.edu.pl**