N2br:

$$I_n = \int \frac{dx}{(1+x^2)^n}, \quad n \geq 2$$

$$I_n = I_{n-1} + \frac{1}{2n-2} \cdot \frac{x}{(x^2+1)^{n-1}} - \frac{1}{2n-2} \cdot I_{n-1}$$

To samo: $\boxed{I_n = \frac{2n-3}{2n-2} I_{n-1} + \frac{1}{2n-2} \cdot \frac{x}{(x^2+1)^{n-1}}}$

$$\int \frac{dx}{(x^2+1)^n} = \int \frac{x^2+1-x^2}{(x^2+1)^n}\,dx = \int \frac{dx}{(x^2+1)^{n-1}} - \int \frac{x^2\,dx}{(x^2+1)^n}$$

$$I_n = I_{n-1} - \underbrace{\int \frac{x^2}{(x^2+1)^n}\,dx}_{\Delta} = \boxed{\ldots}$$

$$\Delta = \int \frac{x^2}{(x^2+1)^n} = \int x \cdot \frac{x}{(x^2+1)^n}\,dx = \left\{ \begin{array}{l|l} u = x & u' = 1 \\ v' = \frac{x}{(x^2+1)^n} & v = \frac{-1}{2(n-1)(x^2+1)^{n-1}} \end{array} \right\} = C$$

$$V = \int \frac{x\,dx}{(x^2+1)^n} = \frac{1}{2}\int \frac{dt}{t^n} = \frac{1}{2}\,\frac{t^{-n+1}}{-n+1} = \frac{1}{2}\,\frac{(x^2+1)^{1-n}}{1-n} = \frac{-1}{2(n-1)(x^2+1)}$$

$t = x^2+1$

$dt = 2x\,dx$

$dx = \frac{dt}{2x}$

$$\circ\ \cancel{A} = \frac{-x}{2(n-1)(x^2+1)^{n-1}} + \int \frac{dx}{2(n-1)(x^2+1)^{n-1}} = \frac{-1}{2n-2}\cdot\frac{x}{(x^2+1)^{n-1}} + \frac{1}{2n-2}\cdot I_{n-1}$$

$$\boxed{\dots} = I_{n-1} + \frac{1}{2n-2}\cdot\frac{x}{(x^2+1)^{n-1}} - \frac{1}{2n-2}\cdot I_{n-1}$$

⇒ (B)

Całki funkcji trygonometrycznych:

Podstawienie:

1. $t=\sin x$, $t=\cos x$
2. $t=\operatorname{tg} x$
3. $t=\operatorname{tg}\frac{x}{2}$ (UNIWERSALNE)

$$R(x,y,z) = \frac{w(x,y,z)}{p(x,y,z)}$$

w, p - wielomiany trzech zmiennych
R - FUNKCJA WYMIERNA

np. $R^{(x,y,z)} = \frac{x^2y + x + xz^2}{x^2+y^3+1}$

(3) $\int R(\sin x, \cos x, \operatorname{tg} x)\,dx =$ (całka funkcji wymiernej jednej zmiennej)

$\boxed{t=\operatorname{tg}\frac{x}{2}}$

$$= \int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}, \frac{2t}{1-t^2}\right)\frac{2\,dt}{1+t^2}$$

WZORY:

$$\cos x = \frac{1-t^2}{1+t^2}$$

$$\sin x = \frac{2t}{1+t^2}$$

$$dx = \frac{2dt}{1+t^2}$$

!

Wyprowadzenie wzoru: dla $t=\operatorname{tg}\frac{x}{2}$

$$dt = \frac{1}{2}\,\frac{1}{\cos^2\frac{x}{2}}\,dx$$

$$dx = 2\cos^2\frac{x}{2}\,dt$$

$$t^2 = \frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}$$

$$t^2\cos^2\frac{x}{2} = \sin^2\frac{x}{2}$$

$$(1+t^2)\cos^2\frac{x}{2} = 1$$

$$\cos^2\frac{x}{2} = \frac{1}{1+t^2}$$

$$dx = \frac{2\,dt}{1+t^2}$$

$$\sin x = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{2t}{1+t^2}$$

$$\cos x = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$$

~~3~~ (2) $\int R(\sin^2 x, \cos^2 x, \sin x \cos x)\,dx = \dots$ | całka funkcji wymiernej jednej zmiennej

$$\boxed{t = \operatorname{tg} x}$$

WYPROWADZENIE :

$$dt = \frac{1}{\cos^2 x}\,dx$$

$$dx = \frac{dt}{1+t^2}$$

$$\sin^2 x = \frac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x} = \frac{t^2}{1+t^2}$$

$$\cos^2 x = \frac{1}{1+\operatorname{tg}^2 x} = \frac{1}{1+t^2}$$

$$\sin x \cos x = \frac{\operatorname{tg} x}{1+\operatorname{tg}^2 x} = \frac{t}{1+t^2}$$

WZORY

$$\cos^2 x = \frac{1}{1+t^2}$$

$$\sin^2 x = \frac{t^2}{1+t^2}$$

$$\sin x \cos x = \frac{t}{1+t^2}$$

$$dx = \frac{dt}{1+t^2}$$

$$\dots = \int R\left(\frac{t^2}{1+t^2}, \frac{1}{1+t^2}, \frac{t}{1+t^2}\right)\frac{dt}{1+t^2}$$

pa 1)

$$\int \frac{1+\sin x\cos x}{(2+\cos^2 x)(1+\sin^2 x)}\,dx \overset{t=\operatorname{tg}x}{=} \int \frac{1+\frac{t}{1+t^2}}{\left(2+\frac{1}{1+t^2}\right)\left(1+\frac{t^2}{1+t^2}\right)}\,\frac{dt}{1+t^2} =$$

$$= \int \frac{1+t^2+t}{(2+2t^2+1)(1+t^2+t^2)}\,dt = \int \frac{t^2+t+1}{(2t^2+3)(2t^2+1)}\,dt = \dots$$

$$\frac{t^2+t+1}{(2t^2+3)(2t^2+1)} = \frac{At+B}{2t^2+3} + \frac{Ct+D}{2t^2+1}$$

$$\frac{t^2+t+1}{(2t^2+3)(2t^2+1)} = \frac{(2t^2+1)(At+B)+(2t^2+3)(Ct+D)}{(2t^2+3)(2t^2+1)}$$

$$t^2+t+1 = (2t^2+1)(At+B)+(2t^2+3)(Ct+D)$$

$A = -\frac{1}{2}$

$B = \frac{1}{4}$

$C = \frac{1}{2}$

$D = \frac{1}{4}$

$$\ldots = \frac{1}{2}\cdot\frac{1}{4}\int\frac{4t\,dt}{2t^2+3} + \frac{1}{4}\int\frac{dt}{2t^2+3} + \frac{1}{2}\cdot\frac{1}{4}\int\frac{4t\,dt}{2t^2+1} + \frac{1}{4}\int\frac{dt}{2t^2+1} =$$

$$= -\frac{1}{8}\ln(2t^2+3) + \frac{1}{8}\ln(2t^2+1) + \frac{1}{4}\underbrace{\int\frac{dt}{2t^2+3}}_{A} + \frac{1}{4}\underbrace{\int\frac{dt}{2t^2+1}}_{B} = \ldots$$

I $A = \int\frac{dt}{2t^2+3} = \frac{1}{2}\int\frac{dt}{t^2+\frac{3}{2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{3}{2}}}\operatorname{arctg}\frac{t}{\sqrt{\frac{3}{2}}} = \frac{1}{\sqrt{6}}\operatorname{arctg}\sqrt{\frac{2}{3}}\,t$

II $B = \int\frac{dt}{2t^2+1} = \frac{1}{2}\int\frac{dt}{t^2+\frac{1}{2}} = \frac{1}{2}\cdot\frac{1}{\sqrt{\frac{1}{2}}}\operatorname{arctg}\frac{t}{\sqrt{\frac{1}{2}}} = \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\,t)$

$$\ldots = -\frac{1}{8}\ln(2\operatorname{tg}^2x+3) + \frac{1}{8}\ln(2\operatorname{tg}^2x+1) + \frac{1}{4\sqrt{6}}\operatorname{arctg}\sqrt{\frac{2}{3}}\operatorname{tg}x +$$
$$+ \frac{1}{4\sqrt{2}}\operatorname{arctg}(\sqrt{2}\operatorname{tg}x) + C$$

A gdyby tak: $t := \operatorname{tg}\frac{x}{2}$

$$\int\frac{1+\sin x\cos x}{(2+\cos^2x)(1+\sin^2x)}\,dx = 2\int\frac{(1+2t^2+t^4+2t-2t^3)(1+t^2)}{(3t^4+2t^2+3)(t^4+6t^2+1)}\,dt =$$

$3t^4+2t^2+3=0$ — Rozkłado się na iloczyn dwóch wielomianów kwadratowych

$t^2 = u \geq 0$

$3u^2+2u+3=0$

$\Delta = 4-36 = -32 < 0$

ale ...

możliwy kąt: 1) — 2) t_1, t_2 3) t_1, t_2, t_3, t_4

zad. 2)

$$\int\left(\frac{\sin^2x\cos x}{\sin x+\cos x}\right)dx \overset{/\cdot\cos x}{=} \int\frac{\sin^2x\cos^2x}{\sin x\cos x+\cos^2x}\,dx = \int\frac{t^2\,dt}{(1+t^2)^2(t+1)} = \ldots$$

$t = \operatorname{tg}x$

$$\frac{t^2}{(1+t^2)^2(t+1)} = \frac{A_1t+B_1}{1+t^2} + \frac{A_2t+B_2}{(1+t^2)^2} + \frac{C}{t+1}$$

$$\ldots = -\frac{1}{4}\int\frac{t\,dt}{1+t^2} + \frac{1}{4}\int\frac{dt}{1+t^2} + \frac{1}{2}\underbrace{\int\frac{t\,dt}{(1+t^2)^2}}_{u} - \frac{1}{2}\underbrace{\int\frac{dt}{(1+t^2)^2}}_{I_2} + \frac{1}{4}\int\frac{dt}{t+1} = \ldots$$

wzór rekurencyjny $I_2 = \int\frac{dt}{(1+t^2)^2} = I_1 + \frac{1}{2}\cdot\frac{x}{x^2+1} - \frac{1}{2}I_1$

$I_1 = \int\frac{dt}{1+t^2} = \operatorname{arctg}t$

A gdyby tak: $t = tg\frac{x}{2}$

$$\int \frac{\sin^2 x \cos x}{\sin x + \cos x}\,dx = -8\int \frac{t^2(1-t^2)}{(1+t^2)^3(t-1+\sqrt{2})(t-1-\sqrt{2})}\,dt \quad \ldots$$

pr. 3)

$$\int \sin^2 x \cos^3 x\,dx = = \int \sin^2 \cos^2 x \underbrace{\cos x\,dx} = \int t^2(1-t^2)\,dt = \ldots$$

$$\begin{cases} t = \sin x \\ \underbrace{dt = \cos x\,dx} \end{cases}$$

* gdyby tak: $t = tg\,x$ nie zadziała

$$t = tg\frac{x}{2} \rightarrow 2\int \frac{4t^2(1-t^2)^3}{(1+t^2)^6}\,dt$$

WZORY:

① $I_n = \int \sin^n x\,dx$, $n \geq 2$ $\quad I_n = -\frac{1}{n}\sin^{n-1}(x)\cdot\cos x + \frac{n-1}{n}\cdot I_{n-2}$

② $\int \cos^n x\,dx = \int \sin^n\underbrace{\left(\frac{\pi}{2}+x\right)}_{u}\,dx \qquad u = \frac{\pi}{2}+x,\ du = dx$

$$\int \cos^n x\,dx = \int \sin^n u\,du$$

Całki funkcji niewymiernych:

(*) $\int R\left(x, \sqrt{ax^2+bx+c}\right)dx$

Podstawienie Eulera — pomijamy!

① $\int \frac{dx}{\sqrt{x^2+k}} = \ln\left|x+\sqrt{x^2+k}\right| + C$, $k \in \mathbb{R}$ $\qquad$ podstawienie $\sqrt{x^2+k} = t - x$

② $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C$, $a > 0$ $\qquad x = at$

③ $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)} + C$

①, ②, ③ $\overset{(*)}{\Longrightarrow} \int \frac{Ax+B}{\sqrt{ax^2+bx+c}}\,dx$ — szczególny przypadek (*)

Prz 1

$$\int \frac{3x+1}{\sqrt{x^2+5x-10}}\,dx = \frac{3}{2}\underbrace{\int \frac{2x+5}{\sqrt{x^2+5x-10}}\,dx}_{\mathrm{I}} - \frac{13}{2}\underbrace{\int \frac{dx}{\sqrt{x^2+5x-10}}}_{\mathrm{II}} =$$

$$= \dots$$

$$\mathrm{I} = 2\sqrt{x^2+5x-10} \leftarrow \text{wzór } ③$$

$$\mathrm{II} = \int \frac{dx}{\sqrt{x^2+5x-10}} = \int \frac{dx}{\sqrt{\left(x+\frac{5}{2}\right)^2 - \frac{65}{4}}} = \int \frac{du}{\sqrt{u^2-\frac{65}{4}}} \overset{\text{wzór } ①}{=}$$

$$u = x + \frac{5}{2}$$
$$du = dx$$

$$= \ln\left|u + \sqrt{u^2 - \frac{65}{4}}\right| = \ln\left|x + \frac{5}{2} + \sqrt{x^2+5x-10}\right|$$

$$\dots = \frac{3}{2}\cdot 2\sqrt{x^2+5x-10} - \frac{13}{2}\ln\left|x+\frac{5}{2}+\sqrt{x^2+5x-10}\right| + C$$

Podstawienie Eulera

$$\int R\left(x, \sqrt{ax^2+bx+c}\right)dx, \quad a \neq 0$$

I $\sqrt{ax^2+bx+c} = \sqrt{a}\,x + t$, $a > 0$

II $\sqrt{ax^2+bx+c} = (x - x_1)t$, $\Delta > 0$, x_1 – jeden z pierwiastków trójmianu kwadratowego

III $\sqrt{ax^2+bx+c} = xt + \sqrt{c}$, $c > 0$

Prz 2

$$\int \frac{dx}{\sqrt{x^2+x+1}} = \dots$$

$$\sqrt{x^2+x+1} = x + t$$
$$\cancel{x^2} + x + 1 = \cancel{x^2} + 2tx + t^2$$
$$x(1-2t) = t^2 - 1$$
$$x = \frac{t^2-1}{1-2t}$$

$$dx = \frac{2t(1-2t) - (t^2-1)(-2)}{(1-2t)^2}\,dt$$

$$dx = \frac{2t - 4t^2 + 2t^2 - 2}{(1-2t)^2}\,dt$$

$$dx = \frac{-2(t^2-t+1)}{(1-2t)^2}\,dt$$

$$\sqrt{x^2+x+1} = \frac{t^2-1}{1-2t} + t = \frac{t^2-1+t-2t^2}{1-2t} =$$
$$= -\frac{t^2-t+1}{1-2t}$$

$$\dots = \int \frac{2\cancel{(t^2-t+1)}\,dt}{(1-2t)^2}\cdot\frac{1-2t}{\cancel{t^2-t+1}} = -\int \frac{-2\,dt}{1-2t} = -\ln|1-2t| + C =$$

$$= -\ln\left|1 - 2\left(\sqrt{x^2+x+1} - x\right)\right| + C$$

Uwaga A można też było tak jak w prz 1, ze wzoru ①