

LEMMA 46.1. Let the quadratic form $\Phi(\lambda) = \Phi(\lambda_1, \dots, \lambda_n) = \sum_{j,k=1}^n a_{jk} \lambda_j \lambda_k$

be positive and the quadratic form $\Psi(\lambda) = \Psi(\lambda_1, \dots, \lambda_n) = \sum_{j,k=1}^n b_{jk} \lambda_j \lambda_k$ be negative; then we have

$$(46.1) \quad \sum_{j,k=1}^n a_{jk} b_{jk} \leq 0.$$

Proof. The form $\Phi(\lambda)$ being positive we have, for suitably chosen coefficients $\tilde{a}_{pq}$ ($p, q = 1, 2, \dots, n$),

$$\Phi(\lambda) = \sum_{j,k=1}^n a_{jk} \lambda_j \lambda_k = \sum_{p=1}^n \left(\sum_{q=1}^n \tilde{a}_{pq} \lambda_q \right)^2;$$

hence

$$a_{jk} = \sum_{p=1}^n \tilde{a}_{pj} \tilde{a}_{pk} \quad (j, k = 1, 2, \dots, n)$$

and consequently

$$(46.2) \quad \sum_{j,k=1}^n a_{jk} b_{jk} = \sum_{p=1}^n \left(\sum_{j,k=1}^n b_{jk} \tilde{a}_{pj} \tilde{a}_{pk} \right) = \sum_{p=1}^n \Psi(\tilde{a}_{p1}, \dots, \tilde{a}_{pn}) \leq 0.$$

DEFINITION OF ELLIPTICITY. Let the function

$$f^i(t, X, U, Q, R) = f^i(t, x_1, \dots, x_n, u^1, \dots, u^m, q_1, \dots, q_n, r_{11}, r_{12}, \dots, r_{nn})$$

be defined for (t, X) belonging to a region $D \subset (t, x_1, \dots, x_n)$ and for arbitrary U, Q, R . Suppose that $U(t, X) = (u^1(t, X), \dots, u^m(t, X))$ is defined and possesses first derivatives with respect to x_j at a point $(\tilde{t}, \tilde{X}) \in D$. Write

$$u_X^i = (u_{x_1}^i, \dots, u_{x_n}^i).$$

Under these assumptions, we say that the function $f^i(t, X, U, Q, R)$ is elliptic with respect to $U(t, X)$ at the point $(\tilde{t}, \tilde{X}) \in D$ if for any two sequences of numbers $R = (r_{11}, r_{12}, \dots, r_{nn})$ and $\tilde{R} = (\tilde{r}_{11}, \tilde{r}_{12}, \dots, \tilde{r}_{nn})$ ($r_{jk} = r_{kj}$, $\tilde{r}_{jk} = \tilde{r}_{kj}$) such that the quadratic form in $\lambda_1, \dots, \lambda_n$

$$(46.3) \quad \sum_{j,k=1}^n (r_{jk} - \tilde{r}_{jk}) \lambda_j \lambda_k \text{ is negative,}$$

we have

$$(46.4) \quad f^i(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u_X^i(\tilde{t}, \tilde{X}), R) \leq f^i(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u_X^i(\tilde{t}, \tilde{X}), \tilde{R}).$$

If the above property holds true for every point $(\tilde{t}, \tilde{X}) \in D$, then we say that $f^i(t, X, U, Q, R)$ is elliptic with respect to $U(t, X)$ in D .

EXAMPL

$$(46.5) \quad u_t =$$

where $a_{jk}(t, X)$ is the coefficient in Equation (46.1) in $\lambda_1, \dots, \lambda_n$.

$$(46.6)$$

Now, b

$$f(t, X, u,$$

of a paraboloid to any function U .

Remark. If f is not elliptic with respect to U at a point of R , then it is not elliptic with respect to U in R .

DEFINITION OF ELLIPTICITY. Let the function $f^i(t, X, U, Q, R)$ be defined for (t, X) belonging to a region $D \subset (t, x_1, \dots, x_n)$ and for arbitrary U, Q, R . Suppose that $U(t, X) = (u^1(t, X), \dots, u^m(t, X))$ is defined and possesses first derivatives with respect to x_j at a point $(\tilde{t}, \tilde{X}) \in D$. Write

$$(46.7) \quad u_t^i =$$

with right-hand side arbitrary. A function $f^i(t, X, U, Q, R)$ is called parabolic with respect to $U(t, X)$ at the point $(\tilde{t}, \tilde{X}) \in D$ if for any two sequences of numbers $R = (r_{11}, r_{12}, \dots, r_{nn})$ and $\tilde{R} = (\tilde{r}_{11}, \tilde{r}_{12}, \dots, \tilde{r}_{nn})$ ($r_{jk} = r_{kj}$, $\tilde{r}_{jk} = \tilde{r}_{kj}$) such that the quadratic form in $\lambda_1, \dots, \lambda_n$

If this condition is satisfied at every point of D , then $f^i(t, X, U, Q, R)$ is called parabolic with respect to $U(t, X)$ in D .

According to the definition (46.5) it follows that a function $f^i(t, X, U, Q, R)$ is parabolic with respect to $U(t, X)$ at a point $(\tilde{t}, \tilde{X}) \in D$ if and only if it is elliptic with respect to $U(t, X)$ at the point $(\tilde{t}, \tilde{X}) \in D$.

Remark. If $f^i(t, X, U, Q, R)$ is not parabolic with respect to $U(t, X)$ at a point of D , then it is not parabolic with respect to $U(t, X)$ in D .

§ 47. Mixed problems for second order differential equations

We are going to consider mixed problems for second order differential equations and as