LEMMA 46.1. *Let the quadratic form* $\Phi(\Lambda) = \Phi(\lambda_1, \ldots, \lambda_n) = \sum_{j,k=1}^{n} a_{jk}\lambda_j\lambda_k$ *be positive and the quadratic form* $\Psi(\Lambda) = \Psi(\lambda_1, \ldots, \lambda_n) = \sum_{j,k=1}^{n} b_{jk}\lambda_j\lambda_k$ *be negative; then we have*

$$\sum_{j,k=1}^{n} a_{jk}b_{jk} \leqslant 0 \,. \tag{46.1}$$

Proof. The form $\Phi(\Lambda)$ being positive we have, for suitably chosen coefficients $\tilde{a}_{pq}$ $(p, q = 1, 2, \ldots, n)$,

$$\Phi(\Lambda) = \sum_{j,k=1}^{n} a_{jk}\lambda_j\lambda_k = \sum_{p=1}^{n}\Big(\sum_{q=1}^{n} \tilde{a}_{pq}\lambda_q\Big)^2;$$

hence

$$a_{jk} = \sum_{p=1}^{n} \tilde{a}_{pj}\tilde{a}_{pk} \qquad (j, k = 1, 2, \ldots, n)$$

and consequently

$$\sum_{j,k=1}^{n} a_{jk}b_{jk} = \sum_{p=1}^{n}\Big(\sum_{j,k=1}^{n} b_{jk}\tilde{a}_{pj}\tilde{a}_{pk}\Big) = \sum_{p=1}^{n}\Psi(\tilde{a}_{p1}, \ldots, \tilde{a}_{pn}) \leqslant 0 \,. \tag{46.2}$$

DEFINITION OF ELLIPTICITY. Let the function

$$f^i(t, X, U, Q, R) = f^i(t, x_1, \ldots, x_n, u^1, \ldots, u^m, q_1, \ldots, q_n, r_{11}, r_{12}, \ldots, r_{nn})$$

be defined for (t, X) belonging to a region $D \subset (t, x_1, \ldots, x_n)$ and for arbitrary U, Q, R. Suppose that $U(t, X) = \big(u^1(t, X), \ldots, u^m(t, X)\big)$ is defined and possesses first derivatives with respect to x_j at a point $(\tilde{t}, \tilde{X}) \in D$. Write

$$u^i_X = (u^i_{x_1}, \ldots, u^i_{x_n}) \,.$$

Under these assumptions, we say that the *function* $f^i(t, X, U, Q, R)$ *is elliptic with respect to* $U(t, X)$ *at the point* $(\tilde{t}, \tilde{X}) \in D$ if for any two sequences of numbers $R = (r_{11}, r_{12}, \ldots, r_{nn})$ and $\tilde{R} = (\tilde{r}_{11}, \tilde{r}_{12}, \ldots, \tilde{r}_{nn})$ $(r_{jk} = r_{kj}, \tilde{r}_{jk} = \tilde{r}_{kj})$ such that the quadratic form in $\lambda_1, \ldots, \lambda_n$

$$\sum_{j,k=1}^{n} (r_{jk} - \tilde{r}_{jk})\lambda_j\lambda_k \text{ is negative ,} \tag{46.3}$$

we have

$$f^i\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^i_X(\tilde{t}, \tilde{X}), R\big) \leqslant f^i\big(\tilde{t}, \tilde{X}, U(\tilde{t}, \tilde{X}), u^i_X(\tilde{t}, \tilde{X}), \tilde{R}\big) \,. \tag{46.4}$$

If the above property holds true for every point $(\tilde{t}, \tilde{X}) \in D$, then we say that $f^i(t, X, U, Q, R)$ *is elliptic with respect to* $U(t, X)$ *in* D.