① $f(x) = \sqrt{-9^x + 4\cdot 3^x - 3} + \ln\left(\left|\arccos\frac{x+1}{2}\right| - \frac{\pi}{4}\right)$

1. $-9^x + 4\cdot 3^x - 3 \geq 0$
2. $-1 \leq \frac{x+1}{2} \leq 1$
3. $\left|\arccos\frac{x+1}{2}\right| - \frac{\pi}{4} > 0$

②

1. $-9^x + 4\cdot 3^x - 3 \geq 0$

$9^x - 4\cdot 3^x + 3 \leq 0$

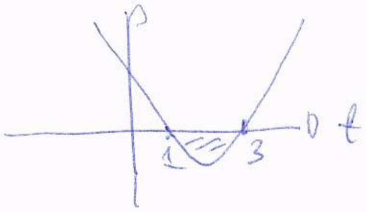
$t = 3^x > 0$

$t^2 - 4t + 3 \leq 0$

$\Delta = 16 - 12 = 4$

$t = \frac{4 \pm 2}{2}$

$t_1 = 1, \ t_2 = 3$



$1 \leq t \leq 3$

$3^0 \leq 3^x \leq 3^1$

$x \in \langle 0, 1 \rangle$ ③

2. $-1 \leq \frac{x+1}{2} \leq 1$

$-2 \leq x+1 \leq 2$

$-3 \leq x \leq 1$

$x \in \langle -3, 1 \rangle$ ①

3. $\left|\arccos\frac{x+1}{2}\right| - \frac{\pi}{4} > 0$

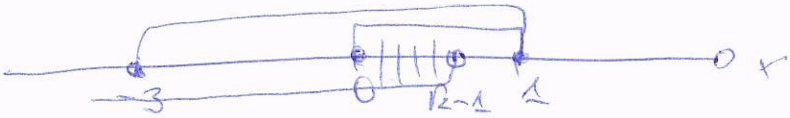
$\left|\arccos\frac{x+1}{2}\right| > \frac{\pi}{4}$

$\arccos\frac{x+1}{2} > \frac{\pi}{4} \ \vee \ \arccos\frac{x+1}{2} < -\frac{\pi}{4}$

Nigdy!

$\frac{x+1}{2} < \frac{\sqrt{2}}{2}$

$x < \sqrt{2} - 1$ ②



$\boxed{D = \langle 0, \sqrt{2}-1)}$ ②

---

② a) $\lim_{n\to\infty} n^2\left(\sqrt{n^4+1} - \sqrt{n^4-1}\right) = \lim \frac{n^2\left(\cancel{n^4}+1-\cancel{n^4}+1\right)}{\sqrt{n^4+1}+\sqrt{n^4-1}} =$ ②

$= \lim \frac{2\cancel{n^2}}{\cancel{n^2}\left(\sqrt{1+\frac{1}{n^4}}+\sqrt{1-\frac{1}{n^4}}\right)} = \boxed{1}$ ② ①

---

b) $\lim_{x\to 0^+} (\arcsin x)^x \overset{0^0}{=} \lim_{x\to 0^+} e^{x \ln \arcsin x} = e^0 = \boxed{1}$ ① ①

$\lim_{x\to 0^+} x \ln \arcsin x \underset{0\cdot(-\infty)}{=} \lim_{x\to 0^+} \frac{\ln \arcsin x}{\frac{1}{x}} \overset{\frac{-\infty}{+\infty}}{\underset{H}{=}} \lim_{x\to 0^+} \frac{\frac{1}{\arcsin x}\cdot\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{x^2}} \ \frac{+0}{-\infty}$

$= -\lim_{x\to 0^+} \underbrace{\frac{x}{\arcsin x}}_{\to 1} \cdot \underbrace{\frac{x}{\sqrt{1-x^2}}}_{\to 0} = 0$ ③

③ $f(x) = \ln(x^2-1) + \frac{8}{x^2-1}$

$x^2 - 1 > 0$ ①

$x^2 > 1$ $\quad \underline{D = (-\infty, -1) \cup (1, \infty)}$

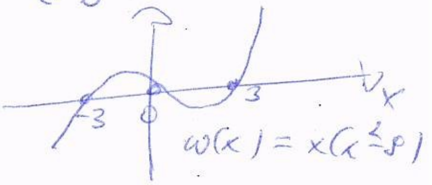
$f'(x) = \frac{2x}{x^2-1} + \frac{-16x}{(x^2-1)^2} = \frac{2x(x^2-1) - 16x}{(x^2-1)^2} = \frac{2x^3 - 18x}{(x^2-1)^2} =$

$= \frac{2x(x^2-9)}{(x^2-1)^2}$

$2x(x^2-9) = 0$ ④

$x = 0 \lor x = \pm 3 \in D$

$\notin D$



$x \in (-\infty, -3)$, $f'(x) < 0$, $f(x) \downarrow$ | $x_1 = -3$ min. lok.

$x \in (-3, -1)$, $f'(x) > 0$, $f(x) \uparrow$

$x \in (1, 3)$, $f'(x) < 0$, $f(x) \downarrow$ | $x_2 = 3$ min. lok.

$x \in (3, \infty)$, $f'(x) > 0$, $f(x) \uparrow$

⑤ Brak uzasadnienia znaków f': -3 pkt!

---

④ $f(x) = x^2 e^{3x} + \frac{1}{9} e^{3x}$ $\quad \underline{D = \mathbb{R}}$ ①

$f'(x) = 2x\, e^{3x} + 3x^2 e^{3x} + \frac{1}{3} e^{3x}$

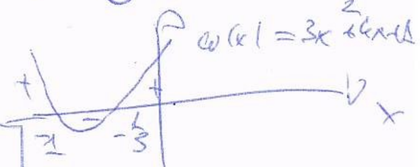
$f''(x) = 2e^{3x} + 6xe^{3x} + 6xe^{3x} + 9x^2 e^{3x} + e^{3x}$

$= 9x^2 e^{3x} + 12x\, e^{3x} + 3e^{3x} = 3e^{3x}(3x^2 + 4x + 1)$

$3x^2 + 4x + 1 = 0$ ④

$\Delta = 16 - 12 = 4$ $\quad x_1 = -1,\ x_2 = -\frac{1}{3} \in D$

$x = \frac{-4 \pm 2}{6}$



$x \in (-\infty, -1)$, $f''(x) > 0$, $f(x) \cup$ | $x_1 = -1$ p.p.

$x \in (-1, -\frac{1}{3})$, $f''(x) < 0$, $f(x) \cap$

$x \in (-\frac{1}{3}, \infty)$, $f''(x) > 0$, $f(x) \cup$ | $x_2 = -\frac{1}{3}$ p.p.

⑤

Brak uzasadnienia znaków f'': -3 pkt!

⑤ $(1+i)^{24} z^3 + (1-\sqrt{3}i)^{12} = 0$

$\underbrace{(1+i)}_{z_1}{}^{24} = 2^{12}(\cos 18\pi + i \sin 18\pi) = 2^{12}$

$|z_1| = \sqrt{2}$

$\varphi_1 = \frac{3\pi}{4}$

$\underbrace{(1-\sqrt{3}i)}_{z_2}{}^{12} = 2^{12}(\cos 20\pi + i \sin 20\pi) = 2^{12}$

$|z_2| = 2$

$\varphi_2 = \frac{5\pi}{3}$

$2^{12} z^3 + 2^{12} = 0$

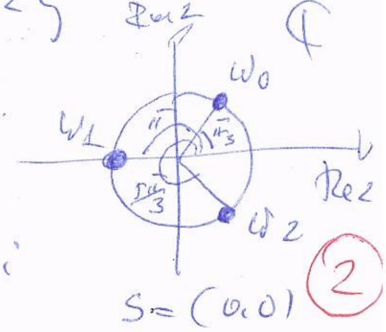
$z^3 = -1$ ④

$z \in \sqrt[3]{-1} = \left\{ w_k = \cos\frac{\pi + 2k\pi}{3} + i \sin\frac{\pi+2k\pi}{3} : k = 0,1,2 \right\}$

$w_0 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i$

$w_1 = \cos\pi + i \sin\pi = -1$ ④

$w_2 = \cos\frac{5\pi}{3} + i \sin\frac{5\pi}{3} = \cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = \frac{1}{2} - \frac{\sqrt{3}}{2} i$



$S = (0,0)$ ②

$R = 1$

---

⑥ a) ② Def. Funkcja $f: \mathbb{R} \supset X \to \mathbb{R}$ jest surjekcją na Y, gdy $\overline{C}_f = Y$.

② b) Def. $f: \mathbb{R} \supset X \to \mathbb{R}$, $x_0 \in X$

f - ciąg. w x_0 $\overset{df.}{\Leftrightarrow}$ $\forall (x_n), x_n \in X \lim\limits_{n\to\infty} x_n = x_0 \Rightarrow \lim\limits_{n\to\infty} f(x_n) = f(x_0)$

② c) Def. $f: \mathbb{R} \supset X \to \mathbb{R}$, $x_0 \in X$ - punkt wewnętrzny X

Punkt $M = (x_0, f(x_0))$ jest punktem przegięcia krzywej $y = f(x)$, gdy istnieje otoczenie $(x_0 - r, x_0 + r)$ punktu x_0 takie, że f jest ściśle wypukła w $(x_0 - r, x_0)$ i ściśle wklęsła w $(x_0, x_0 + r)$ albo na odwrót.

Mały błąd w def => 1 pkt

Duży -//- => 0 pkt

① $f(x)=\sqrt{4^x-5\cdot 2^x+4}+\ln\left(\frac{\pi}{4}-\left|\arcsin\frac{x-1}{2}\right|\right)$

1. $4^x-5\cdot 2^x+4\geq 0$
2. $-1\leq\frac{x-1}{2}\leq 1$ ②
3. $\frac{\pi}{4}-\left|\arcsin\frac{x-1}{2}\right|>0$

1. $4^x-5\cdot 2^x+4\geq 0$

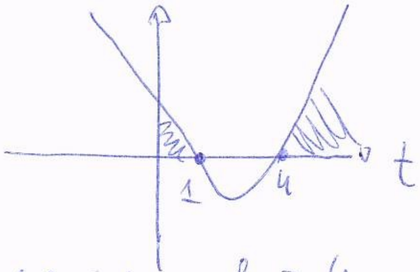
$t=2^x>0$

$t^2-5t+4\geq 0$

$\Delta=9$

$t=\frac{5\pm 3}{2}$

$t_1=1,\ t_2=4$



$0<t\leq 1 \vee t\geq 4$

$0<2^x\leq 1 \vee 2^x\geq 2^2$

Zowne $x\leq 0 \vee x\geq 2$ ③

$x\in(-\infty,0\rangle\cup\langle 2,\infty)$

2. $-1\leq\frac{x-1}{2}\leq 1$

$-2\leq x-1\leq 2$

$-1\leq x\leq 3$

$x\in\langle -1,3\rangle$ ①

3. $\frac{\pi}{4}-\left|\arcsin\frac{x-1}{2}\right|>0$

$\left|\arcsin\frac{x-1}{2}\right|<\frac{\pi}{4}$

$\arcsin\frac{x-1}{2}>-\frac{\pi}{4} \wedge \arcsin\frac{x-1}{2}<\frac{\pi}{4}$

$\frac{x-1}{2}>-\frac{\sqrt{2}}{2} \wedge \frac{x-1}{2}<\frac{\sqrt{2}}{2}$

$x>1-\sqrt{2} \wedge x<1+\sqrt{2}$

$x\in(1-\sqrt{2},1+\sqrt{2})$ ②

$\boxed{D=(1-\sqrt{2},0\rangle\cup\langle 2,1+\sqrt{2})}$ ②

---

② a) $\lim_{n\to\infty}\sqrt[n]{n^2+n+2^n+2^{3n}}=\boxed{8}$ ⑤

$$\underset{\downarrow 8}{8}\leq\sqrt[n]{n^2+n+2^n+2^{3n}}\leq\underbrace{8\cdot\sqrt[n]{4}}_{\downarrow 8}$$

---

b) $\lim_{x\to 0^+}\left(\frac{\pi}{2}-\arccos x\right)^x\overset{0^0}{=}\lim_{x\to 0^+}e^{x\ln\left(\frac{\pi}{2}-\arccos x\right)}=e^0=\boxed{1}$ ① ①

$$\lim_{x\to 0^+}x\ln\left(\frac{\pi}{2}-\arccos x\right)\overset{0(-\infty)}{=}\lim_{x\to 0^+}\frac{\ln\left(\frac{\pi}{2}-\arccos x\right)}{\frac{1}{x}}\overset{\frac{-\infty}{+\infty}}{\underset{H}{=}}$$

$$=\lim_{x\to 0^+}\frac{\frac{1}{\frac{\pi}{2}-\arccos x}\cdot\frac{1}{\sqrt{1-x^2}}}{-\frac{1}{x^2}}=-\lim_{x\to 0^+}\underbrace{\frac{x}{\frac{\pi}{2}-\arccos x}}_{\downarrow 1}\cdot\underbrace{\frac{x}{\sqrt{1-x^2}}}_{\downarrow 0}=0$$ ③

Bo $\lim_{x\to 0^+}\frac{x}{\frac{\pi}{2}-\arccos x}\overset{\frac{0}{0}}{\underset{H}{=}}\lim_{x\to 0^+}\frac{1}{\sqrt{1-x^2}}=1$

③ $f(x) = \ln(1-x^2) - \frac{8}{1-x^2}$

$1-x^2 > 0$ $\qquad$ $\underline{D=(-1,1)}$ ①

$x \in (-1,1)$

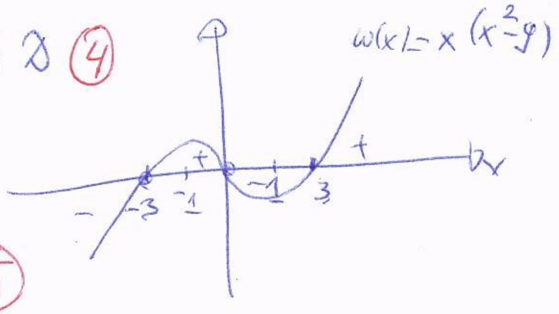
$$f'(x) = \frac{-2x}{1-x^2} - \frac{16x}{(1-x^2)^2} \neq \frac{2x(1-x^2)+16x}{(1-x^2)^2} \neq \frac{-2x^3+18x}{(1-x^2)^2} =$$

$$= \frac{2x(x^2-9)}{(1-x^2)^2}$$

$2x(x^2-9) = 0$

$x = 0 \lor x = \pm 3 \notin D$ ④

$\in D.$



$x \in (-1,0)$, $f'(x) > 0$, $f(x)$ ↑ $\quad$ $x_0 = 0$ maks. lok.

$x \in (0,1)$, $f'(x) < 0$, $f(x)$ ↓

⑤

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④ $f(x) = x^2 e^{4x} - \frac{17}{8} e^{4x}$ $\qquad$ $\underline{D = \mathbb{R}}$ ①

$f'(x) = 2xe^{4x} + 4x^2e^{4x} - \frac{17}{2}e^{4x}$

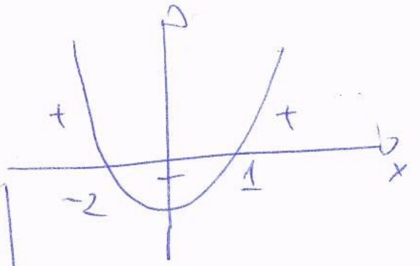
$f''(x) = 2e^{4x} + 8xe^{4x} + 8xe^{4x} + 16x^2e^{4x} - 34e^{4x} =$

$= 16x^2e^{4x} + 16xe^{4x} - 32e^{4x} = 16e^{4x}(x^2+x-2)$

$x^2+x-2 = 0$

$\Delta = 9$ $\qquad$ $x_1 = -2$, $x_2 = 1 \in D$ ④

$x = \frac{-1 \pm 3}{2}$



$x \in (-\infty,-2)$, $f''(x) > 0$, $f(x)$ ∪ $\quad$ $x_1 = -2$ tp

$x \in (-2,1)$, $f''(x) < 0$, $f(x)$ ∩ $\quad$ $x_2 = 1$ tp

$x \in (1,\infty)$, $f''(x) > 0$, $f(x)$ ∪

⑤

⑤ $i(1-i)^{18}z^3-(-1+\sqrt{3}i)^9=0$

$\underbrace{(1-i)}_{z_1}{}^{18}=2^9\left(\cos\frac{63\pi}{2}+i\sin\frac{63\pi}{2}\right)=2^9\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=-2^9i$

$|z_1|=\sqrt{2}$

$\varphi_1=\frac{7\pi}{4}$

$\underbrace{(-1+\sqrt{3}i)}_{z_2}{}^9=2^9(\cos 6\pi+i\sin 6\pi)=2^9$

$|z_2|=2$

$\varphi_2=\frac{2\pi}{3}$

$2^9z^3-2^9=0$

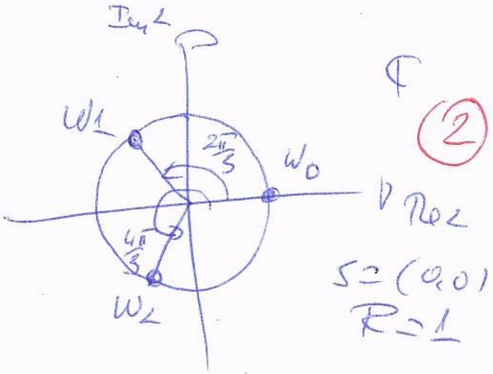
$z^3=1$ ④

$z\in\sqrt[3]{1}=\left\{w_k=\cos\frac{0+2k\pi}{3}+i\sin\frac{0+2k\pi}{3}:k=0,1,2\right\}$

$w_0=1$

$w_1=\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

$w_2=\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}=-\frac{1}{2}-\frac{\sqrt{3}}{2}i$ ④



②

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⑥ a) Df. $f:\mathbb{R}\supset X\to\mathbb{R}$, $A\subset X$

② Funkcja f jest różnowartościowa w A gdy

$\forall x_1,x_2\in A\quad x_1\neq x_2\Rightarrow f(x_1)\neq f(x_2)$

b) Df. $f:\mathbb{R}\supset X\to\mathbb{R}$, $x_0\in X$

② f – ciągła w x_0 $\overset{df}{\iff}$ $\forall\varepsilon>0\ \exists\delta>0\ \forall x\in X\ |x-x_0|<\delta\Rightarrow|f(x)-f(x_0)|<\varepsilon$

c) $f:\mathbb{R}\supset X\to\mathbb{R}$ i $x_0\in X$ – pkt. wewnętrzny X

② Funkcja f ma minimum lokalne w x_0, gdy

$\exists r>0\ \forall x\in(x_0-r,x_0+r)\ f(x_0)\leq f(x)$

Aha: To mają być definicje.
Twierdzeń tutaj absolutnie nie punktujemy 8

~~ZADANIE 4~~