Różniczkowanie złożenia

① $f(u)$, $u(x)$

$F(x) := f(u(x))$

$$\begin{tikzcd} x \arrow[r, bend left, "u"] \arrow[rr, bend right, "F"'] & u(x) \arrow[r, bend left, "f"] & f(u(x)) \end{tikzcd}$$

$$\frac{dF}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$ ← Tutaj u jest funkcją $u(x)$ – skrótowy zapis

Tutaj u jest argumentem funkcji $f(u)$

$$\frac{dF}{dx}(x) = \frac{df}{du}(u(x)) \cdot \frac{du}{dx}(x)$$

$$\boxed{F'(x) = f'(u(x)) \cdot u'(x)}$$

Żeby lepiej zrozumieć, można pisać $f(p)$ zamiast $f(u)$. Wtedy

$$\frac{dF}{dx}(x) = \frac{df}{dp}(u(x)) \cdot \frac{du}{dx}(x).$$

Prz.

$f(u) = 2u^3 + 1$, $u \in \mathbb{R}$

$u(x) = \sin x$, $x \in \mathbb{R}$

- $F(x) = f(u(x)) = f(\sin x) = 2\sin^3 x + 1$

  $f'(u) = 6u^2$, $u'(x) = \cos x$

- $F'(x) = \underbrace{6\sin^2 x}_{f'(\sin x)} \cdot \underbrace{\cos x}_{u'(x)}$

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② $f(u,v,w)$, $u(x,y)$, $v(x,y)$, $w(x,y)$

$F(x,y) := f(u(x,y), v(x,y), w(x,y))$

$$\begin{tikzcd} (x,y) \arrow[r, bend left, "{(u,v,w)}"] \arrow[rr, bend right, "F"'] & (u(x,y), v(x,y), w(x,y)) \arrow[r, bend left, "f"] & f(u(x,y), v(x,y), w(x,y)) \end{tikzcd}$$

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial f}{\partial w} \cdot \frac{\partial w}{\partial y}$$

} skrótowy zapis

$$\boxed{\begin{aligned} \frac{\partial F}{\partial x}(x,y) = {} & \frac{\partial f}{\partial u}(u(x,y), v(x,y), w(x,y)) \cdot \frac{\partial u}{\partial x}(x,y) \\ & + \frac{\partial f}{\partial v}(u(x,y), v(x,y), w(x,y)) \cdot \frac{\partial v}{\partial x}(x,y) \\ & + \frac{\partial f}{\partial w}(u(x,y), v(x,y), w(x,y)) \frac{\partial w}{\partial x}(x,y) \end{aligned}}$$

Żeby lepiej zrozumieć, można pisać $f(p,q,r)$ zamiast $f(u,v,w)$.

$$\boxed{\begin{aligned}\frac{\partial F}{\partial y}(x,y) &= \frac{\partial f}{\partial u}\big(u(x,y), v(x,y), w(x,y)\big)\cdot \frac{\partial u}{\partial y}(x,y)\\ &+ \frac{\partial f}{\partial v}\big(u(x,y), v(x,y), w(x,y)\big)\cdot \frac{\partial v}{\partial y}(x,y)\\ &+ \frac{\partial f}{\partial w}\big(u(x,y), v(x,y), w(x,y)\big)\cdot \frac{\partial w}{\partial y}(x,y)\end{aligned}}$$

Φ12

$f(u,v,w) = u^2 v + v^3 + w^4, \quad u,v,w \in \mathbb{R}$

$u(x,y) = \eta\mu\, xy, \quad v(x,y) = 2x+3y, \quad w(x,y) = x^2+y^2, \quad x,y \in \mathbb{R}$

- $F(x,y) = f\big(u(x,y), v(x,y), w(x,y)\big) = (\eta\mu^2 xy)\cdot(2x+3y) + (2x+3y)^3 + (x^2+y^2)^4$

$\frac{\partial f}{\partial u}(u,v,w) = 2uv, \quad \frac{\partial f}{\partial v}(u,v,w) = u^2+3v^2, \quad \frac{\partial f}{\partial w}(u,v,w) = 4w^3$

$\frac{\partial u}{\partial x}(x,y) = y\cos xy, \quad \frac{\partial v}{\partial x}(x,y) = 2, \quad \frac{\partial w}{\partial x}(x,y) = 2x$

$\frac{\partial u}{\partial y}(x,y) = x\cos xy, \quad \frac{\partial v}{\partial y}(x,y) = 3, \quad \frac{\partial w}{\partial y}(x,y) = 2y$

- $$\begin{aligned}\frac{\partial F}{\partial x}(x,y) &= \underbrace{2(\eta\mu\, xy)(2x+3y)}_{\frac{\partial f}{\partial u}(\eta\mu xy,\, 2x+3y,\, x^2+y^2)}\cdot \underbrace{y\cos xy}_{\frac{\partial u}{\partial x}(x,y)}\\ &+ \underbrace{\left(\eta\mu^2 xy + 3(2x+3y)^2\right)}_{\frac{\partial f}{\partial v}(\eta\mu xy,\, 2x+3y,\, x^2+y^2)}\cdot \underbrace{2}_{\frac{\partial v}{\partial x}(x,y)}\\ &+ \underbrace{4(x^2+y^2)^3}_{\frac{\partial f}{\partial w}(\eta\mu xy,\, 2x+3y,\, x^2+y^2)}\cdot \underbrace{2x}_{\frac{\partial w}{\partial x}(x,y)}\end{aligned}$$

- $$\begin{aligned}\frac{\partial F}{\partial y}(x,y) &= \underbrace{2(\eta\mu\, xy)(2x+3y)}_{\frac{\partial f}{\partial u}(\eta\mu xy,\, 2x+3y,\, x^2+y^2)}\cdot \underbrace{x\cos xy}_{\frac{\partial u}{\partial y}(x,y)}\\ &+ \underbrace{\left(\eta\mu^2 xy + 3(2x+3y)^2\right)}_{\frac{\partial f}{\partial v}(\eta\mu xy,\, 2x+3y,\, x^2+y^2)}\cdot \underbrace{3}_{\frac{\partial v}{\partial y}(x,y)}\\ &+ \underbrace{4(x^2+y^2)^3}_{\frac{\partial f}{\partial w}(\eta\mu xy,\, 2x+3y,\, x^2+y^2)}\cdot \underbrace{2y}_{\frac{\partial w}{\partial y}(x,y)}\end{aligned}$$