1 $x' = \left(1 + \frac{x-1}{2t}\right)^2$

$x' = \left(\frac{2t + x - 1}{2t}\right)^2$

$\begin{cases} 2t + x - 1 = 0 \\ 2t = 0 \end{cases}$ $\quad t = 0,\ x = 1$

$(\alpha, \beta) = (0, 1)$

$t = u$

$x = v + 1$

$\frac{dv}{du} = \left(\frac{2u + v + \cancel{1} - \cancel{1}}{2u}\right)^2$

• $\frac{dv}{du} = \left(1 + \frac{v}{2u}\right)^2$ 1

$z = \frac{v}{u}$, $z = z(u)$

$v = uz$

$\frac{dv}{du} = z + u\frac{dz}{du}$

$z + u\frac{dz}{du} = \left(1 + \frac{1}{2}z\right)^2$

$\cancel{z} + u\frac{dz}{du} = 1 + \cancel{z} + \frac{z^2}{4}$

• $\frac{dz}{du} = \left(1 + \frac{z^2}{4}\right) \cdot \frac{1}{u}$ 1

$\int \frac{dz}{z^2+4} = \frac{1}{4}\int \frac{du}{u}$ 2

$\frac{1}{2}\,\mathrm{arctg}\,\frac{z}{2} = \frac{1}{4}\ln|u| + C$

$\boxed{\frac{1}{2}\,\mathrm{arctg}\,\frac{x-1}{2t} = \frac{1}{4}\ln|t| + C}$ 1

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2 $e^t(t+1)\,dt + (xe^x - te^t)\,dx = 0$

$P \qquad Q$

$\frac{\partial P}{\partial x} = 0$, $\frac{\partial Q}{\partial t} = -e^t - te^t$

2 $\omega = x$

$\frac{-t - te^t}{e^t(t+1)} = -1$ $\quad \mu = Ce^{-\int dx} = Ce^{-x}$

$\boxed{\mu = e^{-x}}$ 1

• $e^{t-x}(t+1)\,dt + (x - te^{t-x})\,dx = 0$

$\begin{cases} \frac{\partial F}{\partial t} = e^{t-x}(t+1) \\ \frac{\partial F}{\partial x} = x - te^{t-x} \end{cases}$ $\to$ $F = \int e^{t-x}(t+1)\,dt = \left\{ \begin{matrix} u' = e^{t-x} \\ v = t+1 \end{matrix} \middle| \begin{matrix} u = e^{t-x} \\ v' = 1 \end{matrix} \right.$

$= e^{t-x}(t+1) - e^{t-x} + \varphi(x) = te^{t-x} + \varphi(x)$

$\cancel{-te^{t-x}} + \varphi'(x) = x - \cancel{te^{t-x}}$

$\varphi(x) = \frac{1}{2}x^2$

$F = te^{t-x} + \frac{1}{2}x^2$ 2

$\boxed{te^{t-x} + \frac{1}{2}x^2 = C}$ 2

$\boxed{1}$ $x - x' \cos t = x^2 (1 - \sin t) \cos t$

$x' \cos t = x - x^2 (1 - \sin t) \cos t$

$x' = \frac{1}{\cos t} x - (1 - \sin t) x^2 \; / \cdot x^{-2}$ — Bernoulli

1° $\boxed{x(t) \equiv 0}$ ok

2° $x \neq 0$

$x' x^{-2} = \frac{1}{\cos t} x^{-1} - (1 - \sin t)$ $\qquad y = x^{-1}, \; y' = -x^{-2} x'$

$-y' = \frac{1}{\cos t} y - (1 - \sin t)$

• $y' = -\frac{1}{\cos t} y + (1 - \sin t)$ (1)

$y_h = C e^{-\int \frac{dt}{\cos t}} = \ldots$

$-\int \frac{dt}{\cos t} = -\int \frac{\cos t \, dt}{1 - \sin^2 t} \underset{\substack{u = \sin t \\ du = \cos t \, dt}}{=} -\int \frac{du}{1 - u^2} = \int \frac{du}{u^2 - 1} = \frac{1}{2} \ln \left| \frac{u-1}{u+1} \right| =$

$= \frac{1}{2} \ln \left(\frac{1 - \sin t}{1 + \sin t} \right)$

$y_h = C e^{\frac{1}{2} \ln \left(\frac{1 - \sin t}{1 + \sin t} \right)} = C \sqrt{\frac{1 - \sin t}{1 + \sin t}} = C \frac{|\cos t|}{1 + \sin t}$

$y_h = C \frac{\cos t}{1 + \sin t}$ (2)

$y_p = C(t) \frac{\cos t}{1 + \sin t}$ $\qquad C'(t) \frac{\cos t}{1 + \sin t} = 1 - \sin t$

$C'(t) = \cos t, \quad C(t) = \sin t$

$y_p = \frac{\sin t \cos t}{1 + \sin t}$

$y = \frac{(C + \sin t) \cos t}{1 + \sin t}$ (1)

$\boxed{x = \frac{1 + \sin t}{(C + \sin t) \cos t}}$ (1)

4 $x' = x^2 + b(t)x + c(t)$, b, c – cg.

x_1, x_2

$$\frac{x_2' - x_1'}{x_2 - x_1} = x_1 + x_2 + b(t)$$

$$x_1' = x_1^2 + b(t)x_1 + c(t)$$
$$x_2' = x_2^2 + b(t)x_2 + c(t)$$

$$x_2' - x_1' = x_2^2 - x_1^2 + b(t)(x_1 - x_2) \quad / : (x_2 - x_1)$$ (Bo różne od zera $\forall\, t \in I$ z jednoznaczności)

$$\frac{x_2' - x_1'}{x_2 - x_1} = x_1 + x_2 + b(t)$$ 3

$$\frac{x_2'(t) - x_1'(t)}{x_2(t) - x_1(t)} = \left[\ln|x_2(t) - x_1(t)| + c\right]' \quad , t \in I$$ 2

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3 $4x' = 4tx'' - (x'')^2$

$y = x'$, $y = y(t)$

$x'' = y'$

$4y = 4ty' - (y')^2$

$\underline{y = ty' - \frac{1}{4}(y')^2}$ – Clairaut 1

$y' = y' + ty'' - \frac{1}{2}y'y''$

$y'' = 0 \quad \vee \quad \begin{cases} t - \frac{1}{2}y' = 0 \\ y = ty' - \frac{1}{4}(y')^2 \end{cases}$

$y' = C$

$\underline{y = Ct - \frac{1}{4}C^2}$

$x' = Ct - \frac{1}{4}C^2$

$\boxed{x = \frac{1}{2}Ct^2 - \frac{1}{4}C^2 t + D}$ 2

$y' = p$

$\begin{cases} t = \frac{1}{2}p \\ y = tp - \frac{1}{4}p^2 \end{cases}$

$p = 2t$

$y = 2t^2 - t^2 = t^2$

$\underline{y = t^2}$

$x' = t^2$

$\boxed{x = \frac{1}{3}t^3 + C}$ 2

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(4) $x' = a(t)\,x^2 + b(t)\,x + c(t)$

$$y = x - x_1$$

$$y' = x' - x_1'$$

$$y' + x_1' = a(t)\cdot(y + x_1)^2 + b(t)\cdot(y + x_1) + c(t)$$

$$y' + x_1' = a(t)\,y^2 + 2a(t)\,y\,x_1 + a(t)\,x_1^2 + b(t)\,y + b(t)\,x_1 + c$$

$$\boxed{y' = a(t)\,y^2 + \big(2a(t)\,x_1 + b(t)\big)\,y}$$

5

(2) $\underbrace{(t \sin x + x)}_{P} dt + \underbrace{(t^2 \cos x + t \ln t)}_{Q} dx = 0$

$P_x = t \cos x + 1, \quad Q_t = 2t \cos x + \ln t + 1 \qquad P_x \neq Q_t$

$w = t\,?$ $\quad \dfrac{Q_x - P_t}{w_x P - w_t Q} = \dfrac{2t \cos x + \ln t + \cancel{1} - t \cos x - \cancel{1}}{-t^2 \cos x - t \ln t} = \dfrac{\cancel{t \cos x + \ln t}}{-t\,(\cancel{t \cos x + \ln t})} =$

$= -\frac{1}{t} = \varphi(t) \quad$ - ok

$\mu = C e^{-\int \frac{dt}{t}} = C e^{-\ln|t|} = C \cdot \frac{1}{|t|} \qquad \boxed{\mu = \frac{1}{t}}$ 1

$\left(\sin x + \frac{x}{t}\right) dt + (t \cos x + \ln t)\, dx = 0$

$\begin{cases} \dfrac{\partial F}{\partial t} = \sin x + \dfrac{x}{t} \\ \dfrac{\partial F}{\partial x} = t \cos x + \ln t \end{cases} \Rightarrow F = \int \left(\sin x + \frac{x}{t}\right) dt = t \sin x + x \ln t + \varphi(x)$

$t \cos x + \ln t + \varphi'(x) = t \cos x + \ln t$

$\varphi'(x) = 0, \quad \underline{\varphi(x) = 0}$

$F = t \sin x + x \ln t$

3

$\boxed{t \sin x + x \ln t = C}$ 1

⊠

3. $x \cdot x'' - (x')^2 = x^2 \cdot x'$

$y = x', \; y = y(x)$

$x'' = y'x' = yy'$

$\boxed{x = C}$

$x\cancel{y}y' - y^{\cancel{2}} = x^2\cancel{y}$

$xy' - y = x^2$

$\bullet\; \underline{y' = \frac{1}{x}y + x}$ 1

$y_h = Ce^{\int \frac{dx}{x}} = Ce^{\ln|x|} = C \cdot x$

$y_p = c(x) \cdot x$

$c'(x)x = x$

$c'(x) = 1$

$c(x) = x$

$y_p = x^2$

$\bullet\; \underline{y = Cx + x^2}$ 2

$1^\circ \quad C = 0$

$\boxed{-\frac{1}{x} = t + C_1}$

$\bullet\; \underline{x' = Cx + x^2}$

$x' = x \cdot (C + x)$

$\int \frac{dx}{x \cdot (C+x)} = \int dt$

2°

$C \neq 0$

$\frac{1}{x(C+x)} = \frac{A}{x} + \frac{B}{C+x}$

$1 = A(C+x) + Bx$

$\begin{cases} AC = 1 \\ A + B = 0 \end{cases} \qquad B = -A \qquad A = \frac{1}{C}, \; B = -\frac{1}{C}$

$\frac{1}{C}\int \frac{dx}{x} - \frac{1}{C}\int \frac{dx}{C+x} = \frac{1}{C}\ln|x| - \frac{1}{C}\ln|C+x|$

$\boxed{\frac{1}{C}\ln|x| - \frac{1}{C}\ln|C+x| = t + C_1}$ 2