1 $f = \arcsin\left(|2 + \log_{\frac{1}{2}}(2x-4)| - 1 \right)$

1) $2x - 4 > 0$ 0.5

$\underline{x > 2}$

2) $-1 \le |2 + \log_{\frac{1}{2}}(2x-4)| - 1 \le 1$

$|\dots| \geq 0 \quad \wedge \quad |2 + \log_{\frac{1}{2}}(2x-4)| \le 2$

$\underline{OK}$

$2 + \log_{\frac{1}{2}}(2x-4) \geq -2 \quad \wedge \quad 2 + \log_{\frac{1}{2}}(2x-4) \le 2$

$\log_{\frac{1}{2}}(2x-4) \geq -4 \qquad\qquad \log_{\frac{1}{2}}(2x-4) \le 0$

$\log_{\frac{1}{2}}(2x-4) \geq \log_{\frac{1}{2}} 16 \qquad\qquad \log_{\frac{1}{2}}(2x-4) \le \log_{\frac{1}{2}} 1$

$2x - 4 \le 16 \qquad\qquad 2x - 4 \geq 1$

$x \le 10 \qquad x \in \langle \frac{5}{2}, 10 \rangle \qquad x \geq \frac{5}{2}$

3.5

$\boxed{D_f = \langle \frac{5}{2}, 10 \rangle}$ 1

---

2 $f(x) = \ln\left| \frac{3^x + 1}{3^x - 1} \right| \qquad \underline{D_f = \mathbb{R} \setminus \{0\}}$ — OK 0.5

$f(-x) = \ln\left| \frac{3^{-x}+1}{3^{-x}-1} \right| = \ln\left| \frac{1+3^x}{1-3^x} \right| = \ln\left| \frac{3^x+1}{-(3^x-1)} \right| = \ln\left| \frac{3^x+1}{3^x-1} \right| = f(x) \quad \forall x \in D_f$

$\boxed{f \text{ - parzysta} \quad , \quad f \text{ - nie nieparzysta}}$ 2.5 2

Bo jedyną parzystą i nieparzystą jest $f(x) \equiv 0$

---

3) $f(x) = \begin{cases} 4x - 1, & x \le 2 \\ x + 3, & x > 2 \end{cases}$

$(f \circ f)(x) = f(f(x)) = \begin{cases} 4f(x) - 1, & f(x) \le 2 \\ f(x) + 3, & f(x) > 2 \end{cases}$ 1

$1^\circ \; f(x) \le 2$

1) $x \le 2$

$\underbrace{4x-1}_{f(x)} \le 2$

$x \le \frac{3}{4}$

$\underline{x \le \frac{3}{4}}$

2) $x > 2$

$\underbrace{x+3}_{f(x)} \le 2$

$x \le -1$

$\underline{x \in \emptyset}$

$2^\circ \; f(x) > 2$

1) $x \le 2$

$\underbrace{4x-1}_{f(x)} > 2$

$x > \frac{3}{4}$

$\underline{x \in (\frac{3}{4}, 2\rangle}$

2) $x > 2$

$\underbrace{x+3}_{f(x)} > 2$

$x > -1$

$\underline{x > 2}$

• $4(4x-1) - 1$

• $(4x-1) + 3$

• $(x+3) + 3$

$\boxed{(f \circ f)(x) = \begin{cases} 16x - 5, & x \le \frac{3}{4} \\ 4x + 2, & x \in (\frac{3}{4}, 2\rangle \\ x + 6, & x > 2 \end{cases}}$ 4

4a $\lim\limits_{n \to \infty} \frac{\sqrt{n^2 + \sqrt{n+1}} - \sqrt{n^2 - \sqrt{n-1}}}{\sqrt{n+1} - \sqrt{n}}$ 1 $= \lim\limits_{n \to \infty} \frac{(\cancel{n^2} + \sqrt{n+1} - \cancel{n^2} + \sqrt{n-1})(\sqrt{n+1} + \sqrt{n})}{(\cancel{n}+1-\cancel{n})(\sqrt{n^2+\sqrt{n+1}} + \sqrt{n^2 - \sqrt{n-1}})}$

$= \lim\limits_{n \to \infty} \frac{\cancel{n}\left(\sqrt{1+\frac{1}{n}} + \sqrt{1 - \frac{1}{n}}\right) \cdot \left(\sqrt{1+\frac{1}{n}} + 1\right)}{\cancel{n}\left(\sqrt{1 + \sqrt{\frac{1}{n^3} + \frac{1}{n^4}}} + \sqrt{1 - \sqrt{\frac{1}{n^3} - \frac{1}{n^4}}}\right)}$ 2 $= \frac{4}{2} = \boxed{2}$

1 0.5 1 1

(1) $f = \arccos\left(1 - \left|2 + \log_{\frac{1}{3}}(2x-1)\right|\right)$

1) $2x-1 > 0$
$\underline{x > \frac{1}{2}}$ (0.5)

2) $-1 \le 1 - \left|2 + \log_{\frac{1}{3}}(2x-1)\right| \le 1$

$\left|2+\log_{\frac{1}{3}}(2x-1)\right| \le 2 \ \wedge \ \left|2+\log_{\frac{1}{3}}(2x-1)\right| \ge 0$ $\underline{OK}$

$2+\log_{\frac{1}{3}}(2x-1) \ge -2 \ \wedge \ 2+\log_{\frac{1}{3}}(2x-1) \le 2$

$\log_{\frac{1}{3}}(2x-1) \ge -4$ $\qquad$ $\log_{\frac{1}{3}}(2x-1) \le 0$

$\log_{\frac{1}{3}}(2x-1) \ge \log_{\frac{1}{3}} 81$ $\qquad$ $\log_{\frac{1}{3}}(2x-1) \le \log_{\frac{1}{3}} 1$

$2x-1 \le 81$ $\qquad$ $2x-1 \ge 1$

$x \le 41$ $\qquad$ $x \ge 1$

$\underline{x \in \langle 1, 41\rangle}$ (3.5)

$\boxed{D_f = \langle 1, 41\rangle}$ (1)

---

(2) $f(x) = \frac{4^x+1}{1-4^x} \cdot \sin 2x$ $\qquad$ $\underline{D_f = \mathbb{R} \setminus \{0\}}$ (0.5)

$f(-x) = \frac{4^{-x}+1}{1-4^{-x}} \cdot \sin(-2x) = \frac{4^x+1}{4^x-1} \cdot \sin 2x = \frac{4^x+1}{1-4^x} \cdot \sin 2x = f(x)$
$\forall x \in D_f$

$\boxed{f - \text{parzysta}, \quad f - \text{nie nieparzysta}}$ (0.5) (2)

Bo jedyna parzysta i nieparzysta jest $f(x) \equiv 0$.

---

(3) $f(x) = \begin{cases} 2x+1, & x \le 1 \\ 3x-5, & x > 1 \end{cases}$

$(f\circ f)(x) = f(f(x)) = \begin{cases} 2f(x)+1, & f(x) \le 1 \\ 3f(x)-5, & f(x) > 1 \end{cases}$ (1)

1° $f(x) \le 1$

1) $x \le 1$
$\underbrace{2x+1}_{f(x)} \le 1$
$x \le 0$
$\underline{x \le 0}$

2) $x > 1$
$\underbrace{3x-5}_{f(x)} \le 1$
$x \le 2$
$\underline{x \in (1, 2\rangle}$

2° $f(x) > 1$

1) $x \le 1$
$\underbrace{2x+1}_{f(x)} > 1$
$x > 0$
$\underline{x \in (0, 1\rangle}$

2) $x > 1$
$\underbrace{3x-5}_{f(x)} > 1$
$x > 2$
$\underline{x > 2}$

• $2(2x+1)+1$ $\qquad$ • $2(3x-5)+1$ $\qquad$ • $3(2x+1)-5$ $\qquad$ • $3(3x-5)-5$

$$\boxed{(f\circ f)(x) = \begin{cases} 4x+3, & x \le 0 \\ 6x-2, & x \in (0,1\rangle \\ 6x-9, & x \in (1,2\rangle \\ 9x-20, & x > 2 \end{cases}}$$ (4)

---

(4) $\lim_{n\to\infty} \frac{\frac{1}{n-1} - \frac{1}{n+1}}{\ln(n^2+1) - \ln(n^2-1)} = \lim_{n\to\infty} \frac{\frac{n+1-n+1}{n^2-1}}{\ln\frac{n^2+1}{n^2-1}} = \lim_{n\to\infty} \frac{2}{(n^2-1)\ln\frac{n^2+1}{n^2-1}}$ (1)

$= \lim_{n\to\infty} \frac{2}{\ln\left(\frac{n^2+1}{n^2-1}\right)^{n^2-1}}$ (1) $= \lim_{n\to\infty} \frac{2}{\ln e^2} = \boxed{1}$ (1) $\qquad$ • $\lim_{n\to\infty} \frac{n^2-1}{2} = \infty$

$\lim_{n\to\infty} \left(\frac{n^2+1}{n^2-1}\right)^{n^2-1} = \lim_{n\to\infty} \left[\left(1+\frac{1}{\frac{n^2-1}{2}}\right)^{\frac{n^2-1}{2}}\right]^2 = e^2$ (2)

(2)

4b) $\lim_{n\to\infty} n\left(\ln\frac{1}{n+1} + \ln(n+3)\right) = \lim_{n\to\infty} \ln\left(\frac{n+3}{n+1}\right)^n = \boxed{\ln e^2 = 2}$ (1) (0.5)

$\lim_{n\to\infty}\left(\frac{n+3}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{(n+1)+2}{n+1}\right)^n = \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{2}{n+1}\right)^{n+1}}_{\to e^2}\right]^{\frac{n}{n+1}} = e^2$ (1)

Bo: $\lim_{n\to\infty}(n+1) = \infty$, $\lim_{n\to\infty}\frac{n}{n+1} = \underline{1}$

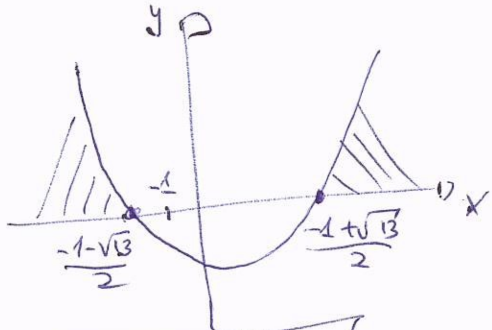
---

3) $\sqrt{x+4} - 1 < x$; $\sqrt{x+4} < x+1$

Z: $x+4 \geq 0$
$\underline{x \geq -4}$ (0.5)

1° $x+1 < 0$
$\underline{x < -1}$ $\underline{\text{Sprz}}$

2° $x+1 \geq 0$
$\underline{x \geq -1}$
$x+4 < x^2+2x+1$
$x^2+x-3 > 0$
$\Delta = 13$
$x = \frac{-1 \pm \sqrt{13}}{2}$ (3.5)



$\boxed{x > \frac{-1+\sqrt{13}}{2}}$ (1)

4a) $\lim_{n\to\infty}\sqrt[n]{\frac{1}{8^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{7^n}} = \boxed{\frac{1}{3}}$ (0.5)

$\underset{\downarrow \frac{1}{3}}{\frac{1}{3}} \leq \sqrt[n]{\dots} \leq \sqrt[n]{4\cdot\frac{1}{3^n}} = \underbrace{\frac{1}{3}\cdot\sqrt[n]{4}}_{\downarrow \frac{1}{3}}$ (2)

4b) $\lim_{n\to\infty} n^2\left(\ln\frac{1}{n^2+2} - \ln\frac{1}{n^2+1}\right) = \lim_{n\to\infty}\ln\left(\frac{n^2+1}{n^2+2}\right)^{n^2} = \boxed{\ln e^{-1} = -1}$ (1) (0.5)

$\lim_{n\to\infty}\left(\frac{n^2+1}{n^2+2}\right)^{n^2} = \lim_{n\to\infty}\left[\underbrace{\left(1+\frac{-1}{n^2+2}\right)^{n^2+2}}_{\to e^{-1}}\right]^{\frac{n^2}{n^2+2}} = e^{-1}$ (1)

Bo: $\lim_{n\to\infty}(n^2+2) = \infty$, $\lim_{n\to\infty}\frac{n^2}{n^2+2} = \underline{1}$

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3) $|x-1| - \sqrt{x} > -2$; $|x-1| > \sqrt{x} - 2$

Z: $\underline{x \geq 0}$ (0.5)

1° $\sqrt{x} - 2 < 0$
$\sqrt{x} < 2$
$\underline{x < 4}$ – OK
$x \in \langle 0, 4)$

2° $\sqrt{x} - 2 \geq 0$
$\underline{x \geq 4}$
$x - 1 \geq \sqrt{x} - 2$
$x + 1 \geq \sqrt{x}$
$x^2+2x+1 \geq x$
$x^2+x+1 \geq 0$ (3.5)

$\Delta = -3 < 0$
$x \in \mathbb{R}$
$x \in \langle 4, \infty)$

$\boxed{x \in \langle 0, \infty)}$ (1)