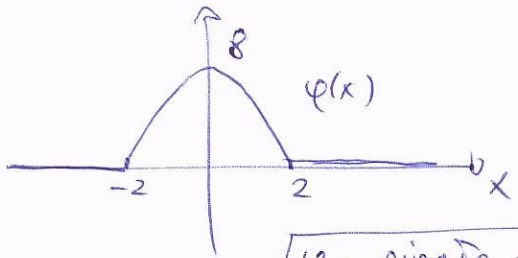
① $\varphi(x) = \begin{cases} 8 - |x|^3, & |x| \le 2 \\ 0, & |x| > 2 \end{cases}$



0.5

$\boxed{\varphi \text{ - [illegible], [illegible]}}$

$\begin{cases} u_t = u_{xx} \\ u(x,0) = \varphi(x) \end{cases}$

$\boxed{0 \le u(x,t) \le 8}$ 1

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} \varphi(y)\,dy = \frac{1}{2\sqrt{\pi t}} \int_{-2}^{2} e^{-\frac{(x-y)^2}{4t}} (8 - |y|^3)\,dy =$$

$$= \frac{1}{2\sqrt{\pi t}} \int_{-2}^{0} e^{-\frac{(x-y)^2}{4t}} (8 + y^3)\,dy + \frac{1}{2\sqrt{\pi t}} \int_{0}^{2} e^{-\frac{(x-y)^2}{4t}} (8 - y^3)\,dy = \boxed{I + II}$$ 1.5

$$I = \frac{1}{2\sqrt{\pi t}} \int_{-2}^{0} e^{-\frac{(x-y)^2}{4t}} (8 + y^3)\,dy = \frac{1 \cdot 8}{2\sqrt{\pi t}} \int_{-2}^{0} e^{-\frac{(x-y)^2}{4t}} dy + \frac{1}{2\sqrt{\pi t}} \int_{-2}^{0} e^{-\frac{(x-y)^2}{4t}} y^3\,dy = \ldots$$

$$w = \frac{x-y}{2\sqrt{t}}, \quad 2\sqrt{t}\,dw = -dy, \quad y = x - 2\sqrt{t}\,w$$

$$\ldots = \frac{4}{\sqrt{\pi t}} \cdot 2\sqrt{t} \int_{\frac{x}{2\sqrt{t}}}^{\frac{x+2}{2\sqrt{t}}} e^{-w^2} dw - \frac{1}{2\sqrt{\pi t}} \cdot 2\sqrt{t} \int_{\frac{x+2}{2\sqrt{t}}}^{\frac{x}{2\sqrt{t}}} e^{-w^2} (x - 2\sqrt{t}\,w)^3\,dy$$

$$= \frac{8}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{t}}}^{\frac{x+2}{2\sqrt{t}}} e^{-w^2} dw + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{2\sqrt{t}}}^{\frac{x+2}{2\sqrt{t}}} e^{-w^2} (x - 2\sqrt{t}\,w)^3\,dw = \ldots$$

$$(a-b)^3 = a^3 - 3a^2 b + 3ab^2 - b^3$$
$$(x - 2\sqrt{t}\,w)^3 = x^3 - 3x^2 2\sqrt{t}\,w + 3x\,4t\,w^2 - 8t\sqrt{t}\,w^3$$

- $\int w e^{-w^2} dw = -\frac{1}{2} e^{-w^2}$
- $\int w^2 e^{-w^2} dw = \left\{ \begin{array}{l|l} u = w & u' = 1 \\ v' = w e^{-w^2} & v = -\frac{1}{2} e^{-w^2} \end{array} \right\} = -\frac{1}{2} w e^{-w^2} + \frac{1}{2} \int e^{-w^2} dw$
- $\int w^3 e^{-w^2} dw = \left\{ \begin{array}{l|l} u = w^2 & u' = 2w \\ v' = w e^{-w^2} & v = -\frac{1}{2} e^{-w^2} \end{array} \right\} = -\frac{1}{2} w^2 e^{-w^2} + \int w e^{-w^2} dw = -\frac{1}{2} w^2 e^{-w^2} - \frac{1}{2} e^{-w^2}$

$$\ldots = \frac{1}{\sqrt{\pi}} (8 + x^3) \int_{\frac{x}{2\sqrt{t}}}^{\frac{x+2}{2\sqrt{t}}} e^{-w^2} dw - \frac{1}{\sqrt{\pi}} \cdot 6x^2\sqrt{t} \left[-\frac{1}{2} e^{-w^2} \right]_{\frac{x}{2\sqrt{t}}}^{\frac{x+2}{2\sqrt{t}}} + \frac{1}{\sqrt{\pi}} \cdot 12xt \cdot$$

$$\cdot \left[-\frac{1}{2} w e^{-w^2} + \frac{1}{2} \int_{\frac{x}{2\sqrt{t}}}^{\frac{x+2}{2\sqrt{t}}} e^{-w^2} dw \right]_{\frac{x}{2\sqrt{t}}}^{\frac{x+2}{2\sqrt{t}}} - \frac{1}{\sqrt{\pi}} \cdot 8t\sqrt{t} \cdot \left[-\frac{1}{2} w^2 e^{-w^2} - \frac{1}{2} e^{-w^2} \right]_{\frac{x}{2\sqrt{t}}}^{\frac{x+2}{2\sqrt{t}}} =$$

$$= \frac{1}{\sqrt{\pi}} (8 + x^3 + 6xt) \int_{\frac{x}{2\sqrt{t}}}^{\frac{x+2}{2\sqrt{t}}} e^{-w^2} dw + \frac{1}{\sqrt{\pi}} \left(3x^2\sqrt{t} + 4t\sqrt{t} \right) \left(e^{-\frac{(x+2)^2}{4t}} - e^{-\frac{x^2}{4t}} \right) +$$

$$-\frac{1}{\sqrt{\pi}}\,6xt\left(\frac{x+2}{2\sqrt{t}}e^{-\frac{(x+2)^2}{4t}}-\frac{x}{2\sqrt{t}}e^{-\frac{x^2}{4t}}\right)+$$

$$+\frac{1}{\sqrt{\pi}}\cdot 4t\sqrt{t}\left(\frac{(x+2)^2}{4t}e^{-\frac{(x+2)^2}{4t}}-\frac{x^2}{4t}e^{-\frac{x^2}{4t}}\right)$$

(2)

•• II – Oblicza się analogicznie – nie ma więc potrzeby obliczać...
("symetria" w rozumowaniu)

---

(2)
$$\begin{cases} u_{tt}=u_{xx}+\sin(t+2x)\\ u(x,0)=e^{-2x}\\ u_t(x,0)=\operatorname{arctg} x\end{cases}$$

$$u(x,t)=\frac{1}{2}\left[\varphi(x-t)+\varphi(x+t)\right]+\frac{1}{2}\int_{x-t}^{x+t}\psi(y)\,dy+\frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+(t-s)}f(y,s)\,dy\,ds$$

$$u(x,t)=\frac{1}{2}\left[e^{-2(x-t)}+e^{-2(x+t)}\right.+\frac{1}{2}\int_{x-t}^{x+2}\operatorname{arctg} y\,dy+\frac{1}{2}\int_0^t\int_{x-(t-s)}^{x+(t-s)}\sin(s+2y)\,dy\,ds=$$

...

• $$\int_{x-t}^{x+t}1\cdot\operatorname{arctg} y\,dy=\left\{\begin{matrix}u'=1 & u=y\\ v=\operatorname{arctg} y & v'=\frac{1}{1+y^2}\end{matrix}\right\}=\left[y\operatorname{arctg} y-\frac{1}{2}\ln(1+y^2)\right]_{x-t}^{x+t}=$$

$$=(x+t)\operatorname{arctg}(x+t)-(x-t)\operatorname{arctg}(x-t)+\frac{1}{2}\ln\left(1+(x+t)^2\right)-\frac{1}{2}\ln\left(1+(x-t)^2\right)$$

(2,5)

•• $$\int_0^t\int_{x-(t-s)}^{x+(t-s)}\sin(s+2y)\,dy\,ds=-\frac{1}{2}\int_0^t\left[\cos(s+2y)\right]_{x-(t-s)}^{x+(t-s)}=$$

$$=-\frac{1}{2}\int_0^t\left[\cos(\underbrace{s+2x+2t-2s}_{2x+2t-s})-\cos(\underbrace{s+2x-2t+2s}_{2x-2t+3s})\right]ds=$$

$$=-\frac{1}{2}\left[-\sin(2x+2t-s)\Big|_0^t-\frac{1}{3}\sin(2x-2t+3s)\Big|_0^t\right]=$$

$$=\frac{1}{2}\sin(2x+t)-\frac{1}{2}\sin(2x+2t)+\frac{1}{6}\sin(2x+t)-\frac{1}{6}\sin(2x-2t)$$

(2,5)

$$\boxed{\begin{aligned}u(x,t)=&\frac{1}{2}\left[e^{-2(x-t)}+e^{-2(x+t)}\right]+\frac{1}{2}(x+t)\operatorname{arctg}(x+t)-\\&-\frac{1}{2}(x-t)\operatorname{arctg}(x-t)+\frac{1}{4}\ln\left(1+(x+t)^2\right)-\frac{1}{4}\ln\left(1+(x-t)^2\right)+\\&+\frac{1}{3}\sin(2x+t)-\frac{1}{4}\sin(2x+2t)-\frac{1}{12}\sin(2x-2t)\end{aligned}}$$

Sprawdziemy, że u spełnia równanie różniczkowe. Na podstawie zadania 47 wystarczy sprawdzić, że
$$\tilde{u}(x,t):=\frac{1}{3}\sin(2x+t)-\frac{1}{4}\sin(2x+2t)-\frac{1}{12}\sin(2x-2t).$$

$$\tilde{u}_{xx} = -\frac{4}{3}\sin(2x+t) + \sin(2x+2t) + \frac{1}{3}\sin(2x-2t)$$

$$\tilde{u}_{tt} = -\frac{1}{3}\sin(2x+t) + \sin(2x+2t) + \frac{1}{3}\sin(2x-2t)$$

- $\tilde{u}_{tt} = \tilde{u}_{xx} + \sin(2x+t)$ — $\boxed{\text{OK}}$

- $u(x,0) = e^{-2x}$ — $\boxed{\text{OK}}$

Sprawdzamy jeszcze $u_t(x,0)$:

$$\tilde{v}(x,t) := \frac{1}{3}\sin(2x+t) - \frac{1}{4}\sin(2x+2t) - \frac{1}{12}\sin(2x-2t)$$

$$\tilde{v}_t = \frac{1}{3}\cos(2x+t) - \frac{1}{2}\cos(2x+2t) + \frac{1}{6}\cos(2x-2t)$$

- $\tilde{v}_t(x,0) = 0$

Na podstawie zadania 47 wystarczy sprawdzić jeszcze

atou

$$\tilde{u}(x,t) := (x+t)\,\mathrm{arctg}(x+t) + \frac{1}{2}\ln\left(1+(x+t)^2\right).$$

$$\tilde{u}_t = \mathrm{arctg}(x+t) - \frac{x+t}{1+(x+t)^2} + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}(x+t)}{1+(x+t)^2} = \mathrm{arctg}(x+t)$$

- $\tilde{u}_t(x,0) = \mathrm{arctg}\, x$ — $\boxed{\text{OK}}$

$\boxed{\text{Sprawdzenie można było pominąć...}}$

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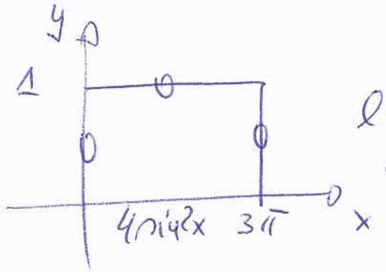
(3)
$$\begin{cases} u_{xx} + u_{yy} = y\sin x \\ u(0,y) = u(3\pi,y) = 0 \\ u(x,0) = 4\sin 2x, \quad u(x,1) = 0 \end{cases}$$



$l = 3\pi$

$\lambda_n = \frac{n}{3}$

$$\sin x = \sum_{n=1}^{\infty} \gamma_n \sin\left(\frac{n}{3}x\right) = \gamma_3 \sin x + 0$$

$$\gamma_3(y) = y$$

$$\sin 2x = \sum_{n=1}^{\infty} \alpha_n \sin\left(\frac{n}{3}x\right) = \alpha_6 \sin 2x$$

$$\alpha_{1,6} = 1$$

$$u(x,y) = w_3(y)\sin x + w_6(y)\sin 2x$$

$$\begin{cases} w_3'' - w_3 = y \\ w_3(0) = 0 \\ w_3(1) = 0 \end{cases}$$

$w_3(y) = C_3 e^{y} + D_3 e^{-y} - y$

$$\begin{cases} C_3 + D_3 = 0 \\ C_3 e + D_3 e^{-1} = 1 \end{cases}$$

$$C_3 = -\frac{1}{e^{-1} - e} = -\frac{e}{1-e^2}$$

$$D_3 = \frac{1}{e^{-1}-e} = \frac{e}{1-e^2}$$

$$\boxed{w_3(y) = -\frac{e}{1-e^2} e^{y} + \frac{e}{1-e^2} e^{-y} - y}$$

(2)

$$\begin{cases} w_6'' - 4w_6 = 0 \\ w_6(0) = 4 \\ w_6(1) = 0 \end{cases}$$

$w_6(y) = C_6 e^{2y} + D_6 e^{-2y}$

$$\begin{cases} C_6 + D_6 = 4 \\ C_6 e^2 + D_6 e^{-2} = 0 \end{cases}$$

$$W = \begin{vmatrix} 1 & 1 \\ e^2 & e^{-2} \end{vmatrix} = e^{-2} - e^2$$

$$W_1 = \begin{vmatrix} 4 & 1 \\ 0 & e^{-2} \end{vmatrix} = 4e^{-2}$$

$$W_2 = \begin{vmatrix} 1 & 4 \\ e^2 & 0 \end{vmatrix} = -4e^2$$

$$C_6 = \frac{4e^{-2}}{e^{-2}-e^2} = \frac{4}{1-e^4}$$

$$D_6 = -\frac{4e^2}{e^{-2}-e^2} = -\frac{4e^4}{1-e^4}$$

$$\boxed{w_6(y) = \frac{4}{1-e^4} e^{2y} - \frac{4e^4}{1-e^4} e^{-2y}}$$

(2)

$$\boxed{u(x,y) = \left(-\frac{e}{1-e^2} e^{y} + \frac{e}{1-e^2} e^{-y} - y\right)\sin x + \left(\frac{4}{1-e^4} e^{2y} - \frac{4e^4}{1-e^4} e^{-2y}\right)\sin 2x}$$

(1)

(4) $\begin{cases} \Delta u + cu = f(x) \\ u(x) = \varphi(x) \end{cases}$

1° $c = 0$ – ma $\boxed{\text{co najwyżej jedno}}$ rozwiązanie : wykład (0.5)

2° $c < 0$ – ma $\boxed{\text{co najwyżej jedno}}$ rozwiązanie : //

u_1, u_2 – rozw., $w := u_1 - u_2$

$$\begin{cases} \Delta w + cw = 0 \\ w = 0 \end{cases} \qquad 0 = \int_\Omega w \Delta w \, dx + c \int_\Omega w^2 \, dx =$$

(3.5)

$$= -\int_\Omega |\nabla w|^2 + \underbrace{\int_{\partial\Omega} w \frac{\partial w}{\partial \nu} \, d\sigma}_{=0} + c \int_\Omega w^2 \, dx \underset{c<0}{\Longrightarrow} \int_\Omega w^2 \, dx = 0$$

$$\Downarrow$$

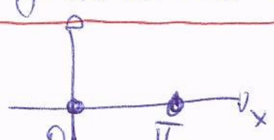
$$w = 0$$

$$\boxed{u_1 = u_2}$$

3° $\boxed{c > 0 \text{ – brak jednoznaczności}}$

Kontrprzykład

$n = 1$



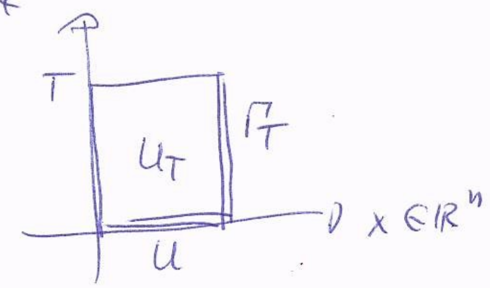
$$\begin{cases} u'' + cu = 0 \\ u(0) = u\left(\frac{\pi}{\sqrt{c}}\right) = 0 \end{cases}$$

(1)

$$\boxed{u(x) = A \sin(\sqrt{c}\, x), \ A \in \mathbb{R} \text{ – rozwiązania}}$$

(Najprościej gdy $c = 1$)

(4) - Druga podgrupa



$$\begin{cases} u_t = \Delta u + cu + f(x,t) \\ u(x,t) = \varphi(x) \end{cases}$$

$\boxed{c \in \mathbb{R} \Rightarrow \text{co najwyżej jedno rozwiązanie}}$

u_1, u_2 - rozw. $\quad w := u_1 - u_2$

$$\begin{cases} w_t = \Delta w + cw \\ w = 0 \end{cases}$$

$$\int_U w_t w \, dx = \int_U w \Delta w \, dx + c \int_U w^2 \, dx = -\int_U |Dw|^2 dx + \underbrace{\int_{\Gamma_T} w \frac{\partial w}{\partial \nu} d\sigma}_{=0} + c\int_U w^2 dx$$

$$\parallel$$

$$\frac{1}{2} \cdot \frac{\partial}{\partial t} \int_U w^2 \, dx$$

Stąd $\frac{\partial}{\partial t} \int_U w^2 dx \leq 2|c| \int_U w^2 dx$.

Oczywiście przez w^2 rozumiemy $w^2(t,x)$.

$$\eta(t) := \int_U w^2(t,x)\, dx \geq 0$$ (5)

Mamy $\begin{cases} \eta'(t) \leq \underbrace{2|c|}_{\geq 0} \cdot \eta(t) \\ \eta(0) = 0 \end{cases}$ - bo dla $t=0$ $\eta(0) = \int_U w^2(0,x)\, dx = 0$

Z nierówności Gronwalla otrzymujemy

$$\eta(t) = 0, \quad t \in [0,T]$$

$$\Updownarrow \int_U w^2 dx = 0 \Rightarrow w^2 = 0 \Rightarrow \boxed{u_1 = u_2}$$
$t \in [0,T]$