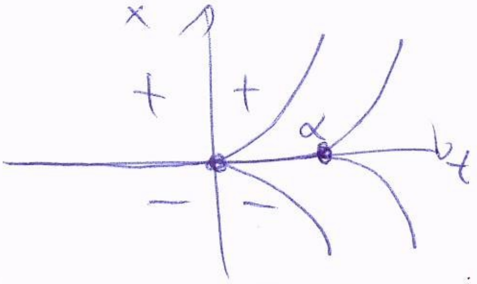
(1)

a) $x' = \sqrt[3]{x}$, $x(0)=0$



$\pm$ - znaki x'

$\boxed{x(t)\equiv 0 - OK}$ (1)

$x\neq 0 \Rightarrow \int \frac{dx}{\sqrt[3]{x}} = \int dt$

$\frac{3}{2}x^{\frac{2}{3}} = t + C$, $t+C>0$

$x = \pm\sqrt{\left(\frac{2}{3} t + C\right)^3}$

$$\boxed{x(t) = \begin{cases} 0, & t\le 0 \\ \sqrt{\left(\frac{2}{3}t\right)^3}, & t>0 \end{cases} \qquad x(t) = \begin{cases} 0, & t\le 0 \\ -\sqrt{\left(\frac{2}{3}t\right)^3}, & t>0 \end{cases}} - OK$$

(2)

Sprawdzamy: $0 = x'(0) \overset{(*)}{=} \sqrt[3]{x(0)} = \sqrt[3]{0} = 0$

$x(0) = 0$ OK

$\alpha \geq 0$ - dowolnie ustalone

$$\boxed{x(t) = \begin{cases} 0, & t\le \alpha \\ \sqrt{\left(\frac{2}{3}(t-\alpha)\right)^3}, & t>\alpha \end{cases} \qquad x(t) = \begin{cases} 0, & t\le \alpha \\ -\sqrt{\frac{2}{3}(t-\alpha)^3}, & t>\alpha \end{cases}} - OK$$

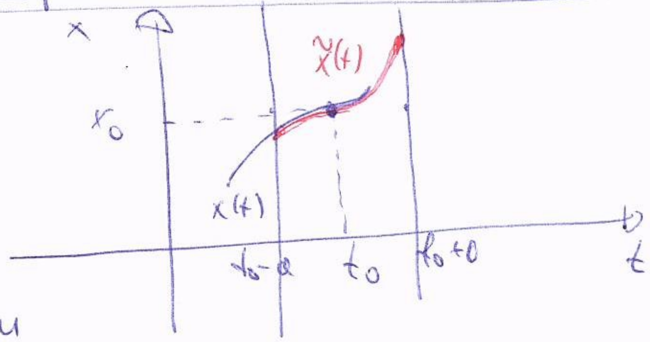
(2)

Sprawdzamy:

$0 = x'(\alpha) \overset{(*)}{=} \sqrt[3]{x(\alpha)} = \sqrt[3]{0} = 0$

$x(\alpha) = 0$ OK

b) $x' = \frac{x+1}{x^{10}+1}$



x(t) dowolne rozw.

(t_0, x_0) - dowolny pkt. z trajektorii $x(t)$

$x' = \frac{x+1}{x^{10}+1}$, $x(t_0) = x_0$ - problem początkowy

$a>0$ - dowolnie ustalone

~~f(t,x)~~ $f: [t_0-a, t_0+a]\times\mathbb{R}\to\mathbb{R}$

$f(t,x) = \frac{x+1}{x^{10}+1}$, $(t,x)\in[t_0-a, t_0+a]\times\mathbb{R}$

(5)

(I) 1) f - cg.

2) f - lokalnie L, bo f - C^1 wzgl. x

3) $|f(t,x)| \le |x|+1$

$\Rightarrow \exists!\ \tilde{x}(t)$ na $[t_0-a, t_0+a]$

Zatem albo $\tilde{x} = x$ na $[t_0-a, t_0+a]$, albo $\tilde{x}$ jest przedłużeniem x, gdyż w każdym pkt. z trajektorii x jest jednoznaczność. Z dowolności a jest OK.

(II) 1) f - cg.

2) f - Lipschitzowska

Bo $\frac{\partial f}{\partial x} = \frac{x^{10}+1-(x+1)10x^9}{(x^{10}+1)^2} = \frac{-9x^{10}-10x^9+1}{(x^{10}+1)^2}$

$\left(\lim_{x\to\pm\infty} \frac{\partial f}{\partial x} = 0 \ \wedge \ f - \text{cg}\right) \Rightarrow \frac{\partial f}{\partial x}$ - ograniczona

Dalej jak w (I).

(III) 1) f - cg

2) f - lokalnie L wzgl. x

3) f - ogr.: Bo f jest cg. i $\lim_{x\to\pm\infty} f(x)=0$

Dalej jak w (I)

3 a) $x''' + x' = \cos t$

$x''' + x' = 0$ $\quad \lambda = 0,\ \lambda = \pm i$

$\lambda^3 + \lambda = 0$

$\lambda(\lambda^2+1) = 0$

• $x_j = C_1 + C_2 \cos t + C_3 \sin t$ (2)

•• $x_s = t(A\cos t + B\sin t)$

$x_s' = A\cos t + B\sin t + t(-A\sin t + B\cos t)$

$x_s'' = -A\sin t + B\cos t - A\sin t + B\cos t + t(-A\cos t - B\sin t) =$

$= -2A\sin t + 2B\cos t + t(-A\cos t - B\sin t)$

$x_s''' = -2A\cos t - 2B\sin t - A\cos t - B\sin t + t(A\sin t - B\cos t) =$

$= -3A\cos t - 3B\sin t + t(A\sin t - B\cos t)$

$-3A\cos t - 3B\sin t + \cancel{At\sin t} - \cancel{Bt\cos t} +$

$+ A\cos t + B\sin t - \cancel{At\sin t} + \cancel{Bt\cos t} = \cos t$

$\begin{cases} -2A = 1 \\ -2B = 0 \end{cases}$ $\quad A = -\frac{1}{2},\ B = 0$ $\quad$ •• $x_s = -\frac{1}{2} t\cos t$ (2)

$\boxed{x = C_1 + C_2\cos t + C_3 \sin t - \frac{1}{2} t\cos t}$ (1)

b) $t^2 x'' + 6t x' + 4x = t^2, \quad t > 0$

$t^2 x'' + 6t x' + 4x = 0$

$x = t^\lambda, \ [\lambda(\lambda-1) + 6\lambda + 4]t^\lambda = 0$

$\lambda^2 + 5\lambda + 4 = 0$

$\Delta = 9$

$\lambda = \frac{-5\pm 3}{2}$

$\lambda_1 = -4,\ \lambda_2 = -1$

• $x_j = C_1 t^{-4} + C_2 t^{-1}$ (2)

•• $x_s = C_1(t) t^{-4} + C_2(t) t^{-1}$

$\begin{bmatrix} t^{-4} & t^{-1} \\ -4t^{-5} & -t^{-2} \end{bmatrix} \cdot \begin{bmatrix} C_1'(t) \\ C_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

ważne V_0

$W = -t^{-6} + 4t^{-6} = 3t^{-6}$

$W_1 = -t^{-1}$

$W_2 = t^{-4}$

$C_1(t) = -\frac{1}{3}\int t^5\,dt = -\frac{1}{18} t^6$

$C_2(t) = \frac{1}{3}\int t^2\,dt = \frac{1}{9} t^3$

(2) •• $x_s = -\frac{1}{18}t^2 + \frac{1}{9}t^2 = \frac{1}{18}t^2$

$\boxed{x = C_1 t^{-4} + C_2 t^{-1} + \frac{1}{18} t^2}$ (1)

2 $\underbrace{(t+t^4+2t^2x^2+x^4)}_{P}\,dt + \underbrace{x}_{Q}\,dx = 0$

$\mu = \mu(t^2+x^2)$, $\omega = t^2+x^2$

$$\frac{\partial P}{\partial x} = 4t^2x + 4x^3, \quad \frac{\partial Q}{\partial t} = 0$$

$$\frac{\frac{\partial Q}{\partial t} - \frac{\partial P}{\partial x}}{\frac{\partial \omega}{\partial x}P - \frac{\partial \omega}{\partial t}Q} = \frac{-(4t^2x+4x^3)}{2x(t+t^4+2t^2x^2+x^4) - 2tx} = \frac{-4\cancel{x}\cancel{(t^2+x^2)}}{2\cancel{x}(t^2+x^2)^2}$$

$$= -\frac{2}{t^2+x^2} = -\frac{2}{\omega}$$

$$\mu = Ce^{-\int\frac{2}{\omega}d\omega} = Ce^{-2\ln|\omega|} = C\cdot\frac{1}{\omega^2} = C\cdot\frac{1}{(t^2+x^2)^2}$$

• $\mu = \frac{1}{(t^2+x^2)^2}$ 3

$$\frac{t+(t^2+x^2)^2}{(t^2+x^2)^2}\,dt + \frac{x}{(t^2+x^2)^2}\,dx = 0$$

• $\left(\frac{t}{(t^2+x^2)^2}+1\right)dt + \frac{x}{(t^2+x^2)^2}\,dx = 0$

$$\begin{cases} \frac{\partial F}{\partial t} = \frac{t}{(t^2+x^2)^2}+1 \\ \frac{\partial F}{\partial x} = \frac{x}{(t^2+x^2)^2} \end{cases}$$

$\rightarrow F = \int\left(\frac{t}{(t^2+x^2)^2}+1\right)dt = -\frac{1}{2}\cdot\frac{1}{t^2+x^2}+t+\varphi(x)$

$\frac{x}{(t^2+x^2)^2} + \varphi'(x) = \frac{x}{(t^2+x^2)^2}$

$\varphi'(x) = 0$

$\varphi(x) = 0$

••• $F = -\frac{1}{2}\cdot\frac{1}{t^2+x^2}+t$ 4

$$\boxed{-\frac{1}{2}\cdot\frac{1}{t^2+x^2}+t = C}$$ 3

~~3 a) $x''' + x' = t^2$~~

~~$x'''+x' = 0$~~

~~$\lambda^3+\lambda = 0$~~

~~$\lambda(\lambda^2+1) = 0$~~

~~$\lambda_1 = 0,\ \lambda_2 = i,\ \lambda_3 = -i$~~

~~• $x_h = C_1 + C_2\cos t + C_3\sin t$~~ 2

~~$x_p = t(at^2+bt+c) = at^3+bt^2+ct$~~

~~$x_p' = 3at^2+2bt+c$~~

~~$x_p'' = 6at+2b$~~

~~$x_p''' = 6a$~~

~~$6a+3at^2+2bt+c = t^2$~~

~~$\begin{cases} 3a = 1 \\ 2b = 0 \\ 6a+c = 0 \end{cases}$ $a = \frac{1}{3},\ b = 0,\ c = -2$~~

~~•• $x_p = \frac{1}{3}t^3 - 2t$~~ 2

1 ~~$\boxed{x = C_1+C_2\cos t+C_3\sin t+\frac{1}{3}t^3-2t}$~~

0

$$x_p = C_1(t)\,t^{-2} + C_2(t)\,t^{3}$$

$$\begin{bmatrix} t^{-2} & t^{3} \\ -2t^{-3} & 3t^{2} \end{bmatrix} \cdot \begin{bmatrix} c_1'(t) \\ c_2'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{5\ln t}{t^2} \end{bmatrix}$$

$$W = 3+2 = 5$$

$$W_1 = \begin{vmatrix} 0 & t^3 \\ 5\frac{\ln t}{t^2} & 3t^2 \end{vmatrix} = -5t\ln t$$

$$W_2 = \begin{vmatrix} t^{-2} & 0 \\ -2t^{-3} & \frac{5\ln t}{t^2} \end{vmatrix} = 5t^{-4}\ln t$$

$$c_1(t) = -\int t\cdot\ln t\,dt = -\left\{\begin{matrix} u' = t \\ v = \ln t \end{matrix}\ \middle|\ \begin{matrix} u = \frac{1}{2}t^2 \\ v' = \frac{1}{t} \end{matrix}\right\} = -\left(\frac{1}{2}t^2\ln t - \frac{1}{2}\int t\,dt\right) =$$

$$= -\frac{1}{2}t^2\ln t + \frac{1}{4}t^2$$

$$c_2(t) = \int t^{-4}\ln t\,dt = \left\{\begin{matrix} u' = t^{-4} \\ v = \ln t \end{matrix}\ \middle|\ \begin{matrix} u = -\frac{1}{3t^3} \\ v' = \frac{1}{t} \end{matrix}\right\} = -\frac{1}{3t^3}\ln t + \int\frac{dt}{3t^4} =$$

$$= -\frac{1}{3t^3}\ln t - \frac{1}{9t^3}$$

$$x_p = -\frac{1}{2}\ln t + \frac{1}{4} - \frac{1}{3}\ln t - \frac{1}{9} = -\frac{5}{6}\ln t + \frac{5}{36}$$

$$\boxed{x = C_1 t^{-2} + C_2 t^3 - \frac{5}{6}\ln t + \frac{5}{36}}$$ 5

4

$$A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 8 - 8 - 4(1-\lambda) - 4(1-\lambda) - 4(1-\lambda) =$$

$$= 1 - 3\lambda + 3\lambda^2 - \lambda^3 - 16 - 12 + 12\lambda = -\lambda^3 + 3\lambda^2 + 9\lambda - 27 = 0$$

$$-\lambda^2(\lambda-3) + 9(\lambda-3) = 0 \qquad -(\lambda-3)^2(\lambda+3) = 0$$

$$(\lambda-3)(9-\lambda^2) = 0$$

$$\lambda_1 = 3,\ k_1 = 2$$

$$\lambda_2 = -3,\ k_2 = 1$$

$m(\lambda) = (\lambda-3)(\lambda+3)$ – OK

$$(A-3I)(A+3I) = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & -2 & 2 \\ -2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$m(\lambda) = (\lambda-3)(\lambda+3)$ 2

$f(A) = f(3)\, B_1 + f(-3)\, B_2$

$f_1(\lambda) = 1, \quad f_2(\lambda) = \lambda - 3$

$$\begin{cases} I = B_1 + B_2 \\ A - 3I = -6B_2 \end{cases}$$

$$B_2 = -\frac{1}{6}(A-3I) = -\frac{1}{6}\begin{bmatrix} -2 & -2 & 2 \\ -2 & -2 & 2 \\ 2 & 2 & -2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$B_1 = I - B_2 = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$e^{At} = e^{3t} \cdot \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} + e^{-3t}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$ 4

$$c'(t) = e^{-At} \cdot \begin{bmatrix} 3e^{3t} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 + e^{6t} \\ -1 + e^{6t} \\ 1 - e^{6t} \end{bmatrix}, \quad c(t) = \begin{bmatrix} 2t + \frac{1}{6}e^{6t} \\ -t + \frac{1}{6}e^{6t} \\ t - \frac{1}{6}e^{6t} \end{bmatrix}$$ 2

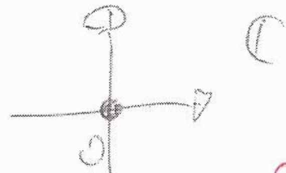
$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \left(e^{3t} \cdot \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix} + e^{-3t}\begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{bmatrix} \right) \cdot \left(\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} 2t + \frac{1}{6}e^{6t} \\ -t + \frac{1}{6}e^{6t} \\ t - \frac{1}{6}e^{6t} \end{bmatrix} \right)}$$

2

5. $\begin{cases} x' = xy^2 \\ y' = -x^2 y \end{cases}$ $(0,0),\ (0,1)$

$$J(x,y) = \begin{bmatrix} y^2 & 2xy \\ -2xy & -x^2 \end{bmatrix}$$

a) $(0,0)$



$$|J(0,0) - \lambda I| = \begin{vmatrix} -\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$$

$$\lambda = 0$$

• G-H – nie wystarcza (to jest układ nieliniowy!) 1

•• Proponujemy funkcję Lapunowa $V(x,y) := x^2 + y^2$ I rodzaju:

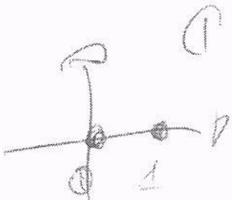
$V \in C^1(\mathbb{R}^2)$ (prostszych nie ma ...)

$V(0,0) = 0$, $V(x,y) > 0$ dla $(x,y) \neq (0,0)$

$(2x, 2y) \circ (xy^2, -x^2 y) = 2x^2y^2 - 2x^2y^2 = 0 \leq 0.$

$\boxed{\text{A zatem } (0,0) \text{ jest stabilny.}}$ 4

b) $(0,1)$



$$|J(0,1) - \lambda I| = \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} = -\lambda(1-\lambda) = 0$$

$$\lambda = 0 \lor \lambda = 1$$

$\boxed{\text{A zatem } (0,1) \text{ jest niestabilny}}$ 5