1. $f = \sqrt{\arcsin \log_{\frac{1}{3}} \frac{x-2}{x+4}}$

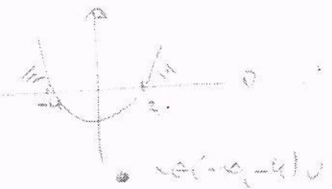
1) $\frac{x-2}{x+4} > 0$

2) $-1 \leq \log_{\frac{1}{3}} \frac{x-2}{x+4} \leq 1$

3) $\arcsin \log_{\frac{1}{3}} \frac{x-2}{x+4} \geq 0$

2

1) $\frac{x-2}{x+4} > 0$, $(x-2)(x+4) > 0$



$x \in (-\infty, -4) \cup (2, \infty)$

1

2) $\log_{\frac{1}{3}} \frac{x-2}{x+4} \geq -1 \quad \wedge \quad \log_{\frac{1}{3}} \frac{x-2}{x+4} \leq 1$

$\log_{\frac{1}{3}} \frac{x-2}{x+4} \geq \log_{\frac{1}{3}} 3$

$\frac{x-2}{x+4} \leq 3$

$\frac{x-2-3x-12}{x+4} \leq 0$

$(-2x-14)(x+4) \leq 0$

$\log_{\frac{1}{3}} \frac{x-2}{x+4} \leq \log_{\frac{1}{3}} \frac{1}{3}$

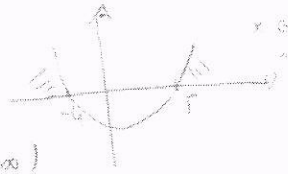
$\frac{x-2}{x+4} \geq \frac{1}{3}$

$\frac{3x-6-x-4}{x+4} \geq 0$

$(2x-10)(x+4) \geq 0$

$x \in (-\infty, -7\rangle \cup (-4, \infty)$

3



$x \in (-\infty, -4) \cup \langle 5, \infty)$

$x \in (-\infty, -7\rangle \cup \langle 5, \infty)$

3) $\arcsin \log_{\frac{1}{3}} \frac{x-2}{x+4} \geq 0$

$\log_{\frac{1}{3}} \frac{x-2}{x+4} \geq 0 = \log_{\frac{1}{3}} 1$

$\frac{x-2}{x+4} \leq 1$

2

$\frac{x-2-x-4}{x+4} \leq 0$

$x+4 > 0$

$x > -4$

$\boxed{D_f = \langle 5, \infty)}$

2

---

2. $f = \begin{cases} \dfrac{x\left(\sqrt{x^2+x+1}-1\right)}{\sqrt{x^2+4}-2}, & x \neq 0 \\ 4^a + 2^{a+1} - 1, & x = 0 \end{cases}$

$f(0) = 4^a + 2^{a+1} - 1$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x\left(\sqrt{x^2+x+1}-1\right)}{\sqrt{x^2+4}-2} = \lim_{x \to 0} \frac{x\left(x^2+x+1-1\right)\left(\sqrt{x^2+4}+2\right)}{\left(x^2+4-4\right)\left(\sqrt{x^2+x+1}+1\right)}$

$= \lim_{x \to 0} \frac{\cancel{x^2}(x+1)\left(\sqrt{x^2+4}+2\right)}{\cancel{x^2}\left(\sqrt{x^2+x+1}+1\right)} = \frac{4}{2} = 2$

$4^a + 2^{a+1} - 1 = 2$

$(2^a)^2 + 2 \cdot 2^a - 3 = 0$

$t = 2^a > 0$

$t^2 + 2t - 3 = 0$

$D = 4 + 12 = 16$

$t_1 = \frac{-2-4}{2} = -3$

$t_2 = 1$

$2^a = 1$

$\boxed{a = 0}$

1

(2) a) $\lim_{n\to\infty}\left(\frac{n^4}{n^4+n^2+3}\right)^n \overset{1^\infty}{=} \lim_{n\to\infty}\left(\frac{n^4+n^2+3-n^2-3}{n^4+n^2+3}\right)^n =$

$= \lim_{n\to\infty}\left[\left(1+\frac{1}{\frac{n^4+n^2+3}{-n^2-3}}\right)^{\frac{n^4+n^2+3}{-n^2-3}}\right]^{\frac{-n^2-3}{n^4+n^2+3}\cdot n} = e^0 = \boxed{1}$ ~~3~~ 1

2

Bo: $\lim_{n\to\infty}\frac{n^4+n^2+3}{-n^2-3} = -\infty$; $\lim_{n\to\infty}\frac{-n^2-3}{n^4+n^2+3}\cdot n = 0$ 2

Za wypisanie: $\ldots = \lim_{n\to\infty} e^{\frac{-n^2-3}{n^4+n^2+3}} = \ldots$ odejmujemy 1 pkt !

---

b) $\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{x+9}-3} \overset{\frac{0}{0}}{=} \lim_{x\to 0^+}\frac{\sqrt{x}\,(\sqrt{x+9}+3)}{x+9-9} = \lim_{x\to 0^+}\frac{\sqrt{x+9}+3}{\sqrt{x}} \overset{\frac{6}{0^+}}{=} \boxed{+\infty}$ 2 2 1

Albo z de l'Hospitala:

$\lim_{x\to 0^+}\frac{\sqrt{x}}{\sqrt{x+9}-3} \underset{H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0^+}\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{x+9}}} = \lim_{x\to 0^+}\frac{\sqrt{x+9}}{\sqrt{x}} \overset{\frac{3}{0^+}}{=} \boxed{+\infty}$

---

(3) • $f(0) = \sin^2 a + \cos a + 1$

•• $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{1}{x}\left[\ln\left(\frac{1}{x^2}+1\right) + 2\ln x\right] \overset{\infty-\infty}{=}$ 4 ~~5~~

$= \lim_{x\to 0^+}\ln(1+x^2)^{\frac{1}{x}} = \lim_{x\to 0^+}\ln\left[\left(1+\frac{1}{\frac{1}{x^2}}\right)^{\frac{1}{x^2}}\right]^x = \ln e^0 = \ln 1 = 0$

$\to e$ Bo: $\lim_{x\to 0^+}\frac{1}{x^2} = +\infty$ 1

Albo z de l'Hospitala:

$\lim_{x\to 0^+}\frac{\ln(1+x^2)}{x} \underset{H}{\overset{\frac{0}{0}}{=}} \lim_{x\to 0^+}\frac{\frac{2x}{1+x^2}}{1} = 0$

Za wypisanie: $\ldots = \lim_{x\to 0^+}\ln e^x = \ldots$ odejmujemy 1 pkt. !

••• $\sin^2 a + \cos a + 1 = 0$ 1

$1-\cos^2 a + \cos a + 1 = 0$

$\cos^2 a - \cos a - 2 = 0$

$t = \cos a \in \langle -1, 1\rangle$ 2

$t^2 - t - 2 = 0$

$\Delta = 9$

$t = \frac{1\pm 3}{2}$

$t_1 = -1$, ~~$t_2 = 2$~~

$\cos a = -1$

$\boxed{a = \pi + 2k\pi,\ k\in\mathbb{Z}}$ 2

1) $f(x) = \left| 2\arccos(x-1) - \frac{\pi}{2} \right|$

D_f: $x-1 \in \langle -1, 1 \rangle$ (1)

$-1 \le x-1 \le 1$

$0 \le x \le 2$

$\boxed{D_f = \langle 0, 2 \rangle}$ (1)

Z_f: $\arccos(x-1) \in \langle 0, \pi \rangle$

$2\arccos(x-1) \in \langle 0, 2\pi \rangle$

$2\arccos(x-1) - \frac{\pi}{2} \in \langle -\frac{\pi}{2}, \frac{3\pi}{2} \rangle$

(4) $\boxed{Z_f = \langle 0, \frac{3\pi}{2} \rangle}$ ← Nawet bez wykresu jest OK!

$\left| 2\arccos(x-1) - \frac{\pi}{2} \right| = 0$

$2\arccos(x-1) - \frac{\pi}{2} = 0$

$\arccos(x-1) = \frac{\pi}{4}$

$x-1 = \cos\frac{\pi}{4}$

$\boxed{x = 1 + \frac{\sqrt{2}}{2}}$ (4)

---

2) $f(x) = \begin{cases} \frac{x^2}{\sqrt{x^2+1} - 1}, & x \neq 0 \\ 4^a - 2^a, & x = 0 \end{cases}$

$f(0) = 4^a - 2^a$ (2)

$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2}{\sqrt{x^2+1} - 1} \overset{\left[\frac{0}{0}\right]}{=} \lim_{x\to 0} \frac{x^2(\sqrt{x^2+1}+1)}{x^2+1-1} = 2$ (4)

z reguły de l'Hospitala też OK!

$4^a - 2^a = 2$

$4^a - 2^a - 2 = 0$

$t = 2^a > 0$

$t^2 - t - 2 = 0$

$\Delta = 1 + 8 = 9$

$t_1 = \frac{1-3}{2} = -1, \quad t_2 = \frac{1+3}{2} = 2$ (2)

$2^a = 2$

$\boxed{a = 1}$ (4)

---

4) $f(x) = \frac{\sqrt{x}}{\ln x}$ $\qquad D = (0, \infty) \setminus \{1\}$ (1)

$f'(x) = \frac{\frac{1}{2\sqrt{x}}\ln x - \sqrt{x}\cdot\frac{1}{x}}{\ln^2 x} = \frac{\frac{1}{\sqrt{x}}\left(\frac{1}{2}\ln x - 1\right)}{\ln^2 x}$ (2)

$\frac{1}{2}\ln x - 1 = 0 \quad / \cdot 2$

$\ln x = 2$ (2)

$x = e^2$

$x \in (0, 1)$, $f'(x) < 0$, $f \searrow$

$x \in (1, e^2)$, $f'(x) < 0$, $f \searrow$

$x \in (e^2, \infty)$, $f'(x) > 0$, $f \nearrow$

$x_0 = e^2$ min lok. (5)

Jeśli jest ciężki błąd w pochodnej, to za całe zadanie max = 2 pkt.!

Bez uzasadnienia znaków: − 3 pkt. !!!

3

(4) $f(x) = \frac{1}{x^2 - x}$

$x^2 - x \neq 0$

$x(x-1) \neq 0$

$x \neq 0 \wedge x \neq 1$

$D = \mathbb{R} \setminus \{0, 1\}$ (1)

$f'(x) = \frac{-2x+1}{(x^2-x)^2}$

$f''(x) = \frac{-2(x^2-x)^2 - (-2x+1)\,2(x^2-x)(2x-1)}{(x^2-x)^{4\,3}} = \frac{-2\left(x^2 - x - (4x^2 - 4x + 1)\right)}{(x^2-x)^3} =$

$= \frac{-2(-3x^2+3x-1)}{x^3(x-1)^3} = \frac{2(3x^2-3x+1)}{x^3(x-1)^3}$ $>0\ \forall x$ (2)

$3x^2 - 3x + 1 = 0$

$\Delta = 9 - 12 = -3 < 0$

$\boxed{x \in \emptyset \quad \text{Brak PP}}$ (0)

$x \in (-\infty, 0)$; $f''(x) > 0$; $f \cup$

$x \in (0, 1)$; $f''(x) < 0$; $f \cap$

$x \in (1, \infty)$; $f''(x) > 0$; $f \cup$

(0)

Bez uzasadnienia znaku – 3 pkt. !!!

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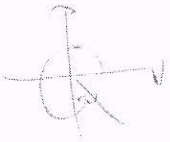
(5)

a) $(1-i)^{2013}$

$z = 1 - i$

$|z| = \sqrt{2}$



$\varphi = \frac{7\pi}{4}$

(2)

$(1-i)^{2013} = 2^{1006}\sqrt{2}\left(\cos \frac{7\pi}{4}\cdot 2013 + i \sin \frac{7\pi}{4}\cdot 2013\right) =$

$= 2^{1006}\sqrt{2}\left[\cos\left(3522\pi + \frac{3\pi}{4}\right) + i \sin\left(3522\pi + \frac{3\pi}{4}\right)\right] =$

$= 2^{1006}\sqrt{2}\left[\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}\right] = 2^{1006}\sqrt{2}\left(-\cos\frac{\pi}{4} + i \sin \frac{\pi}{4}\right) =$

$= 2^{1006}\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2}\right) = 2^{1006}(-1+i) = \boxed{-2^{1006} + i\, 2^{1006}}$

(3)

b) $\operatorname{Re}(2z-1) \leq \operatorname{Im}(z^2)$

$z = x + iy$, $z^2 = x^2 - y^2 + 2xyi$

$2z - 1 = 2x + 2iy - 1$

(2) $2x - 1 \leq 2xy$

3° $x = 0$

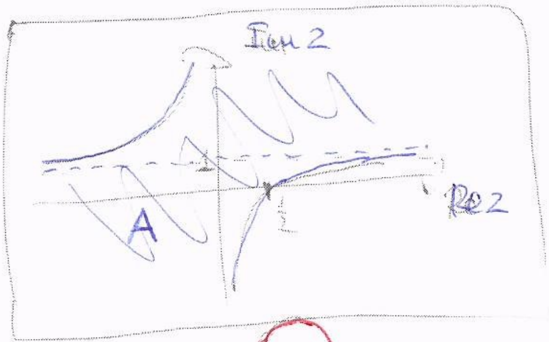
$\Rightarrow y \in \mathbb{R}$

1° $x > 0$

$y \geq 1 - \frac{1}{2x}$

2° $x < 0$

$y \leq 1 - \frac{1}{2x}$



(3)

(6) a) Def. $f: \mathbb{R} \supset X \to \mathbb{R}$, $x_0 \in X$ – pkt. skupienia zb. X
Pierwszą pochodną funkcji f w punkcie x_0 nazywamy liczbę (2)
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}.$$

b) Def. $f: \mathbb{R} \supset X \to \mathbb{R}$, $A \subset X$
Funkcja f jest malejąca na zbiorze A, gdy (2)
$$\forall x_1, x_2 \in A \quad x_1 < x_2 \Rightarrow f(x_1) > f(x_2).$$

c) Def. $e := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$ (2)

Za napisanie tylko, że $e \approx 2.71$ – 0 pkt!

Jeśli w a) i b) brak założeń, to po 1 pkt.!