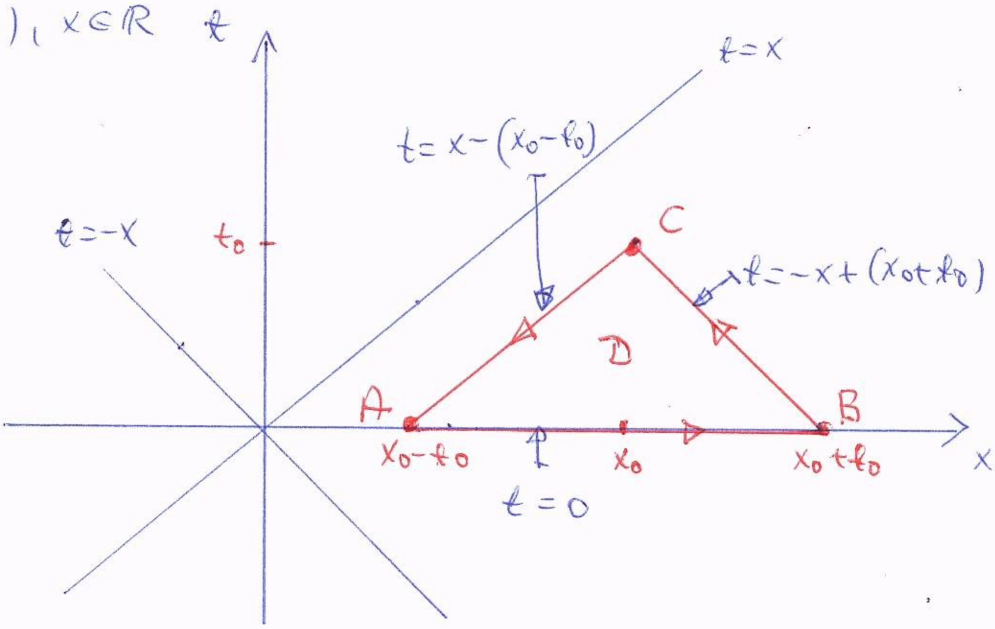
(23) $$\begin{cases} u_{tt} = u_{xx} + f(x,t), & x\in\mathbb{R},\ t>0 \\ u(x,0)=\varphi(x), & x\in\mathbb{R} \\ u_t(x,0)=\psi(x), & x\in\mathbb{R}\end{cases}$$



$t=x$, $t=-x$: charakterystyki równania $u_{tt}=u_{xx}$

(*) $$u(x,t)=\frac{1}{2}\left[\varphi(x-t)+\varphi(x+t)\right]+\frac{1}{2}\int_{x-t}^{x+t}\psi(y)\,dy+\frac{1}{2}\int_{x-(t-s)}^{x+(t-s)} f(y,s)\,dy\,ds$$

(x_0,t_0) – dowolnie ustalony

$$\iint_D \big(\underbrace{u_{xx}-u_{tt}}_{-f(x,t)}\big)\,dx\,dt \underset{\text{Tw. Greena}}{=} \int_{\partial D} \underbrace{u_t}_{P(x,t)}\,dx + \underbrace{u_x}_{Q(x,t)}\,dt =$$

$(P(x,t),Q(x,t))$, $Q_x=u_{xx}$, $P_t=u_{tt}$

$$= \int_A^B u_t\,dx + u_x\,\underset{dt=0}{dt} + \int_B^C u_t\,dx + u_x\,\underset{dt=-dx}{dt} + \int_C^A u_t\,dx + u_x\,\underset{dt=dx}{dt} =$$

$$= \int_A^B u_t\,dx - \int_B^C u_x\,dx + u_t\,dt + \int_C^A u_x\,dx + u_t\,dt =$$

bo (u_x,u_t) jest potencjalne, bo $u_{tx}=u_{xt}$. Potencjałem jest u.

$$= \int_A^B u_t\,dx - \left[u(C)-u(B)\right] + \left[u(A)-u(C)\right]$$

A stąd

$$-\iint_D f(x,t)\,dx\,dt = \int_A^B u_t\,dx + \left[u(A)+u(B)\right] - 2u(C)$$

$$u(C) = \frac{1}{2}\left[u(A) + u(B)\right] + \frac{1}{2}\int_A^B u_t \, dx + \frac{1}{2}\iint_D f(x,t)\, dx\, dt$$

$$u(x_0, t_0) = \frac{1}{2}\left[\varphi(x_0 - t_0) + \varphi(x_0 + t_0)\right] + \frac{1}{2}\int_{x_0 - t_0}^{x_0 + t_0} \psi(x)\, dx + \frac{1}{2}\int_0^{t_0} \int_{t + (x_0 - t_0)}^{-t + (x_0 + t_0)} f(x,t)\, dx\, dt$$

Z dowolności (x_0, t_0) przechodzimy na (x, t) i mamy

$$u(x,t) = \frac{1}{2}\left[\varphi(x-t) + \varphi(x+t)\right] + \frac{1}{2}\int_{x-t}^{x+t} \psi(y)\, dy + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(y,s)\, dy\, ds$$