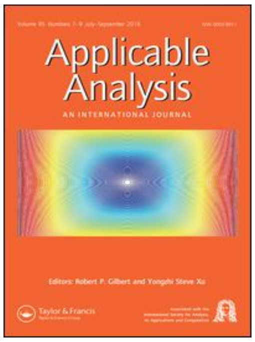

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On local weak solutions to Nernst–Planck–Poisson system

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ABSTRACT
We study the one-dimensional nonlinear Nernst–Planck–Poisson system of partial differential equations with the class of nonlinear boundary conditions which cover the Chang–Jaffé conditions. The system describes certain physical and biological processes, for example ionic diffusion in porous media, electrochemical and biological membranes, as well as electrons and holes transport in semiconductors. The considered boundary conditions allow the physical system to be not only closed but also open. Theorems on existence, uniqueness, and nonnegativity of local weak solutions are proved. The main tool used in the proof of the existence result is the Schauder–Tychonoff fixed point theorem.

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1. Introduction and preliminaries

Transport of charged particles (ions, electrons, holes, or colloids) plays an important role in many disciplines of science. In electrical engineering the motion of electrons and holes in semiconductors in the electric field is essential for the functioning of modern electronic components such as diodes and transistors. In electrochemistry the motion of inorganic ions (for example Na^+, K^+, Ca^{2+}, Cl^-) in electrolytes and how this motion is influenced when the potential is applied to electrodes is the basis for modeling of very diverse phenomena ranging from industrial electrolysis processes to the construction and prediction of properties of important class of chemical sensors – the potentiometric ion sensors (ion selective electrodes, ISE). Especially the last example is very important because ISEs have prominent features (small size, portability, low-energy consumption, and low cost) which make them attractive for practical applications (in electroanalysis and medicine). ISEs based on polymeric membranes containing neutral or charged carriers are commercially produced for the quantitative determination of a large number of inorganic and organic ions. Another important area of ion transport applications is concerned with the mechanisms of deterioration of reinforced concrete and corrosion of rebars. This is a wide topic, but again one of the basic processes involved here is the transport of ions (in particular chlorides) and molecules (e.g. CO_2) through the concrete to the reinforcement. In the above applications (see for example Figure 1), we can study without loss of physical generality one-dimensional models.

Moreover, transport of ions and molecules (or macro-molecules) is fundamental for many mechanisms in biological systems. The prominent example here is the movement of ions through the

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biological membranes via the ion channels. It is well known that the identification of potassium and sodium currents in the behavior of a nerve system was one of the milestones in the history of electrophysiology which earned their discoverers the Nobel Prize in Physiology and Medicine. But, these models are generally three-dimensional because of additional sources of potential on the boundary of ion channels.[1,2]

The driving force of ionic motion involves at least two parts: the gradient of concentration (diffusion part) and the gradient of electric potential (migration part), that is the electric field generated by electric charges. It is important to underline that electric field component comes here via two mechanisms, namely as the externally applied potential (e.g. by the voltage difference between electrodes immersed in the solution) and as electrostatic interactions between the ions present in the system. The mathematical equations that govern ion transport in the contexts described above are well established.[3–5] The model is expressed by a system of partial differential equations. The formula for the Nernst–Planck flux $\mathbf{J}_i \in \mathbb{R}^3$ of the ith species, $i = 1, \ldots, m$ is a sum of two contributions

$$\mathbf{J}_i = -D_i \nabla c_i + u_i c_i \mathbf{E}, \tag{1.1}$$

where $c_i(t, x, y, z)$ is the concentration of i-th species as a function of time t and location $(x, y, z) \in \Omega \subset \mathbb{R}^3$, D_i – the diffusion coefficient, u_i – the mobility of the ion, and $\mathbf{E} \in \mathbb{R}^3$ – the electric field.[6, Chapter 4, p.138] To associate the flux with the concentration the obvious choice is the law of mass conservation (also called the continuity equation)

$$\frac{\partial c_i}{\partial t} = -\operatorname{div} \mathbf{J}_i \tag{1.2}$$

[7, Chapter 12, p.347]. The charge density ϱ is related to the electric field $\mathbf{E}$ by Gauss' law

$$\operatorname{div} \mathbf{E} = \frac{\varrho}{\varepsilon_r \varepsilon_0}, \tag{1.3}$$

where ε_0 is the permittivity of free space (the electric constant) and ε_r is the relative permittivity of the medium.[8, Chapters 3 and 6, Section 6.5.1] In a usual chemical context, the concentration is measured in moles per volume so the expression of charge density by concentration shall contain the Faraday constant F, $\varrho = F \sum_{j=1}^{m} z_j c_j$, where z_j is the charge number of the jth ion, and the sum is over all species taken into account in the model. Now Equation (1.3) becomes

$$\operatorname{div} \mathbf{E} = \frac{F}{\varepsilon_r \varepsilon_0} \sum_{j=1}^{m} z_j c_j. \tag{1.4}$$

In most applications the electric field is replaced with the electric potential φ according to the relation $\mathbf{E} = -\nabla \varphi$ which simply states that the electric field is a conservative field (this is true in the absence of magnetic fields).[8, Chapter 3, Section 3.3] Additionally, two material constants (the diffusion coefficient and mobility) are connected by the Einstein–Smoluchowski relation $u_i = (z_i F/RT) D_i$, where R is the gas constant and T is the absolute temperature of the medium in Kelvin temperature scale.[9, Chapter 18] Now Equations (1.1) and (1.4) put together give

$$\begin{cases} \frac{\partial c_i}{\partial t} = D_i \left(\Delta c_i + z_i \frac{F}{RT} \nabla \cdot (c_i \nabla \varphi) \right), \quad i = 1, \ldots, m, \\ \Delta \varphi = -\frac{F}{\varepsilon_r \varepsilon_0} \sum_{j=1}^{m} z_j c_j. \end{cases} \tag{1.5}$$

This system of equations constitutes a mathematical framework for the deterministic modeling of electrodiffusion in continuum media approximation.[10,11] In the literature it is known as the Nernst–Planck–Poisson (NPP) or Poisson–Nernst–Planck (PNP) system (Figure 1).

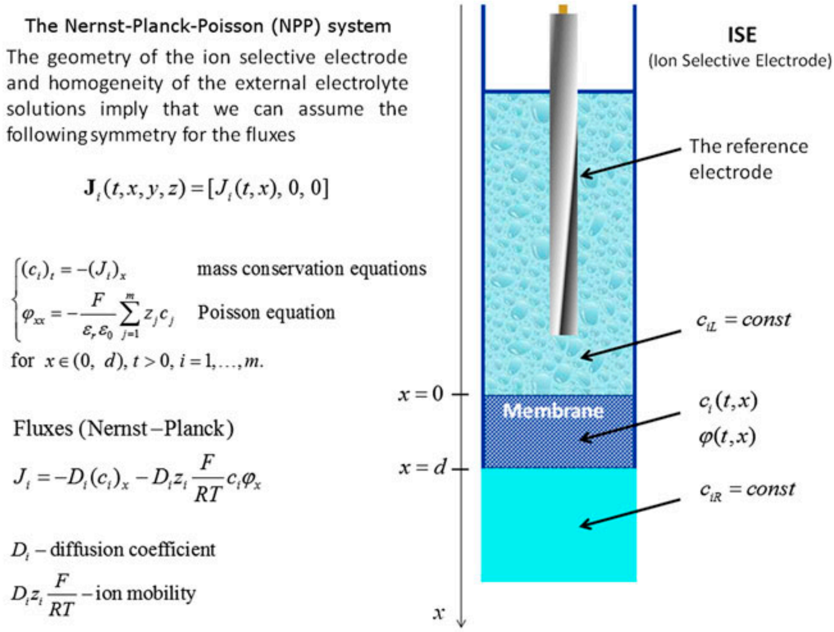

Figure 1. NPP system.

We specify the boundary conditions. Here we do not just use "any reasonable" conditions that would satisfy the mathematical necessity but rather pursue for conditions that are applicable for modeling of real systems of practical importance that we alluded to above and which are consistent with reasonable experimental situations.[3,12] From this perspective, the so-called Chang–Jaffé (CJ) boundary conditions are particularly important (Figure 2). The basic idea is that the flux on the boundary (or more precisely, the normal component of the flux) is proportional to the weighted difference between concentration inside and outside of the region where the process takes place

$$\mathbf{n}(x,y,z)\cdot\mathbf{J}_i(t,x,y,z)=-k_{i,f}c_{i,out}+k_{i,b}c_i(t,x,y,z) \tag{1.6}$$

for $(x,y,z)\in\partial\Omega$, where $\mathbf{n}\in\mathbb{R}^3$ is the normal vector to $\partial\Omega$, $k_{i,f}$, $k_{i,b}$ are the material constants (the so called heterogeneous rate constants) which describe the permeability of the boundary, and $c_{i,out}$ is the concentration of the ith species outside Ω – it is assumed that $c_{i,out}$ is constant.

In the case of one dimension these types of boundary conditions were first used by Chang and Jaffé in 1951 in their paper on the polarization in electrolytic solutions.[13] Since then the CJ boundary conditions have been used extensively in the field of potentiometric sensors modeling as proposed by Brumleve and Buck in their seminal paper [12] (in 1D)

$$\begin{cases} J_i(t,0)=k_{iL,f}c_{iL}-k_{iL,b}c_i(t,0),\\ J_i(t,d)=-k_{iR,f}c_{iR}+k_{iR,b}c_i(t,d),\end{cases} \tag{1.7}$$

where the domain $\Omega=(0,d)\subset\mathbb{R}$ is an interval, the subscript L refers to the "left end" ($x=0$) and the subscript R to the "right end" ($x=d$), respectively.

Another boundary conditions that are nonlinear and play an important role in electrolysis with two species ($m=2$) have the form

$$\mathbf{n}(x,y,z)\cdot\mathbf{J}_1(t,x,y,z)=k(t,x,y,z)c_1(t,x,y,z)e^{-a\,\mathbf{n}(x,y,z)\cdot\nabla\varphi(t,x,y,z)}, \tag{1.8}$$

$$\mathbf{n}(x,y,z)\cdot\mathbf{J}_2(t,x,y,z)=-k(t,x,y,z)c_2(t,x,y,z)e^{b\,\mathbf{n}(x,y,z)\cdot\nabla\varphi(t,x,y,z)}, \tag{1.9}$$

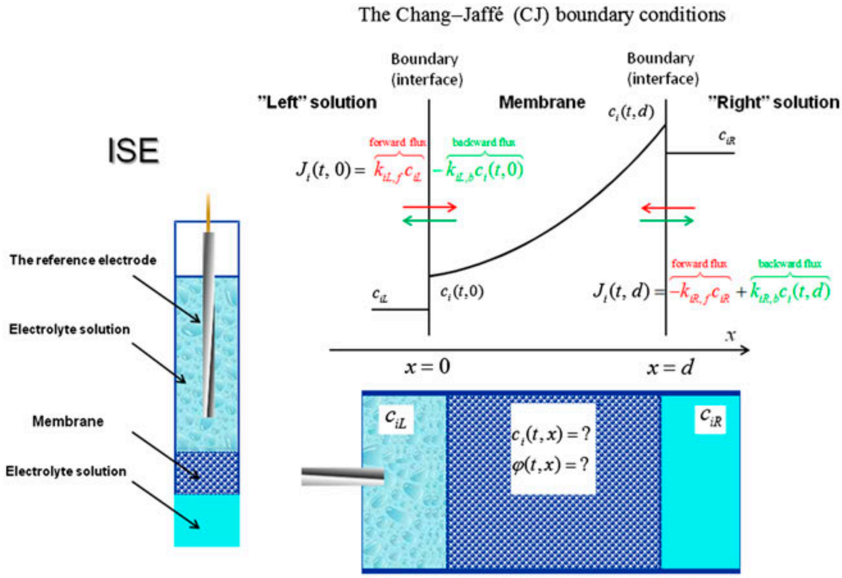

Figure 2. CJ boundary conditions.

where $k > 0$ is the standard rate function and $a, b > 0$ are the transfer coefficients. They are known in the electrode kinetics theory as the Butler–Volmer boundary conditions.[6, Chapter 3]

Although the formulation is straightforward, the numerical solution is quite a challenging task. One reason for this difficulty is that the values of parameters that appear in (1.5), the permittivity of free space ε_0 is of the order of 10^{-12} and Faraday constant is of the order of 10^5, thus the coefficient in Poisson equation is of the order of 10^{17}. This leads to the phenomenon of boundary layers and requires special discretization meshes.[14]

The mathematical theory of the NPP system is quite extensive but still it lacks the main results (local and global existence and uniqueness) in the case of relevant boundary conditions. Krzywicki and Nadzieja [15] consider system (1.5) with one component ($m = 1$), while Biler et al. [16] with two components ($m = 2$) and prove the global existence and uniqueness, and the convergence to the steady-state solution as time advances to infinity. However, the boundary conditions they use are simpler than (1.6), (1.7) and have the form of null fluxes on the boundary ($\mathbf{J}_i = 0$ on $\partial\Omega$). This has a simple physical interpretation meaning the closed system. But as we motivated above, real electrodiffusion applications of practical importance are almost always open systems which interact with surroundings through boundary fluxes. There is a handful of mathematical papers that address the NPP system but in the steady-state variant.[17–21] Because we are here interested only in the time-dependent system we shall not go into details but stress the fact that none of these papers uses the CJ boundary conditions or its extensions.

There exists vast literature on existence and regularity of solutions for both parabolic and elliptic problems, cf., e.g. [22–25]. The aim of this paper is to give theorems on existence, uniqueness, and nonnegativity of local weak solutions to the one-dimensional nonlinear parabolic-elliptic NPP (1.5) system with the initial conditions and the class of nonlinear boundary conditions that cover the CJ conditions. In the existence proof we use the Schauder–Tychonoff fixed point theorem instead of the Schauder fixed point theorem as in [16,26] because of compact embeddings in the boundary spaces. We expect that it is possible to prove the existence of global weak solutions by the entropy method with the use of the logarithmic Lyapunov function (see [16,26]), but unfortunately, to this day, our result is not complete. We expect it to be the topic of our further research.

The paper is organized in the following way. In Section 2, the initial-boundary differential problem is formulated, and in Section 3 its weak version is given together with the assumptions that will be used in the further part. Sections 4–6 deal with the existence, uniqueness, and nonnegativity of local weak solutions of the problem studied, respectively.

2. Problem formulation

Denote for simplicity $\Omega = (0,1)$. The study of the general case $\Omega = (0,d)$ is the same. Let functions $u_0, v_0 : \Omega \to \mathbb{R}$, $f_i, g_i : [0,T] \times \mathbb{R} \to \mathbb{R}$, $h_i : [0,T] \to \mathbb{R}$ and constants $\alpha_i, \beta_i, \lambda > 0$ for $i = 1,2$ be given, where $T > 0$. We consider the nonlinear parabolic–elliptic system of equations in one-dimensional case

$$\begin{cases} u_t = \alpha_1 u_{xx} - \alpha_2 (u\varphi_x)_x, \\ v_t = \beta_1 v_{xx} + \beta_2 (v\varphi_x)_x, \\ \varphi_{xx} = \lambda(u - v) \end{cases} \tag{2.1}$$

for $(t,x) \in [0,T] \times \Omega$ with the initial conditions

$$u(0,x) = u_0(x) \quad \text{and} \quad v(0,x) = v_0(x) \tag{2.2}$$

for $x \in \Omega$ and the nonlinear boundary conditions

$$\begin{cases} \alpha_1 u_x(t,0) - \alpha_2 u(t,0)\varphi_x(t,0) = f_1(t,u(t,0)), \\ \alpha_1 u_x(t,1) - \alpha_2 u(t,1)\varphi_x(t,1) = f_2(t,u(t,1)), \\ \beta_1 v_x(t,0) + \beta_2 v(t,0)\varphi_x(t,0) = g_1(t,v(t,0)), \\ \beta_1 v_x(t,1) + \beta_2 v(t,1)\varphi_x(t,1) = g_2(t,v(t,1)), \\ \varphi(t,0) = h_1(t), \\ \varphi(t,1) = h_2(t) \end{cases} \tag{2.3}$$

for $t \in [0,T]$. We study two evolution equations for simplicity only. If we put $m = 2$, $z_1 = -z$, $z_2 = z$, $(z \in \mathbb{N})$, $u = c_1$, $v = c_2$, $\alpha_1 = D_1$, $\alpha_2 = -D_1 z_1 \frac{F}{RT}$, $\beta_1 = D_2$, $\beta_2 = D_2 z_2 \frac{F}{RT}$, $\lambda = \frac{F}{\varepsilon_r \varepsilon_0}$, then system (2.1) is a special case of (1.5).

The NPP system (2.1) and the boundary conditions (2.3) can be written in a more lucid physical notation as follows

$$\begin{cases} u_t = -(J_1)_x, \\ v_t = -(J_2)_x, \\ \varphi_{xx} = \lambda(u - v), \end{cases} \tag{2.4}$$

$$\begin{cases} J_1(t,0) = -f_1(t,u(t,0)), \\ J_1(t,1) = -f_2(t,u(t,1)), \\ J_2(t,0) = -g_1(t,v(t,0)), \\ J_2(t,1) = -g_2(t,v(t,1)), \\ \varphi(t,0) = h_1(t), \\ \varphi(t,1) = h_2(t), \end{cases} \tag{2.5}$$

where J_1, J_2 are the Nernst–Planck fluxes defined by the formulas

$$J_1 = -\alpha_1 u_x + \alpha_2 u\varphi_x \quad \text{and} \quad J_2 = -\beta_1 v_x - \beta_2 v\varphi_x. \tag{2.6}$$

The boundary conditions (2.5) (see also (2.3)) cover as a special case the well-known CJ conditions

$$\begin{cases} J_1(t,0) = a_{11} - a_{21}u(t,0), \\ J_1(t,1) = -a_{12} + a_{22}u(t,1), \\ J_2(t,0) = b_{11} - b_{21}v(t,0), \\ J_2(t,1) = -b_{12} + b_{22}v(t,1), \\ \varphi(t,0) = h_1(t), \\ \varphi(t,1) = h_2(t), \end{cases} \tag{2.7}$$

where $a_{ij} > 0$, $b_{ij} > 0$, $i,j = 1,2$, are given constants.

It follows from the initial-boundary conditions that a suitable physical system can be closed or open.

Remark 2.1: If $h_1(t) \not\equiv 0$ or $h_2(t) \not\equiv 0$, then the substitution $\varphi(t,x) = (h_2(t) - h_1(t))x + h_1(t) + \psi(t,x)$ transforms problem (2.1)–(2.3) to the equivalent one with the homogeneous boundary conditions on ψ. To avoid cumbersome computations we will consider only the case $h_1(t) \equiv h_2(t) \equiv 0$ in the sequel, however, the corresponding results can be easily generalized to the case $h_1, h_2 \in L^2(0,T)$. We add that from a physical point of view the charge numbers of ions such that $z_1 < 0$ and $z_2 > 0$ are interesting only, and our results can be extended immediately to this general case.

3. Assumptions and a weak formulation

Denote $V = H^1(\Omega)$ and $H = L^2(\Omega)$. Then $V \subset H \subset V^*$ constitute an evolution triple with the embeddings being dense, continuous, and compact. By H^+ we denote the cone of nonnegative functions in H, that is

$$H^+ = \{u \in H : \ u(x) \geq 0 \quad \text{a.e. in} \quad \Omega\}.$$

In the paper by $C > 0$ we will always denote a generic constant dependent only on the problem data.

We assume the following conditions on u_0, v_0 and f_i, g_i.

Assumption H

(H_0) $u_0 \in H$ and $v_0 \in H$.

(H_1) f_i, g_i, $i = 1,2$, satisfy the Caratheodory conditions: $f_i(\cdot,u)$ and $g_i(\cdot,u)$ are measurable, and $f_i(t,\cdot)$ and $g_i(t,\cdot)$ are continuous.

(H_2) The following growth conditions hold

$$|f_i(t,u)| \leq a_{1i} + a_{2i}|u| \quad \text{and} \quad |g_i(t,u)| \leq b_{1i} + b_{2i}|u|,$$

for a.e. $t \in (0,T)$ and all $u \in \mathbb{R}$, with the constants $a_{1i}, a_{2i}, b_{1i}, b_{2i} \geq 0$, $i = 1,2$.

(H_3) The following one sided Lipschitz conditions hold

$$\begin{aligned} f_1(t,u_1) - f_1(t,u_2) &\geq -L_{f_1}(u_1 - u_2),\\ g_1(t,u_1) - g_1(t,u_2) &\geq -L_{g_1}(u_1 - u_2),\\ f_2(t,u_1) - f_2(t,u_2) &\leq L_{f_2}(u_1 - u_2),\\ g_2(t,u_1) - g_2(t,u_2) &\leq L_{g_2}(u_1 - u_2), \end{aligned}$$

for a.e. $t \in (0,T)$ and all $u_1, u_2 \in \mathbb{R}$, $u_1 \geq u_2$, with the constants $L_{f_i}, L_{g_i} \geq 0$, $i = 1,2$.

Assumption H$^+$

(H_0^+) $u_0 \in H^+$ and $v_0 \in H^+$.

(H_1^+) For $u < 0$ and a.e. $t \in (0,T)$

$$\begin{aligned} f_1(t,u) \leq 0 \quad &\text{and} \quad g_1(t,u) \leq 0,\\ f_2(t,u) \geq 0 \quad &\text{and} \quad g_2(t,u) \geq 0. \end{aligned}$$

Remark 3.1: Note that, as the solutions u, v of the considered problem which will be substituted in the place of the second variable in f_i, g_i are supposed to be nonnegative, it is possible to modify arbitrarily the functions f_i, g_i for the negative values of the second argument. For example we can take

$$f_i(t,u) = f_i(t,0) \text{ and } g_i(t,u) = g_i(t,0) \text{ for } u < 0 \text{ and a.e. } t \in (0,T), \ i = 1,2.$$

Hence, it is enough to assume, instead of (H_1^+), that $f_1(t,0) \leq 0$, $g_1(t,0) \leq 0$ and $f_2(t,0) \geq 0$, $g_2(t,0) \geq 0$.

Remark 3.2: It follows from Remark 3.1 that Assumptions **H** and $\mathbf{H}^+$ are fulfilled for the CJ boundary conditions (2.7). In this case $f_1(t,u) = -a_{11} + a_{21}u$, $f_2(t,u) = a_{12} - a_{22}u$, $g_1(t,u) = -b_{11} + b_{21}u$, $g_2(t,u) = b_{12} - b_{22}u$ for $t \in [0,T]$, $u \in \mathbb{R}$.

The original initial-boundary value problem (2.1)–(2.3) has the following weak version.

Problem PE. Find $u, v \in L^2(0,T;\, V)$ and $\varphi \in L^2(0,T;\, H^3(\Omega) \cap H_0^1(\Omega))$ such that $u_t, v_t \in L^2(0,T;\, V^*)$ and for a.e. $t \in (0,T)$

$$\langle u_t, \eta\rangle_{V^*\times V} + \int_\Omega (\alpha_1 u_x - \alpha_2 u\varphi_x)\eta_x \mathrm{d}x \tag{3.1}$$
$$= f_2(t,u(t,1))\eta(1) - f_1(t,u(t,0))\eta(0) \quad \text{for each} \quad \eta \in V,$$

$$\langle v_t, \zeta\rangle_{V^*\times V} + \int_\Omega (\beta_1 v_x + \beta_2 v\varphi_x)\zeta_x \mathrm{d}x \tag{3.2}$$
$$= g_2(t,v(t,1))\zeta(1) - g_1(t,v(t,0))\zeta(0) \quad \text{for each} \quad \zeta \in V,$$

$$\int_\Omega \varphi_x\xi_x \mathrm{d}x + \lambda \int_\Omega (u-v)\xi \mathrm{d}x = 0 \quad \text{for each} \quad \xi \in H_0^1(\Omega), \tag{3.3}$$

and the initial conditions (2.2) hold.

4. Existence of local weak solutions

We will use the following version of the Schauder–Tychonoff fixed point theorem which is a simple consequence of [27, Theorem 1].

Theorem 4.1: *Let X be a reflexive Banach space and let $C \subset X$ be a closed, bounded, convex and nonempty set. If the function $\Lambda : C \to C$ is sequentially weakly continuous, then it must have a fixed point.*

To study the existence of a weak solution for Problem **PE**, we split it into two auxiliary problems, an elliptic one, and a parabolic one.

Problem E. Given $w, z \in L^2(0,T;\, V)$ find $\psi \in L^2(0,T;\, H^3(\Omega) \cap H_0^1(\Omega))$ such that for a.e. $t \in (0,T)$

$$\int_\Omega \psi_x\xi_x \mathrm{d}x + \lambda \int_\Omega (w-z)\xi \mathrm{d}x = 0 \quad \text{for each} \quad \xi \in H_0^1(\Omega). \tag{4.1}$$

Problem P. Given $w, z \in L^2(0,T;\, V)$ and $\psi \in L^2(0,T;\, H^3(\Omega) \cap H_0^1(\Omega))$ find $u, v \in L^2(0,T;\, V)$ such that $u_t, v_t \in L^2(0,T;\, V^*)$ and for a.e. $t \in (0,T)$

$$\langle u_t, \eta\rangle_{V^*\times V} + \int_\Omega (\alpha_1 u_x - \alpha_2 u\psi_x)\eta_x \mathrm{d}x \tag{4.2}$$
$$= f_2(t,w(t,1))\eta(1) - f_1(t,w(t,0))\eta(0) \quad \text{for each} \quad \eta \in V,$$

$$\langle v_t, \zeta\rangle_{V^*\times V} + \int_\Omega (\beta_1 v_x + \beta_2 v\psi_x)\zeta_x \mathrm{d}x \tag{4.3}$$
$$= g_2(t,z(t,1))\zeta(1) - g_1(t,z(t,0))\zeta(0) \quad \text{for each} \quad \zeta \in V,$$

and the initial conditions (2.2) hold.

Define for a fixed $T > 0$ the space of vector valued functions

$$X_T = \{(u,v) \in L^2(0,T;\, V) \times L^2(0,T;\, V) \,:\, u_t, v_t \in L^2(0,T;\, V^*)\} \tag{4.4}$$

normed by

$$\|(u,v)\|_{X_T} = \|u\|_{L^2(0,T;V)} + \|v\|_{L^2(0,T;V)} + \|u_t\|_{L^2(0,T;V^*)} + \|v_t\|_{L^2(0,T;V^*)},$$
$$\|u\|^2_{L^2(0,T;V)} = \int_0^T \|u(t)\|^2_V \, \mathrm{d}t, \quad \|u_t\|^2_{L^2(0,T;V^*)} = \int_0^T \|u_t(t)\|^2_{V^*} \, \mathrm{d}t.$$

We will use two topologies in this space, namely the strong topology and the weak topology. We define the set

$$B = B(T, Q_0, Q_1, Q_2, R_0, R_1, R_2)$$

parameterized by the time $T > 0$ and the constants $Q_0, Q_1, Q_2, R_0, R_1, R_2 > 0$

$$\begin{aligned} B = \{(w,z) \in X_T \ : \ & \|w\|^2_{L^2(0,T;H)} \le Q_0, \ \|w_x\|^2_{L^2(0,T;H)} \le Q_1, \\ & \|z\|^2_{L^2(0,T;H)} \le R_0, \ \|z_x\|^2_{L^2(0,T;H)} \le R_1, \\ & \|w_t\|^2_{L^2(0,T;V^*)} \le Q_2, \ \|z_t\|^2_{L^2(0,T;V^*)} \le R_2\}. \end{aligned} \tag{4.5}$$

Note that the set B is convex and strongly closed in X_T, so it is also weakly closed. Since it is also strongly bounded, it follows that it is weakly compact. Define the operator

$$\Lambda_E : B \to L^2(0,T; H^3(\Omega) \cap H^1_0(\Omega)),$$

which maps any pair $(w,z) \in B$ to the unique solution $\psi \in L^2(0,T; H^3(\Omega) \cap H^1_0(\Omega))$ of Problem **E**, and the operator

$$\Lambda_P : B \times L^2(0,T; H^3(\Omega) \cap H^1_0(\Omega)) \to X_T,$$

which maps any pair $(w,z) \in B$ and function $\psi \in L^2(0,T; H^3(\Omega) \cap H^1_0(\Omega))$ to the unique solution $(u,v) \in X_T$ of Problem **P**. Composing the two operators we define

$$\Lambda : B \to X_T, \ \ \Lambda(w,z) = \Lambda_P(w,z,\Lambda_E(w,z)).$$

Obviously, (u,v,φ) is a solution of Problem **PE** if and only if (u,v) is a fixed point of Λ, and $\varphi = \Lambda_E(u,v)$. We establish several lemmas on the properties of Λ_E, Λ_P, Λ which imply the correctness of their definitions and will be useful in the local existence result.

Lemma 4.1: *Problem **E** has the unique solution, and the following estimate holds*

$$\|\psi(t)\|_{H^2(\Omega)} \le C\lambda \|w(t) - z(t)\|_H \quad \textit{for a.e.} \quad t \in (0,T) \quad \textit{with} \quad C > 0. \tag{4.6}$$

Proof: The existence for a.e. $t \in (0,T)$ of the unique weak solution $\psi = \psi(t,\cdot) \in H^1_0(\Omega)$ follows from [22, Chapter 6.2, Theorem 3]. Then, using [22, Chapter 6.3, Theorem 5] we have for a.e. $t \in (0,T)$ that $\psi(t) \in H^3(\Omega)$ and

$$\|\psi(t)\|_{H^3(\Omega)} \le C\|\lambda(w(t) - z(t))\|_V \quad \text{for a.e.} \quad t \in (0,T). \tag{4.7}$$

Note that ψ depends measurably on t, because of measurability of w, z on t, linearity of (4.1) and inequality (4.7). Hence,

$$\int_0^T \|\psi(t)\|^2_{H^3(\Omega)} \, \mathrm{d}t \le C \int_0^T \|\lambda(w(t) - z(t))\|^2_V \, \mathrm{d}t < \infty \tag{4.8}$$

and in consequence $\psi \in L^2(0,T; H^3(\Omega) \cap H^1_0(\Omega))$. The estimate (4.6) follows from [22, Chapter 6.3, Theorem 4]. □

Remark 4.1: It follows from [22, Chapter 5.6, Theorem 6] and [28, Corollary 26] that $H^s(\Omega) \subset C(\bar{\Omega})$ for $s \in (\frac{1}{2}, 1]$, with continuous embeddings.

Lemma 4.2: *If assumptions* (H_0), (H_1), (H_2) *hold, then Problem* ***P*** *has the unique solution.*

Proof: Note that $\psi_x \in L^2(0,T;H^2(\Omega))$ and $f_i(\cdot, w(\cdot,0))$, $f_i(\cdot, w(\cdot,1))$, $g_i(\cdot, z(\cdot,0))$, $g_i(\cdot, z(\cdot,1)) \in L^2(0,T)$, $i = 1,2$ are all given functions. The proof of existence and uniqueness of the solutions for the linear problem is standard and can be done for example by the Galerkin method. The proof that uses the Galerkin method follows the steps of the proof of [23, Theorem 11.7]. □

Lemma 4.3: *If assumptions* (H_0), (H_1), (H_2) *are fulfilled, then there exists* $T > 0$ *such that* $\Lambda : B(T, Q_0, Q_1, Q_2, R_0, R_1, R_2) \to B(T, Q_0, Q_1, Q_2, R_0, R_1, R_2)$ *for certain* $Q_0, Q_1, Q_2, R_0, R_1, R_2 > 0$.

Proof: Let $(w,z) \in B(T, Q_0, Q_1, Q_2, R_0, R_1, R_2)$, where Q_0, R_0 are arbitrary, and the choice of T, Q_1, R_1, Q_2, R_2 will be specified later. Denote $\psi = \Lambda_E(w,z)$ and $(u,v) = \Lambda_P(w,z,\psi)$. We will derive *a priori* estimates for Problem **P**. Taking $\eta = u(t)$ in (4.2) and $\zeta = v(t)$ in (4.3) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_H^2 + \alpha_1\|u_x(t)\|_H^2 - \alpha_2\int_\Omega u(t)\psi_x(t)u_x(t)\,\mathrm{d}x = f_2(t, w(t,1))u(t,1) - f_1(t, w(t,0))u(t,0), \tag{4.9}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v(t)\|_H^2 + \beta_1\|v_x(t)\|_H^2 + \beta_2\int_\Omega v(t)\psi_x(t)v_x(t)\,\mathrm{d}x = g_2(t, z(t,1))v(t,1) - g_1(t, z(t,0))v(t,0), \tag{4.10}$$

for a.e. $t \in (0,T)$. We will only prove, using the Equation (4.9), the estimates $\|u\|_{L^2(0,T;H)} \le Q_0$, $\|u_x\|_{L^2(0,T;H)} \le Q_1$, and $\|u_t\|_{L^2(0,T;V^*)} \le Q_2$. The proof of the estimates $\|v\|_{L^2(0,T;H)} \le R_0$, $\|v_x\|_{L^2(0,T;H)} \le R_1$, and $\|v_t\|_{L^2(0,T;V^*)} \le R_2$ uses (4.10) and is analogous, so we omit it here. First we estimate the term $\alpha_2\int_\Omega |u(t)\psi_x(t)u_x(t)|\,\mathrm{d}x$. We have

$$\alpha_2\int_\Omega |u(t)\psi_x(t)u_x(t)|\,\mathrm{d}x \le \frac{\alpha_1}{4}\|u_x(t)\|_H^2 + C\|\psi_x(t)\|_{L^\infty(\Omega)}^2\|u(t)\|_H^2.$$

As $H^1(\Omega) \subset C(\overline{\Omega})$ (cf. Remark 4.1), we further have

$$\alpha_2\int_\Omega |u(t)\psi_x(t)u_x(t)|\,\mathrm{d}x \le \frac{\alpha_1}{4}\|u_x(t)\|_H^2 + C\|\psi(t)\|_{H^2(\Omega)}^2\|u(t)\|_H^2, \tag{4.11}$$

and, by (4.6)

$$\alpha_2\int_\Omega |u(t)\psi_x(t)u_x(t)|\,\mathrm{d}x \le \frac{\alpha_1}{4}\|u_x(t)\|_H^2 + C\|w(t) - z(t)\|_H^2\|u(t)\|_H^2.$$

To estimate the boundary terms let $s \in (\frac{1}{2}, 1)$. Define $Z = H^s(\Omega)$. By Remark 4.1 this space embeds continuously in $C(\overline{\Omega})$. Consider the triple of spaces $V \subset Z \subset H$. The embedding $V \subset Z$ is compact (see [29, Theorem 2.80]) and the embedding $Z \subset H$ is continuous. We can use the Ehrling lemma (see e.g. [24, Lemma 7.6]) to conclude that for any $\varepsilon > 0$ we can find $C(\varepsilon) > 0$ such that

$$|y(1)| \le \varepsilon\|y_x\|_H + C(\varepsilon)\|y\|_H \quad \text{and} \quad |y(0)| \le \varepsilon\|y_x\|_H + C(\varepsilon)\|y\|_H,$$

for all $y \in V$. We estimate the boundary terms in (4.9). By (H_3) we have

$$\begin{aligned}
&|f_2(t, w(t,1))u(t,1)| + |f_1(t, w(t,0))u(t,0)| \\
&\quad \le (a_{12} + a_{22}|w(t,1)|)|u(t,1)| + (a_{11} + a_{21}|w(t,0)|)|u(t,0)| \\
&\quad \le |u(t,1)|^2 + |u(t,0)|^2 + C + C|w(t,1)|^2 + C|w(t,0)|^2
\end{aligned}$$

$$\leq \varepsilon\|u_x(t)\|_H^2 + C(\varepsilon)\|u(t)\|_H^2 + C + \varepsilon_1\|w_x(t)\|_H^2 + C(\varepsilon_1)\|w(t)\|_H^2,$$

where the constants $\varepsilon, \varepsilon_1 > 0$ are at this point arbitrary and will be specified later. We take $\varepsilon = \frac{\alpha_1}{4}$ in the last estimate, and using this estimate together with (4.11) in (4.9) we get

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\|u(t)\|_H^2 + \alpha_1\|u_x(t)\|_H^2 \\ &\quad \leq C(\|w(t)\|_H^2 + \|z(t)\|_H^2 + 1)\|u(t)\|_H^2 + C + \varepsilon_1\|w_x(t)\|_H^2 + C(\varepsilon_1)\|w(t)\|_H^2.\end{aligned} \tag{4.12}$$

The Gronwall lemma implies that for all $t \in [0, T]$ we have

$$\begin{aligned}\|u(t)\|_H^2 &\leq e^{\int_0^t C(\|w(s)\|_H^2+\|z(s)\|_H^2+1)\,ds} \\ &\quad \times \left[\|u_0\|_H^2 + \int_0^t (C + \varepsilon_1\|w_x(s)\|_H^2 + C(\varepsilon_1)\|w(s)\|_H^2)\,ds\right] \\ &\leq e^{CT + C\int_0^T (\|w(t)\|_H^2+\|z(t)\|_H^2)\,\mathrm{d}t} \\ &\quad \times \left[\|u_0\|_H^2 + CT + \int_0^T (\varepsilon_1\|w_x(t)\|_H^2 + C(\varepsilon_1)\|w(t)\|_H^2)\,\mathrm{d}t\right] \\ &\leq e^{C(T+Q_0+R_0)}(\|u_0\|_H^2 + CT + C(\varepsilon_1)Q_0 + \varepsilon_1 Q_1).\end{aligned} \tag{4.13}$$

Integrating over the interval $(0, T)$ we get

$$\|u\|_{L^2(0,T;H)}^2 \leq CTe^{C(T+Q_0+R_0)}(\|u_0\|_H^2 + T + C(\varepsilon_1)Q_0 + \varepsilon_1 Q_1). \tag{4.14}$$

Integrating (4.12) from 0 to T we get

$$\begin{aligned}\alpha_1\int_0^T \|u_x(t)\|_H^2\,\mathrm{d}t &\leq C\|u\|_{L^\infty(0,T;H)}^2 \int_0^T (\|w(t)\|_H^2 + \|z(t)\|_H^2 + 1)\,\mathrm{d}t \\ &\quad + CT + \int_0^T (\varepsilon_1\|w_x(t)\|_H^2 + C(\varepsilon_1)\|w(t)\|_H^2)\,\mathrm{d}t + \|u_0\|_H^2.\end{aligned}$$

Using (4.13) in the last inequality we get after cumbersome but straightforward computation

$$\begin{aligned}&\|u_x\|_{L^2(0,T;H)}^2 \\ &\quad \leq C(\|u_0\|_H^2 + T + C(\varepsilon_1)Q_0 + \varepsilon_1 Q_1)(1 + (T + Q_0 + R_0)e^{C(T+Q_0+R_0)}) \\ &\quad \leq \varepsilon_1 CQ_1(1 + (T + Q_0 + R_0)e^{C(T+Q_0+R_0)}) \\ &\qquad + C(\|u_0\|_H^2 + T + C(\varepsilon_1)Q_0)(1 + (T + Q_0 + R_0)e^{C(T+Q_0+R_0)}).\end{aligned}$$

Without loss of generality we may assume that $T \leq 1$, thus

$$\begin{aligned}\|u_x\|_{L^2(0,T;H)}^2 &\leq \varepsilon_1 CQ_1(1 + (1 + Q_0 + R_0)e^{C(1+Q_0+R_0)}) \\ &\quad + C(\|u_0\|_H^2 + 1 + C(\varepsilon_1)Q_0)(1 + (1 + Q_0 + R_0)e^{C(1+Q_0+R_0)}).\end{aligned} \tag{4.15}$$

Let $Q_0, R_0 >$ be fixed. We put

$$\varepsilon_1 = \frac{1}{2C(1 + (1 + Q_0 + R_0)e^{C(1+Q_0+R_0)})}$$

and

$$Q_1 = 2C(\|u_0\|_H^2 + 1 + C(\varepsilon_1)Q_0)(1 + (1 + Q_0 + R_0)e^{C(1+Q_0+R_0)}).$$

The inequality (4.15) yields $\|u_x\|^2_{L^2(0,T;H)} \le Q_1$. From (4.14), assuming that $T \le 1$, we get

$$\|u\|^2_{L^2(0,T;H)} \le CTe^{C(1+Q_0+R_0)}(\|u_0\|_H^2 + 1 + C(\varepsilon_1)Q_0 + \varepsilon_1 Q_1) = TF(Q_0, R_0),$$

where $F(Q_0, R_0)$ depends only on Q_0, R_0, but not on T. Hence if we take $T = \min\{1, \frac{Q_0}{F(Q_0,R_0)}\}$, we obtain $\|u\|^2_{L^2(0,T;H)} \le Q_0$. It remains to obtain the estimate for u_t. We have

$$\begin{aligned}\int_0^T \langle u_t(t), \eta(t)\rangle_{V^*\times V}\,\mathrm{d}t &\le \alpha_1\|u_x\|_{L^2(0,T;H)}\|\eta_x\|_{L^2(0,T;H)}\\ &+ \alpha_2\|\psi_x\|_{L^2(0,T;L^\infty(\Omega))}\|u\|_{L^\infty(0,T;H)}\|\eta_x\|_{L^2(0,T;H)}\\ &+ \int_0^T (a_{11} + a_{12} + (a_{21} + a_{22})\|w(t)\|_V)\|\eta(t)\|_V\,\mathrm{d}t. \end{aligned} \tag{4.16}$$

Moreover,

$$\begin{aligned}\int_0^T (a_{11} + a_{12} + (a_{21} + a_{22})\|w(t)\|_V)\|\eta(t)\|_V\,\mathrm{d}t\\ \le \|\eta\|_{L^2(0,T;V)}(C_1 + C_2\|w\|_{L^2(0,T;V)}),\end{aligned}$$

whence

$$\begin{aligned}\|u_t\|_{L^2(0,T;V^*)} \le \alpha_1\|u_x\|_{L^2(0,T;H)} + \alpha_2\|\psi_x\|_{L^2(0,T;L^\infty(\Omega))}\|u\|_{L^\infty(0,T;H)}\\ + C_1 + C_2\|w\|_{L^2(0,T;V)}.\end{aligned}$$

But we know that $\|\psi_x(t)\|_{L^\infty(\Omega)} \le C\|w(t) - z(t)\|_H$, whereas, using (4.13) to estimate the term $\|u\|_{L^\infty(0,T;H)}$, we can write

$$\|u_t\|_{L^2(0,T;V^*)} \le G(T, Q_0, Q_1, R_0),$$

with a constant $G(T, Q_0, Q_1, R_0)$. It is enough to take $Q_2 = G^2(T, Q_0, Q_1, R_0)$. The proof is complete. □

Remark 4.2: Without the use of the Ehrling lemma the proof would still be possible with additional bounds on the constants present in the model.

We will denote $B = B(T, Q_0, Q_1, Q_2, R_0, R_1, R_2)$ found in Lemma 4.3.

Lemma 4.4: *If Assumptions* (H_0), (H_1), (H_2) *hold, then the mapping* $\Lambda : B \to B$ *is weakly sequentially continuous.*

Proof: Consider sequences $w_n \to w$ and $z_n \to z$ weakly in $L^2(0, T; V)$ with $(w_n)_t \to w_t$ and $(z_n)_t \to z_t$ weakly in $L^2(0, T; V^*)$ such that $w_n, z_n, w, z \in B$. Let $(u_n, v_n) = \Lambda(w_n, z_n)$ and $\psi_n = \Lambda_E(w_n, z_n)$. We must prove that $u_n \to u$ and $v_n \to v$ weakly in $L^2(0, T; V)$, and $(u_n)_t \to u_t$ and $(v_n)_t \to v_t$ weakly in $L^2(0, T; V^*)$ for $(u, v) = \Lambda(w, z)$. As $(u_n, v_n) \in B$, a bounded set in X_T, for a subsequence, not renumbered we must have $u_n \to u$ and $v_n \to v$ weakly in $L^2(0, T; V)$. Moreover $(u_n)_t \to \bar{u}$ and $(v_n)_t \to \bar{v}$ weakly in $L^2(0, T; V^*)$ where it must be $\bar{u} = u_t$ and $\bar{v} = v_t$. If we are able to show that $(u, v) = \Lambda(w, z)$ then, by the uniqueness of the limit, the convergence will hold for the whole sequence and the proof will be complete. For any $\eta \in L^2(0, T; V)$, $\zeta \in L^2(0, T; V)$ and $\xi \in L^2(0, T; H_0^1(\Omega))$ we have

$$\int_0^T \langle (u_n)_t(t), \eta(t)\rangle_{V^*\times V}\,\mathrm{d}t + \int_0^T\int_\Omega (\alpha_1(u_n)_x(t) - \alpha_2 u_n(t)(\psi_n)_x(t))\eta_x(t)\,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^T (f_2(t, w_n(t,1))\eta(t,1) - f_1(t, w_n(t,0))\eta(t,0))\,dt, \tag{4.17}$$

$$\int_0^T \langle (v_n)_t(t), \zeta(t)\rangle_{V^*\times V}\,dt + \int_0^T \int_\Omega (\beta_1 (v_n)_x(t) + \beta_2 v_n(t)(\psi_n)_x(t))\zeta_x(t)dx$$
$$= \int_0^T (g_2(t, z_n(t,1))\zeta(t,1) - g_1(t, z_n(t,0))\zeta(t,0))\,dt, \tag{4.18}$$

and

$$\int_0^T \int_\Omega (\psi_n)_x(t)\xi_x(t)\,dx\,dt + \lambda \int_0^T \int_\Omega (w_n(t) - z_n(t))\xi(t)\,dx\,dt = 0. \tag{4.19}$$

From (4.19), by taking $\xi = \psi_n$ we obtain the estimate

$$\|\psi_n\|^2_{L^2(0,T;H^1_0(\Omega))} \le \lambda(\|w_n\|_{L^2(0,T;H)} + \|z_n\|_{L^2(0,T;H)})\|\psi_n\|_{L^2(0,T;H)}.$$

Using the Poincaré inequality it follows that the sequence ψ_n is bounded in $L^2(0,T;H^1_0(\Omega))$ and hence for a subsequence, denoted again by n, we must have $\psi_n \to \psi$ weakly in $L^2(0,T;H^1_0(\Omega))$. We can pass to the limit in (4.19) which implies that

$$\int_0^T \int_\Omega \psi_x(t)\xi_x(t)\,dx\,dt + \lambda \int_0^T \int_\Omega (w(t) - z(t))\xi(t)\,dx\,dt = 0, \tag{4.20}$$

whence $\psi = \Lambda_E(w,z)$. We need to pass to the limit in (4.17). For (4.18) the proof is analogous.

Passing to the limit in the terms with time derivative and α_1 is clear. Using the fact that the Nemytskii trace operators

$$\{y \in L^2(0,T;V) : y_t \in L^2(0,T;V^*)\} \ni y \to y(\cdot,1) \in L^2(0,T),$$
$$\{y \in L^2(0,T;V) : y_t \in L^2(0,T;V^*)\} \ni y \to y(\cdot,0) \in L^2(0,T)$$

are compact it follows that

$$w_n(\cdot,1) \to w(\cdot,1) \quad \text{and} \quad w_n(\cdot,0) \to w(\cdot,0) \quad \text{strongly in} \quad L^2(0,T).$$

By the growth conditions (H_2) we are allowed to use the dominated convergence theorem, whereas by the continuity of f_i with respect to the second variable in (H_1) we get

$$\int_0^T (f_2(t, w_n(t,1))\eta(t,1) - f_1(t, w_n(t,0))\eta(t,0))\,dt$$
$$\to \int_0^T (f_2(t, w(t,1))\eta(t,1) - f_1(t, w(t,0))\eta(t,0))\,dt.$$

It remains to pass to the limit in the term with $(\psi_n)_x$ in (4.17). We have

$$\int_0^T \int_\Omega (u_n(t)(\psi_n)_x(t) - u(t)\psi_x(t))\eta_x(t)\,dx\,dt \tag{4.21}$$
$$= \int_0^T \int_\Omega (u_n(t) - u(t))(\psi_n)_x(t)\eta_x(t)\,dx\,dt$$
$$+ \int_0^T \int_\Omega ((\psi_n)_x(t) - \psi_x(t))u(t)\eta_x(t)\,dx\,dt.$$

Clearly, $(\psi_n)_x - \psi_x \to 0$ weakly in $L^2(0,T;H)$, and

$$\int_0^T \int_\Omega (u(t)\eta_x(t))^2 \,\mathrm{d}x\,\mathrm{d}t \le \|u\|_{L^\infty(0,T;H)} \|\eta_x\|_{L^2(0,T;L^\infty(\Omega))},$$

whereas $u\eta_x \in L^2(0,T;H)$. It follows that

$$\int_0^T \int_\Omega ((\psi_n)_x(t) - \psi_x(t))u(t)\eta_x(t)\,\mathrm{d}x\,\mathrm{d}t \to 0.$$

By the Aubin–Lions compactness theorem we have

$$u_n \to u \quad \text{strongly in} \quad L^2(0,T;H).$$

We must prove that $(\psi_n)_x\eta_x$ is bounded in $L^2(0,T;H)$. Lemma 4.1 implies that

$$\|(\psi_n)_x\|_H \le C(\|w_n\|_H + \|z_n\|_H),$$

whereas

$$\|(\psi_n)_x\|_{L^\infty(0,T;H)} \le C(\|w_n\|_{L^\infty(0,T;H)} + \|z_n\|_{L^\infty(0,T;H)}).$$

We have

$$\begin{aligned}\int_0^T \int_\Omega ((\psi_n)_x(t)\eta_x(t))^2 \,\mathrm{d}x\,\mathrm{d}t &\le \|(\psi_n)_x(t)\|_{L^\infty(0,T;H)} \|\eta_x\|_{L^2(0,T;L^\infty(\Omega))} \\ &\le C(\|w_n\|_{L^\infty(0,T;H)} + \|z_n\|_{L^\infty(0,T;H)})\|\eta_x\|_{L^2(0,T;L^\infty(\Omega))}.\end{aligned}$$

Hence $(\psi_n)_x\eta_x$ is bounded in $L^2(0,T;H)$, whence

$$\int_0^T \int_\Omega (u_n(t) - u(t))(\psi_n)_x(t)\eta_x(t)\,\mathrm{d}x\,\mathrm{d}t \to 0.$$

It follows that the integral on the left-hand side of (4.21) also converges to zero and we can pass to the limit in the term

$$\int_0^T \int_\Omega \alpha_2 u_n(t)(\psi_n)_x(t)\eta_x(t)\,\mathrm{d}x\,\mathrm{d}t$$

in (4.17). The proof is complete. □

Theorem 4.2: *Let Assumptions* (H_0), (H_1), (H_2) *be satisfied. Then there exists* $T > 0$ *such that Problem* ***PE*** *has a solution.*

Proof: The assertion follows immediately by Theorem 4.1 and Lemmas 4.1–4.4. □

5. Uniqueness of weak solutions

In this section, we prove that Problem **PE** cannot have more than one weak solution.

Theorem 5.1: *Let Assumption* **H** *be true. Then Problem* ***PE*** *has at most one solution on* $[0,T]$ *for an arbitrary* $T > 0$.

Proof: Suppose that Problem **PE** has two solutions (u_1, v_1, φ_1), (u_2, v_2, φ_2) on $[0,T]$. We will show that they must be equal. By putting $\eta = u_1(t) - u_2(t)$ and $\zeta = v_1(t) - v_2(t)$ in (3.1), (3.2), respectively, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u_1 - u_2)(t)\|_H^2 + \alpha_1\|(u_1 - u_2)_x(t)\|_H^2$$

$$
\begin{aligned}
&- \alpha_2 \int_\Omega u_1(t)(\varphi_1 - \varphi_2)_x(t)(u_1 - u_2)_x(t)\,\mathrm{d}x \\
&- \alpha_2 \int_\Omega \varphi_{2x}(t)(u_1 - u_2)(t)(u_1 - u_2)_x(t)\,\mathrm{d}x \\
&= (f_2(t, u_1(t,1)) - f_2(t, u_2(t,1)))(u_1(t,1) - u_2(t,1)), \\
&- (f_1(t, u_1(t,0)) - f_1(t, u_2(t,0)))(u_1(t,0) - u_2(t,0)),
\end{aligned} \tag{5.1}
$$

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(v_1 - v_2)(t)\|_H^2 + \beta_1\|(v_1 - v_2)_x(t)\|_H^2 \\
&- \beta_2 \int_\Omega v_1(t)(\varphi_1 - \varphi_2)_x(t)(v_1 - v_2)_x(t)\,\mathrm{d}x \\
&- \beta_2 \int_\Omega \varphi_{2x}(t)(v_1 - v_2)(t)(v_1 - v_2)_x(t)\,\mathrm{d}x \\
&= (g_2(t, v_1(t,1)) - g_2(t, v_2(t,1)))(v_1(t,1) - v_2(t,1)), \\
&- (g_1(t, v_1(t,0)) - g_1(t, v_2(t,0)))(v_1(t,0) - v_2(t,0)),
\end{aligned} \tag{5.2}
$$

for a.e. $t \in (0, T)$.

Note that the equation

$$
\int_\Omega \varphi_x \xi_x \mathrm{d}x + \lambda \int_\Omega ((u_1 - u_2) - (v_1 - v_2))\xi\,\mathrm{d}x = 0 \quad \text{for each} \quad \xi \in H_0^1(\Omega), \tag{5.3}
$$

has the unique solution $\varphi = \varphi_1 - \varphi_2 \in L^2(0, T; H^3(\Omega) \cap H_0^1(\Omega))$ by the same arguments as in the proof of Lemma 4.1, and moreover

$$
\|(\varphi_1 - \varphi_2)(t)\|_{H^2(\Omega)} \le C\lambda\|(u_1 - u_2)(t) - (v_1 - v_2)(t)\|_H \quad \text{for a.e.} \quad t \in (0, T) \tag{5.4}
$$

with $C > 0$. Analogously

$$
\|\varphi_2(t)\|_{H^2(\Omega)} \le C\lambda\|(u_2 - v_2)(t)\|_H \quad \text{for a.e.} \quad t \in (0, T) \quad \text{with} \quad C > 0. \tag{5.5}
$$

By the similar argument as in the proof of Lemma 4.3, using (5.4) and (5.5) we obtain the following estimates of the integral and boundary terms in (5.1)

$$
\begin{aligned}
&\alpha_2 \int_\Omega |u_1(t)(\varphi_1 - \varphi_2)_x(t)(u_1 - u_2)_x(t)|\,\mathrm{d}x \\
&\quad \le \frac{\alpha_1}{4}\|(u_1 - u_2)_x(t)\|_H^2 + C\|u_1(t)\|_H^2\|(\varphi_1 - \varphi_2)(t)\|_{H^2(\Omega)}^2 \\
&\quad \le \frac{\alpha_1}{4}\|(u_1 - u_2)_x(t)\|_H^2 + C\|u_1(t)\|_H^2(\|(u_1 - u_2)(t)\|_H^2 + \|(v_1 - v_2)(t)\|_H^2), \\
&\alpha_2 \int_\Omega |\varphi_{2x}(t)(u_1 - u_2)(t)(u_1 - u_2)_x(t)|\,\mathrm{d}x \\
&\quad \le \frac{\alpha_1}{4}\|(u_1 - u_2)_x(t)\|_H^2 + C\|(\varphi_2)(t)\|_{H^2(\Omega)}^2\|(u_1 - u_2)(t)\|_H^2 \\
&\quad \le \frac{\alpha_1}{4}\|(u_1 - u_2)_x(t)\|_H^2 + C(\|u_2(t)\|_H^2 + \|v_2(t)\|_H^2)\|(u_1 - u_2)(t)\|_H^2, \\
&(f_2(t, u_1(t,1)) - f_2(t, u_2(t,1)))(u_1(t,1) - u_2(t,1)) \\
&\quad \le L_{f_2}(u_1(t,1) - u_2(t,1))^2 \\
&\quad \le \frac{\alpha_1}{4}\|(u_1 - u_2)_x(t)\|_H^2 + C\|(u_1 - u_2)(t)\|_H^2, \\
&(f_1(t, u_1(t,0)) - f_1(t, u_2(t,0)))(u_1(t,0) - u_2(t,0))
\end{aligned}
$$

$$
\begin{aligned}
&\le L_{f_1}(u_1(t,0)-u_2(t,0))^2\\
&\le \frac{\alpha_1}{4}\|(u_1-u_2)_x(t)\|_H^2 + C\|(u_1-u_2)(t)\|_H^2.
\end{aligned}
$$

In consequence

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|(u_1-u_2)(t)\|_H^2 &\le C(\|u_1(t)\|_H^2+\|u_2(t)\|_H^2+\|v_2(t)\|_H^2+1)\\
&\quad\times(\|(u_1-u_2)(t)\|_H^2+\|(v_1-v_2)(t)\|_H^2).
\end{aligned}
\tag{5.6}
$$

Making similar estimates as above for (5.2) we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\|(v_1-v_2)(t)\|_H^2 &\le C(\|u_2(t)\|_H^2+\|v_1(t)\|_H^2+\|v_2(t)\|_H^2+1)\\
&\quad\times(\|(u_1-u_2)(t)\|_H^2+\|(v_1-v_2)(t)\|_H^2).
\end{aligned}
\tag{5.7}
$$

Adding (5.6) and (5.7) we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}(\|(u_1-u_2)(t)\|_H^2+\|(v_1-v_2)(t)\|_H^2)\\
&\quad\le C(\|u_1(t)\|_H^2+\|u_2(t)\|_H^2+\|v_1(t)\|_H^2+\|v_2(t)\|_H^2+1)\\
&\quad\quad\times(\|(u_1-u_2)(t)\|_H^2+\|(v_1-v_2)(t)\|_H^2).
\end{aligned}
$$

Hence the Gronwall lemma implies that for all $t\in[0,T]$

$$
\begin{aligned}
&\|(u_1-u_2)(t)\|_H^2+\|(v_1-v_2)(t)\|_H^2\\
&\quad\le e^{\int_0^T C(\|u_1(t)\|_H^2+\|u_2(t)\|_H^2+\|v_1(t)\|_H^2+\|v_2(t)\|_H^2+1)\,\mathrm{d}t}\\
&\quad\quad\times(\|(u_1-u_2)(0)\|_H^2+\|(v_1-v_2)(0)\|_H^2).
\end{aligned}
\tag{5.8}
$$

Therefore $u_1=u_2$, $v_1=v_2$, because the right-hand side in (5.8) is equal zero. It follows from the uniqueness of the solution to (5.3) that $\varphi_1=\varphi_2$. □

6. Nonnegativity of local weak solutions

In this section we prove that, provided the initial conditions u_0, v_0 are nonnegative for a.e. $x\in\Omega$, the concentrations $u(t), v(t)$ must also remain nonnegative.

Theorem 6.1: *Let Assumptions* **H**, $\mathbf{H}^+$ *hold. Then for all $t\in[0,T]$ such that the solution of Problem* ***PE*** *exists on the interval $[0,T]$ we have $u(t)\in H^+$ and $v(t)\in H^+$.*

Proof: The proof follows the lines of the corresponding part of the proof of [30, Lemma 4.1], see also [26, Proposition 1]. Consider the following auxiliary problem, which differs from Problem **PE** by replacement of u, v with u^+, v^+, where $u^+=\max\{u,0\}$, $v^+=\max\{v,0\}$ in the terms representing electrostatic forces in (3.1) and (3.2).

Problem PE$^+$. Find $u,v\in L^2(0,T;V)$ and $\varphi\in L^2(0,T;H^3(\Omega)\cap H_0^1(\Omega))$ such that $u_t,v_t\in L^2(0,T;V^*)$ and for a.e. $t\in(0,T)$

$$
\begin{aligned}
&\langle u_t,\eta\rangle_{V^*\times V}+\int_\Omega(\alpha_1u_x-\alpha_2u^+\varphi_x)\eta_x\mathrm{d}x \qquad (6.1)\\
&\quad=f_2(t,u(t,1))\eta(1)-f_1(t,u(t,0))\eta(0)\quad\text{for each}\quad \eta\in V,\\
&\langle v_t,\zeta\rangle_{V^*\times V}+\int_\Omega(\beta_1v_x+\beta_2v^+\varphi_x)\zeta_x\mathrm{d}x \qquad (6.2)
\end{aligned}
$$

$$= g_2(t, v(t,1))\zeta(1) - g_1(t, v(t,0))\zeta(0) \quad \text{for each} \quad \zeta \in V,$$
$$\int_\Omega \varphi_x \xi_x \mathrm{d}x + \lambda \int_\Omega (u-v)\xi \mathrm{d}x = 0 \quad \text{for each} \quad \xi \in H_0^1(\Omega), \tag{6.3}$$

and the initial conditions (2.2) hold.

By the same argument as in the proof of Theorem 4.2 based on the Schauder–Tychonoff fixed point theorem it follows that Problem $\mathbf{PE}^+$ has a local in time weak solution (u, v, φ). We will prove that (u, v, φ) solves Problem $\mathbf{PE}^+$ on a certain interval $[0, T]$ if and only if it solves Problem **PE** on this interval.

Indeed, assume that (u, v, φ) solves Problem $\mathbf{PE}^+$ on $[0, T]$. We will prove that $u(t) \in H^+$ and $v(t) \in H^+$ for all $t \in [0, T]$. As the calculations for u and v are analogous, they will be done only for u. Taking $\eta = u^-(t) = -\min\{u(t), 0\}$ in (6.1) we get

$$-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^-(t)\|_H^2 - \alpha_1\|u_x^-(t)\|_H^2 = f_2(t, u(t,1))u^-(t,1) - f_1(t, u(t,0))u^-(t,0),$$

for a.e. $t \in (0, T)$. By (H_1^+) the right-hand side in the above equation is nonnegative, which gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u^-(t)\|_H^2 + \alpha_1\|u_x^-(t)\|_H^2 \le 0,$$

for a.e. $t \in (0, T)$. After integration on the interval $(0, t)$ for $t \in [0, T]$ we get

$$\frac{1}{2}\|u^-(t)\|_H^2 + \alpha_1 \int_0^t \|u_x^-(s)\|_H^2\, ds \le \frac{1}{2}\|u^-(t,0)\|_H^2,$$

for all $t \in [0, T]$, which, in view of (H_0^+), yields that $u^-(t) = 0$, and $u^+(t) = u(t)$ for all $t \in (0, T)$, whereas u satisfies (3.1). Note that we have also proved, that for every solution of Problem $\mathbf{PE}^+$ we must have $u(t) \in H^+$ and $v(t) \in H^+$ for all t in the interval of solution existence.

Now assume that (u, v, φ) solves Problem **PE** on an interval $[0, T]$. We know that there exists a solution $(\bar{u}, \bar{v}, \bar{\varphi})$ on a certain time interval $[0, T_0]$ of Problem $\mathbf{PE}^+$ and $(\bar{u}, \bar{v}, \bar{\varphi})$ must also solve Problem **PE** on this time interval. If $T \le T_0$, then the uniqueness Theorem 5.1 implies that $(\bar{u}, \bar{v}, \bar{\varphi}) = (u, v, \varphi)$ on $[0, T]$ and the assertion is proved. If $T_0 < T$ we will use the barrier method. Indeed, denote by $\bar{T}_0$ the supremum of all times T_0 such that Problem $\mathbf{PE}^+$ has a solution $(\bar{u}, \bar{v}, \bar{\varphi})$ in $[0, T_0]$ (it may be that $\bar{T}_0 = +\infty$). If $\bar{T}_0 > T$ we arrive at the previous case $T \le T_0$. We will prove that the case $\bar{T}_0 \le T$ leads to a contradiction. Observe that $(\bar{u}, \bar{v}, \bar{\varphi})$ must also solve Problem **PE** on each interval $[0, \bar{T}_0 - \varepsilon]$, which, by the uniqueness of solution to Problem **PE** implies that $(\bar{u}, \bar{v}, \bar{\varphi}) = (u, v, \varphi)$ on the interval $[0, \bar{T}_0)$. But, as $u, v \in C([0, T]; H)$, the values $\lim_{t \nearrow \bar{T}_0} \|\bar{u}(t)\|_H$ and $\lim_{t \nearrow \bar{T}_0} \|\bar{v}(t)\|_H$ are well defined and finite, and hence we can continue the solution $(\bar{u}, \bar{v}, \bar{\varphi})$ of Problem $\mathbf{PE}^+$, starting from $\bar{T}_0$, which contradicts its maximality.

As both problems are equivalent, and for the solution of Problem **PE** we must have $u(t) \in H^+$ and $v(t) \in H^+$ for all t, the assertion is proved. □

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No potential conflict of interest was reported by the authors.

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