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Parabolic-elliptic system modeling biological ion channels

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Abstract

The mathematical model of the transport and diffusion of ions in biological channels is considered. It is described by the three-dimensional nonlinear evolution classical Poisson–Nernst–Planck (cPNP) system of partial differential equations with nonlinear coupled boundary conditions. In particular the Chang–Jaffé (CJ) conditions are given on the input and output of a channel. The Robin boundary conditions on a potential are taken. Theorems on the existence, uniqueness and nonnegativity of local weak solutions, in the suitable Sobolev spaces, are proved. The main tool used in the proof of the existence result is the Schauder–Tychonoff fixed point theorem.

MSC: primary 35M33, 35D30, 35A16; secondary 35A01, 35A02, 35B09

Keywords: Biological ion channel; Parabolic-elliptic system; Existence, uniqueness and nonnegativity; Weak solution; Strong and weak topologies; Fixed point theorem

1. Introduction

The transport of ions and molecules is a fundamental process in biological systems. The relevant example here is the identification of potassium and sodium currents in the behavior of a nerve system [24]. The work was a milestone in the history of electrophysiology for which Hodgkin and Huxley were awarded the 1963 Nobel Prize in Medicine. The intensive research

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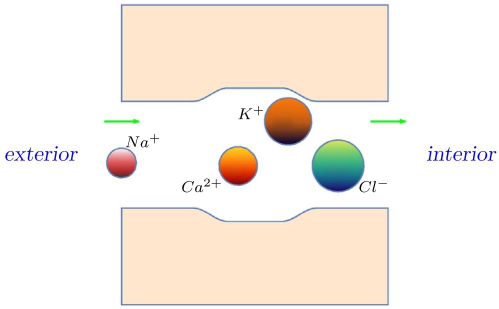

Fig. 1. The simplified biological ion channel.

is devoted toward understanding the functions of protein channels in the living cell membranes [1,14,40]. They separate the interior of a cell from the exterior, fluctuate between open and closed states and mediate the transport of specific inorganic ions (Fig. 1). In an open state, they work as selective filters, permitting some ions to pass, but limiting the rate of passing of the others. The mobile ions diffuse downhill the gradient of an electrochemical potential, without coupling to an energy source. The cell channels open in response to a specific stimulus, e.g., the change of the voltage across the membrane. Most interesting is however their ability to work as selective filters in the open conformation. The fact is that the selectivity mechanism is extremely rich and provides challenge for comprehensive understanding [16,45]. The big activity in understanding the generic phenomena and processes occurring in living organisms is partially due to a growing interest in bio-mimetic materials designed based on knowledge of biology. It is believed that learning and understanding of the selective transport of ions through the channels will give rise towards real-world applications and will provide guidelines to design and fabricate synthetic nanopores that can be applied in sensing, purifying or energy conversion.

Mathematical modeling serves to bridge the gap between the fundamental physics behind device operation and experimental results. The driving force in the Nernst–Planck flux involves two indispensable parts: the gradients of the concentrations c_i and the electrical potential φ generated by electric charges

$$J_i = -D_i \left(\nabla c_i + \alpha z_i c_i \nabla \varphi \right) \tag{1.1}$$

in the case of s components, $i = 1, \ldots, s$, where D_i and z_i are the diffusion coefficient and the charge number of the ith ion, respectively, and $\alpha = \frac{F}{RT}$ is a constant with the Faraday constant F, the gas constant R and the absolute temperature of the medium T. The electric field component comes here via two mechanisms, namely as the externally applied potential and as the electrostatic interaction between the ions present in the system. We will study the nonlinear evolution classical Poisson–Nernst–Planck (cPNP) system of equations

$$\begin{cases} \partial_t c_i + \operatorname{div} J_i = 0, \quad i = 1, \ldots, s, \\ -\triangle \varphi = \lambda \sum_{j=1}^{s} z_j c_j, \end{cases} \tag{1.2}$$

on $[0, T] \times \Omega$, $\Omega \subset \mathbb{R}^3$ is an open bounded set. The Debye constant is given by $\lambda = \frac{F}{\varepsilon_0 \varepsilon_r}$, where F denotes the Faraday constant, ε_0 - the vacuum permittivity and ε_r - the relative permittivity of the medium. This system constitutes a mathematical framework for the deterministic modeling of electrodiffusion in continuum media approximation [29,30,44,45,47].

The investigations of different types of boundary conditions show efficiency of the Chang-Jaffé boundary conditions. The basic idea is that the flux on the boundary (or more precisely, the normal component of the flux) is proportional to the weighed difference between the concentration inside and outside of the region where the process takes place

$$J_i(t,x) \circ n(x) = -k_{i,f}c_{i,out} + k_{i,b}c_i(t,x) \tag{1.3}$$

for $x \in \partial\Omega$, where $n \in \mathbb{R}^3$ is the outside normal unit vector to $\partial\Omega$, $k_{i,f}, k_{i,b} \geq 0$ are the material constants (the so called heterogeneous rate constants) which describe the permeability of the boundary, and $c_{i,out}$ is the concentration of the i-th species outside Ω – it is assumed that $c_{i,out}$ is constant. The symbol "$\circ$" means the standard scalar product in $\mathbb{R}^3$. In the case where the ions have different mobilities, the exact impedance spectra using the Chang-Jaffé-Nernst–Planck-Poisson response model were obtained and the conditions providing essentially exact or approximate numerical correspondence of different models were determined [33,39]. Impedance is a complex quantity characterizing the relationship between the current intensity and voltage in alternating current (sinusoidal alternating) circuits. In one dimension this type of boundary conditions were first used by Chang and Jaffé in 1951 in their paper on the polarization in electrolytic solutions [12]. This phenomenon consists in the change of the electrode potential with respect to the solution as a result of the current flow during electrolysis. Since then, the CJ boundary conditions have been used extensively in the field of potentiometric sensors modeling as proposed by Brumleve and Buck in their seminal paper [10]. These sensors measure the equilibrium electrical potential of an electrode when no current is present.

The real materials commonly have the high concentration of internal bounded charges. By subtracting the effect of internal charges, the equations for free charges follow. They have a form of the original Maxwell equations but with $E = -\nabla\varphi$ replaced by the supplementary field $D = -\varepsilon\nabla\varphi$, $\varepsilon = \varepsilon_0\varepsilon_r$ and charges q by the free charges q_{free}. A similar argument holds for the magnetic flux density B and the magnetic field intensity H. Upon introducing those supplementary D and H, the Maxwell equations get a geometric, particularly useful in electrochemistry, connotation. Let $\Sigma \subset \partial\Omega$ be a fixed part of $\partial\Omega$ with the closed boundary $\partial\Sigma$. The fields E, D, B and H allow expressing the electromagnetism by the system of integral equations. In this work, the total current formula is of interest

$$\oint_{\partial\Sigma} H \circ dl - \frac{\partial}{\partial t}\int_{\Sigma} D \circ dS = \int_{\Sigma} I_\Sigma \circ dS, \tag{1.4}$$

where $I_\Sigma = I + F\sum_{i=1}^{s} z_i J_i$ is the overall current density at Σ, I and $F\sum_{i=1}^{s} z_i J_i$ are the densities of the electron and ionic currents. In ionic solutions at low frequencies it might be simplified

$$\frac{\partial}{\partial t}\int_{\Sigma} D \circ dS = -\int_{\Sigma} I_\Sigma \circ dS, \tag{1.5}$$

and in the local form

$$\varepsilon(t,x)\frac{\partial\varphi}{\partial n}(t,x) = I_\Sigma(t,x) \circ n(x), \tag{1.6}$$

where n is the outside normal unit vector to Σ. A channel wall can be formed by different compounds. It implies different values of ε at the boundary. In particular $\varepsilon = \varepsilon(t, x)$. In order to formulate the boundary conditions for the whole Ω we postulate the Robin type ones

$$a(t,x)\frac{\partial \varphi}{\partial n} + b(t,x)\varphi = h(t,x). \tag{1.7}$$

The processes inside Ω refer to the electrolyte (solution). The processes at the boundary $\partial\Omega$ expressed by (1.7) allow inventing appropriate boundary conditions on φ for the Poisson equation in (1.2). Our partial differential equations (1.2) describe everything except a very thin double contact layer of ions due to the Stern effect or the permanent charge $Q(t, x)$, which may be included in (1.7) by putting $h(t, x) := Q(t, x)$. The "electrode", we refer to the electrically conducting phase on the wall of the channel in contact with the electrolyte inside of its, which does not include either the diffuse or contact parts of the double layer. The Maxwell equations are valid in mediums where gradients of scalar potentials are not too large and the continuum approximation is allowed, e.g., the local densities are defined.

The boundary models should, in principle, be treated using a quantum mechanical treatment of the boundary. So far such satisfactory quantum mechanical models have not been developed for aqueous solutions considered here. A common simplification, valid if the electric field does not change very quickly [26], assumes the local and instantaneous material response. The Langevin function [22] models such dielectric saturation in a polar substance, i.e. such a substance that contains polar molecules, the chemical species in which the distribution of electrons is not homogenous. Thus the a, b and h coefficients in (1.7) can be approximated based on experimental data and simplified boundary models. The boundary conditions considered in this paper are much more physical and perspective than those in [6,7,15,44,46]. We suppose that the difference methods formulated in [8,42] will be appropriate to make numerical simulations for the problems studied. It will be a subject of our future articles.

Let us stress that this paper considers only the relatively simplest model that treats ions as point-charges and no specific ion effects are touched. Also, the model is primitive in the sense that it treats water as a dielectric medium through the dielectric coefficient. A one-dimensional steady-state model with the richer Nernst–Planck fluxes

$$J_i = -r(x)\frac{1}{RT}D_i c_i \frac{d\mu_i}{dx} \tag{1.8}$$

and the Poisson equation

$$-\frac{1}{r(x)}\frac{d}{dx}\Big(r(x)\frac{d\varphi}{dx}\Big) = \lambda\Big(\sum_{j=1}^{s} z_j c_j + Q(x)\Big), \tag{1.9}$$

where $r(x)$ represents the cross-section area of the channel over the longitudinal and $Q(x)$ is the distribution of the permanent charge along the interior wall of the channel, was developed in [18,28]. The electrochemical potential μ_i for the ith ion species consists of the ideal component μ_i^{id}, the excess component μ_i^{ex} and the concentration-independent component μ_i^0: $\mu_i = \mu_i^{id} + \mu_i^{ex} + \mu_i^0$, where

$$\mu_i^{id} = F z_i \varphi + RT \ln \frac{c_i}{c_0} \tag{1.10}$$

with some characteristic number density c_0 which without loss of generality is normalized to one in the model. The cPNP system (1.2) takes the ideal component μ_i^{id} only. This component reflects the collision between ion particles and the water molecules. It has been accepted that the cPNP system is a reasonable model in, for example, the dilute case under which the ions can be treated as point charges and the ion-to-ion interaction can be ignored. As remarked in [28], D_i's involve the ionic radii through the Einstein relation so that the cPNP does not completely ignore ion sizes. Two critical potential values at the entry of the channel that characterize some size effects on current-voltage relation are identified with the use of a combination of the geometric singular perturbation and the density functional theory, DFT, in [28]. This work generalizes results from [43]. It is determined in [18], using the geometric singular perturbation theory, when the permanent charge $Q(x)$ produces current reversal. A singular orbit of a one-dimensional cPNP boundary value problem for two species is identified based on the dynamics of limiting fast and slow steady-state systems in [36]. An application of the geometric singular perturbation theory gives rise to the existence and (local) uniqueness of the boundary value problem. These ideas are extended to systems with multiple regions of the permanent charge and with multiple ion species in [37]. The connections between attractors of three-dimensional systems with permanent charges for two species and theirs one-dimensional limiting reductions are studied in [17,38], with the use of the geometric singular perturbation theory.

In recent years some more general two- or three-dimensional ionic electrodiffusion models have been studied also, but with boundary conditions simpler than CJ and Robin's. The Poisson–Nernst–Planck–Navier–Stokes cross-diffusion system describing the compressible viscous conductive fluid with the crowded charged particles of two species has been considered in [48], while in [15], the similar system for the incompressible conductive fluid with diluted multiple species has been studied. The Poisson–Nernst–Planck–Fourier system for two species with a variable temperature has been developed in [25]. In these papers, the existence and stability of global weak solutions has been proven. The work [34] concerns electrostatic properties of an ionic solution with multiple ionic species of possibly different ionic sizes. Such properties are described by the minimization of an electrostatic free-energy functional of ionic concentrations.

This work presents the model of representing the dynamical operation of the nanochannel. The three-dimensional differential problem models electrodiffusion of s-mobile ions of different charges and mobilities, and the electrode effects, e.g. the permanent charge effect. It contains new advances in two major expects, namely it allows more than two ion species and it allows nonhomogenous boundary conditions. The CJ boundary conditions for the normal fluxes can realize a new blocking or selective mechanism for different types of ions. It allows modeling a range of dynamics and behavior which have not been studied previously, and explore the numerical challenges required when adding more complexity to a model. Furthermore, the basic transport equations can in the future accommodate the inclusion of additional physics (see (1.8), (1.9), (1.10)), and coupling to more complex boundary conditions that incorporate two-dimensional surface phenomena and higher order reactions. In mathematical models and numerical simulations of electrochemical systems, it is particularly important that all Maxwell's equations are satisfied. If not it leads to highly not physical phenomena like the superluminal propagation of waves and failure to conserve energy or momentum. The model takes into account nonhomogeneity of the wall of the channel. Namely by including dielectric polarization effects by introducing different values of the dielectric permittivity in different parts of the boundary due to adsorption. Advection and convection effects are not modeled but could in future be incorporated.

The mathematical theory of the cPNP and PNP systems is quite extensive but still it lacks the main results on the local or global existence and uniqueness of solutions in the case of relevant boundary conditions in dimensions bigger than one. Well-posedness of the one-dimensional cPNP system for two components with the class of nonlinear boundary conditions which cover the CJ on fluxes and the Dirichlet on potential boundary conditions is studied in [20]. But even in this case there are no theorems about the global in time existence. Biler et al. in [4,5] consider the cPNP system describing chemotaxis in different dimensions for one or two components and prove the global existence and uniqueness, and the convergence to the steady-state solution as time advances to infinity. However, the boundary conditions they use are simpler than CJ and have the form of null normal fluxes on the whole boundary. Those boundary conditions imply the law of mass conservation and in consequence a construction of the Lapunov function. This function is crucial in the proof of the global existence. This has a simple physical interpretation meaning the closed system. But as we motivated above, real electrodiffusion applications of practical importance are almost always open systems which interact with surroundings through boundary fluxes. There is a handful of mathematical papers that address the cPNP and PNP systems but in the steady-state variant [3,18,23,27,28,32,34,36,37,43,49]. Because we are here interested only in the time dependent system we shall not go into details but stress the fact that none of these papers uses the CJ boundary conditions or their extensions.

There exists vast literature on the existence and regularity of solutions for both parabolic and elliptic problems, cf., e.g. [9,11,13,19,31,35,41,50]. The aim of this paper is to give theorems on the existence, uniqueness and nonnegativity of weak solutions, in the suitable Sobolev spaces, to the three-dimensional cPNP system. We have two parts of boundary conditions on fluxes, i.e. null normal fluxes on the wall of the channel and the Chang–Jaffé (CJ) conditions on the input and output. The Robin boundary conditions on the potential are taken. In the existence proof we use the Schauder–Tychonoff fixed point theorem instead of the Schauder fixed point theorem as in [4,5] because of compact embeddings in the boundary spaces.

The paper is organized in the following way. In Section 2, the initial-boundary differential problem is formulated and its weak version is given together with the assumptions that will be used in the further parts. Section 3 is concerned the regularity and estimate of weak solutions to the auxiliary elliptic Robin boundary value problem. Sections 4, 5 and 6 deal with the existence, uniqueness and nonnegativity of weak solutions of the problem studied, respectively.

2. Problem formulation

We define a simplified tubular-like membrane channel $\Omega \subset \mathbb{R}^3$ with the boundary $\partial\Omega$ belonging to class C^∞ of the form

$$\Omega = \Big\{(x_1, x_2, x_3) \in \mathbb{R}^3 : \ 0 < x_1 < 1, \ x_2^2 + x_3^2 < g^2(x_1)\Big\},$$

where $g \in C^\infty([0, 1], \mathbb{R})$ (see Fig. 2). The boundary $\partial\Omega$ will be divided into three portions as follows:

$$\partial_1\Omega = \Big\{(x_1, x_2, x_3) \in \partial\Omega : \ x_1 = 0\Big\},$$

$$\partial_2\Omega = \Big\{(x_1, x_2, x_3) \in \partial\Omega : \ x_1 = 1\Big\},$$

$$\partial_3\Omega = \Big\{(x_1, x_2, x_3) \in \partial\Omega : \ 0 < x_1 < 1, \ x_2^2 + x_3^2 = g^2(x_1)\Big\}.$$

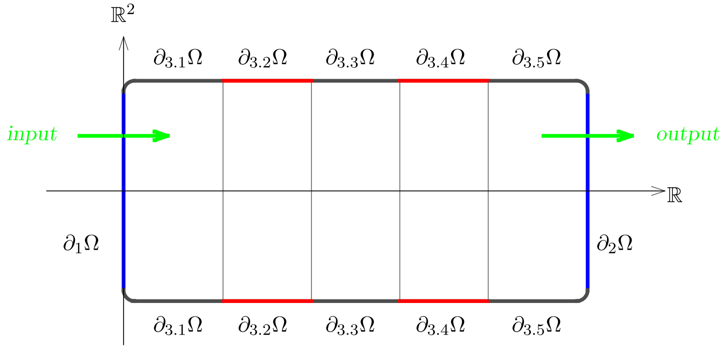

Fig. 2. The domain Ω and its sample boundary $\partial\Omega$.

Thus, $\partial_1\Omega$ and $\partial_2\Omega$ are viewed as the input and output of the channel, respectively and $\partial_3\Omega$ - the wall of the channel. Let functions $c_{0i} : \Omega \to \mathbb{R}$, $a, b, h : [0, T] \times \partial\Omega \to \mathbb{R}$ and constants $D_i, \alpha, \lambda > 0$, $a_{ji}, b_{ji} \geq 0$, $z_i \in \mathbb{R}$, $i = 1, ..., s$, $j = 1, 2$ be given, where $T > 0$ is arbitrary.

We consider the nonlinear parabolic-elliptic system of equations

$$\begin{cases} \partial_t c_i + \operatorname{div}\big(-D_i\big(\nabla c_i + \alpha z_i c_i \nabla\varphi\big)\big) = 0 & \text{on} \quad \big[0, T\big] \times \Omega, \\ -\Delta\varphi = \lambda \sum_{j=1}^{s} z_j c_j & \text{on} \quad \big[0, T\big] \times \Omega, \end{cases} \tag{2.1}$$

with the initial condition

$$c_i(0, x) = c_{0i}(x) \quad \text{on} \quad \Omega, \tag{2.2}$$

and the nonlinear boundary conditions

$$\begin{cases} -D_i\big(\frac{\partial c_i}{\partial n} + \alpha z_i c_i \frac{\partial\varphi}{\partial n}\big) = -a_{1i} + b_{1i} c_i & \text{on} \quad [0, T] \times \partial_1\Omega, \\ -D_i\big(\frac{\partial c_i}{\partial n} + \alpha z_i c_i \frac{\partial\varphi}{\partial n}\big) = -a_{2i} + b_{2i} c_i & \text{on} \quad [0, T] \times \partial_2\Omega, \\ -D_i\big(\frac{\partial c_i}{\partial n} + \alpha z_i c_i \frac{\partial\varphi}{\partial n}\big) = 0 & \text{on} \quad [0, T] \times \partial_3\Omega, \\ a(t, x)\frac{\partial\varphi}{\partial n} + b(t, x)\varphi = h(t, x) & \text{on} \quad [0, T] \times \partial\Omega, \end{cases} \tag{2.3}$$

for $i = 1, ..., s$. The first and second conditions in (2.3) are called the Chang–Jaffé boundary conditions.

Define the Sobolev spaces $V = H^1(\Omega)$ and $H = L^2(\Omega)$. Then $V \subset H \subset V^*$ constitute an evolution triple with the embeddings being dense, continuous and compact. By H^+ we denote the cone of nonnegative functions in H, that is

$$H^+ = \big\{u \in H : \ u(x) \geq 0 \quad \text{a.e. in} \quad \Omega\big\}.$$

In the paper by $C > 0$ we will always denote a generic constant dependent only on the problem data.

We assume the following conditions on c_{0i}, $i = 1, ..., s$ and a, b, h.

Assumption H.

(H_0) $c_{0i} \in H$, $i = 1, ..., s$.
(H_1) $\frac{b}{a} \in L^\infty(0, T; C^\infty(\partial\Omega))$, $\frac{h}{a} \in L^\infty(0, T; H^{\frac{1}{2}}(\partial\Omega))$.
(H_2) $\frac{b}{a}(t, x) \geq p_0 > 0$ for a.e. $t \in (0, T)$ and all $x \in \partial\Omega$, $p_0 = \text{const}$.

The original initial-boundary value problem (2.1)–(2.3) has the following weak version.

Problem PE. Find $c_i \in L^2(0, T; V)$ and $\varphi \in L^2(0, T; H^2(\Omega))$ such that $\partial_t c_i \in L^2(0, T; V^*)$ and for a.e. $t \in (0, T)$

$$\langle \partial_t c_i, \eta_i \rangle_{V^* \times V} + \int_\Omega D_i (\nabla c_i + \alpha z_i c_i \nabla\varphi) \circ \nabla \eta_i \, dx \tag{2.4}$$

$$= \int_{\partial_1\Omega} \big(a_{1i} - b_{1i} c_i\big) \eta_i \, d\sigma + \int_{\partial_2\Omega} \big(a_{2i} - b_{2i} c_i\big) \eta_i \, d\sigma \quad \text{for each} \quad \eta_i \in V,$$

$$\int_\Omega \nabla\varphi \circ \nabla\xi \, dx + \int_{\partial\Omega} \frac{b}{a} \varphi \xi \, d\sigma = \lambda \sum_{j=1}^{s} \int_\Omega z_j c_j \xi \, dx + \int_{\partial\Omega} \frac{h}{a} \xi \, d\sigma \quad \text{for each} \quad \xi \in V, \tag{2.5}$$

and the initial condition (2.2) holds.

Remark 2.1. We postulate that the differences of the potential φ and its normal derivative $\frac{\partial\varphi}{\partial n}$ on different parts of the boundary of the channel are a stimulus that causes selective ion flow. For numerical simulations, we can divide the wall of the channel $\partial_3\Omega$ into several rings, for example as in Fig. 2. Then we may put $a(t, x) \approx \chi_{\partial_{3.1}\Omega \cup \partial_{3.3}\Omega \cup \partial_{3.5}\Omega}(t, x)$ and $b(t, x) \approx \chi_{\partial_{3.2}\Omega \cup \partial_{3.4}\Omega}(t, x)$ for $(t, x) \in \partial\Omega_3 := \bigcup_{k=1}^{5} \partial_{3.k}\Omega$, where χ means the characteristic function. Moreover, we can test different functions h, e.g. $h(t, x) := Q(t, x)$, where Q means the distribution of the permanent charge along the interior wall of the channel. Numerical calculations for such a boundary $\partial\Omega$ with much simpler boundary conditions were made in [6,7,46]. Now it is a challenge task and we have not finished it yet.

Remark 2.2. We assumed the constants $a_{ji}, b_{ji} \geq 0$, $i = 1, ..., s$, $j = 1, 2$ in the CJ boundary conditions in (2.3). This assumption implies the physically desirable positivity property of solutions given in Theorem 6.1. Moreover, putting $a_{1i} = b_{1i} = 0$ we block inflow of ith ions into the channel and putting $a_{2i} = b_{2i} = 0$ we block outflow. In this way, the boundary conditions (2.3) give a blocking or selective mechanism for different types of ions. Another kind of selectivity is proposed in [15], where blocking and unblocking of flow of ith ions through $\partial_1\Omega$ or $\partial_2\Omega$ is realized by setting $J_i \circ n = 0$ and $c_i > 0$, respectively. It is different in [6,7,46], where $\partial_1\Omega$ and $\partial_2\Omega$ are permeable, which means that the ions enter and exit the channel, and it is realized by setting $c_i > 0$.

3. Regularity of weak solutions to the elliptic problem

We will formulate below two theorems regarding the auxiliary elliptic Robin boundary value problem and then we will come back to our main Problem **PE**. Suppose that $\Omega \subset \mathbb{R}^d$, $d \in \mathbb{N}$

is an open bounded set with the boundary $\partial\Omega$ of class C^2 (see [9], p. 272). Let $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, $p \in L^\infty(\partial\Omega)$ and $p(x) \geq p_0 > 0$ for all $x \in \partial\Omega$, $p_0 = \text{const}$. We consider the Robin boundary value problem

$$\begin{cases} -\triangle\varphi = f(x) & \text{on} \quad \Omega, \\ \frac{\partial\varphi}{\partial n} + p(x)\varphi = g(x) & \text{on} \quad \partial\Omega. \end{cases} \tag{3.1}$$

We say that $\varphi \in H^1(\Omega)$ is a weak solution of (3.1) if

$$\int_\Omega \nabla\varphi \circ \nabla\xi \, dx + \int_{\partial\Omega} p\varphi\xi \, d\sigma = \int_\Omega f\xi \, dx + \int_{\partial\Omega} g\xi \, d\sigma \quad \text{for each} \quad \xi \in H^1(\Omega). \tag{3.2}$$

The space $H^1(\Omega)$ is equipped with the standard Sobolev norm

$$\|\varphi\|_{H^1(\Omega)} = \Big(\int_\Omega \big(\varphi^2 + |\nabla\varphi|^2\big)\, dx\Big)^{\frac{1}{2}} \tag{3.3}$$

and with the equivalent norm

$$\|\varphi\|_* = \Big(\int_\Omega |\nabla\varphi|^2 \, dx + \int_{\partial\Omega} \varphi^2 \, d\sigma\Big)^{\frac{1}{2}} \tag{3.4}$$

(see [50, Theorem 21.A]).

Theorem 3.1. *Under the assumptions above, the Robin boundary value problem (3.1) has a unique weak solution $\varphi \in H^1(\Omega)$. Moreover*

$$\|\varphi\|_{H^1(\Omega)} \leq C\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\big), \tag{3.5}$$

where C is a positive constant depending on p_0.

Proof. The existence of a unique solution $\varphi \in H^1(\Omega)$ follows from [50, Proposition 22.16]. To prove the estimate (3.5) note that, by (3.2)

$$\int_\Omega |\nabla\varphi|^2 \, dx + \int_{\partial\Omega} p\varphi^2 \, d\sigma \leq C\big(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\partial\Omega)}\big)\|\varphi\|_{H^1(\Omega)}.$$

But on the other hand

$$C\|\varphi\|^2_{H^1(\Omega)} \leq C_1\|\varphi\|^2_* \leq \int_\Omega |\nabla\varphi|^2 \, dx + p_0\int_{\partial\Omega} \varphi^2 \, d\sigma \leq \int_\Omega |\nabla\varphi|^2 \, dx + \int_{\partial\Omega} p\varphi^2 \, d\sigma.$$

Using these relations and the Young with ε inequality, we obtain

$$C\|\varphi\|^2_{H^1(\Omega)} \le C(\varepsilon)\big(\|f\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(\partial\Omega)}\big) + \varepsilon\|\varphi\|^2_{H^1(\Omega)}$$

and consequently (3.5), by putting $\varepsilon = \frac{C}{2}$. □

Theorem 3.2. *Under the assumptions of Theorem 3.1 and adding that $\partial\Omega$ belongs to class C^∞, $p \in C^\infty(\partial\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$, the Robin boundary value problem (3.1) has a weak solution $\varphi \in H^2(\Omega)$ which is unique in $H^1(\Omega)$. Moreover*

$$\|\varphi\|_{H^2(\Omega)} \le C\big(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\partial\Omega)}\big), \tag{3.6}$$

where C is a positive constant depending on p_0.

Proof. Consider the linear operator

$$P : H^2(\Omega) \to L^2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega), \quad P\varphi = \big(A\varphi, B_0\varphi\big),$$

where $A\varphi = -\Delta\varphi$ on Ω and $B_0\varphi = \frac{\partial\varphi}{\partial n} + p(x)\varphi$ on $\partial\Omega$. Obviously the conjugate operator A^* is equal A. Put $S_0\varphi = \varphi$, $C_0 = B_0$ and $T_0 = S_0$ on $\partial\Omega$. We have for any $\varphi, v \in H^2(\Omega)$ the following relation, by the Green's formula

$$\int_\Omega (A\varphi)v\,dx - \int_\Omega \varphi A^* v\,dx = \int_{\partial\Omega} S_0\varphi C_0 v\,d\sigma - \int_{\partial\Omega} B_0\varphi T_0 v\,d\sigma.$$

It follows from Theorem 3.1 that the homogeneous problem $A^*v = 0$, $C_0 v = 0$ has a unique solution $v = 0$ in $C^\infty(\overline{\Omega})$ because $C^\infty(\overline{\Omega}) \subset H^1(\Omega)$. Theorem 5.3 in [35] implies that P is surjective. In consequence problem (3.1) has a solution $\varphi \in H^2(\Omega)$, and it is unique in $H^1(\Omega)$ from Theorem 3.1. We get from [35, Theorem 5.1] the estimate

$$\|\varphi\|_{H^2(\Omega)} \le C\big(\|f\|_{L^2(\Omega)} + \|g\|_{H^{\frac{1}{2}}(\partial\Omega)} + \|\varphi\|_{H^1(\Omega)}\big).$$

Then the use of (3.5) in Theorem 3.1 together with the continuous imbedding $H^{\frac{1}{2}}(\partial\Omega) \subset L^2(\partial\Omega)$ given by [11, Theorems 2.80] finishes the proof. □

4. Existence of local weak solutions

We will use the following version of the Schauder–Tychonoff fixed point theorem which is a simple consequence of [2, Theorem 1].

Theorem 4.1. *Let X be a reflexive Banach space and let $C \subset X$ be a closed, bounded, convex and nonempty set. If the function $\Lambda : C \to C$ is sequentially weakly continuous, then it must have a fixed point.*

To study the existence of a weak solution to Problem **PE**, we split it into two auxiliary problems, an elliptic one, and a parabolic one.

Problem E. Given $w_i \in L^2(0,T;V)$ find $\psi \in L^2(0,T;H^2(\Omega))$ such that for a.e. $t \in (0,T)$

$$\int_\Omega \nabla\psi \circ \nabla\xi \, dx + \int_{\partial\Omega} \frac{b}{a}\psi\,\xi\, d\sigma = \lambda \sum_{j=1}^{s} \int_\Omega z_j w_j \xi\, dx + \int_{\partial\Omega} \frac{h}{a}\xi\, d\sigma \quad \text{for each} \quad \xi \in V. \tag{4.1}$$

Problem P. Given $w_i \in L^2(0,T;V)$ and $\psi \in L^2(0,T;H^2(\Omega))$ find $c_i \in L^2(0,T;V)$ such that $\partial_t c_i \in L^2(0,T;V^*)$ and for a.e. $t \in (0,T)$

$$\langle \partial_t c_i, \eta_i \rangle_{V^*\times V} + \int_\Omega D_i\big(\nabla c_i + \alpha z_i c_i \nabla\psi\big)\circ\nabla\eta_i dx \tag{4.2}$$
$$= \int_{\partial_1\Omega} \big(a_{1i} - b_{1i}w_i\big)\eta_i d\sigma + \int_{\partial_2\Omega} \big(a_{2i} - b_{2i}w_i\big)\eta_i d\sigma \quad \text{for each} \quad \eta_i \in V,$$

and the initial condition (2.2) holds.

Define for a fixed $T > 0$ the space of vector valued functions

$$X_T = \big\{c = \big(c_1, ..., c_s\big) \colon c_i \in L^2(0,T;V),\ \partial_t c_i \in L^2(0,T;V^*)\big\} \tag{4.3}$$

normed by

$$\|c\|_{X_T} = \sum_{i=1}^{s}\Big(\|c_i\|_{L^2(0,T;V)} + \|\partial_t c_i\|_{L^2(0,T;V^*)}\Big), \tag{4.4}$$

where

$$\|c_i\|^2_{L^2(0,T;V)} = \int_0^T \|c_i(t)\|_V^2\, dt,$$
$$\|\partial_t c_i\|^2_{L^2(0,T;V^*)} = \int_0^T \|\partial_t c_i(t)\|_{V^*}^2\, dt.$$

The space $\big(X_T, \|\ \|_{X_T}\big)$ is a reflexive Banach space. We will use two topologies in this space, namely the strong topology and the weak topology. We define the set

$$B = B(T, Q_0, Q_1, Q_2, Q_3)$$

parameterized by the time $T > 0$ and the constants $Q_0, Q_1, Q_2, Q_3 > 0$,

$$B = \big\{w \in X_T : \|w_i\|^2_{L^2(0,T;H)} \le Q_0,\ \|\nabla w_i\|^2_{L^2(0,T;H)} \le Q_1,\ \|\partial_t w_i\|^2_{L^2(0,T;V^*)} \le Q_2, \tag{4.5}$$
$$\|w_i\|^4_{L^4(0,T;H)} \le Q_3\big\}.$$

Note that the set B is convex and strongly closed in X_T, so it is also weakly closed. Since it is also strongly bounded, it follows that it is weakly compact. Define the operator

$$\Lambda_E : B \to L^2(0,T;H^2(\Omega)),$$

which maps any $w \in B$ to the unique solution $\psi \in L^2(0,T;H^2(\Omega))$ of Problem **E**, and the operator

$$\Lambda_P : B \times L^2(0,T;H^2(\Omega)) \to X_T,$$

which maps any pair $w \in B$ and function $\psi \in L^2(0,T;H^2(\Omega))$ to the unique solution $c \in X_T$ of Problem **P**. Composing the two operators we define

$$\Lambda : B \to X_T, \quad \Lambda(w) = \Lambda_P(w, \Lambda_E(w)). \tag{4.6}$$

Obviously, (c,φ) is a solution of Problem **PE** if and only if c is a fixed point of Λ, and $\varphi = \Lambda_E(c)$. We establish several lemmas on the properties of Λ_E, Λ_P, Λ which imply the correctness of their definitions and will be useful in the local existence result.

Lemma 4.1. *If assumptions (H_1), (H_2) hold, then Problem* **E** *has a unique solution, and the following estimate is true*

$$\|\psi(t)\|_{H^2(\Omega)} \le C\Big(\sum_{j=1}^{s} \|w_j(t)\|_H + 1\Big) \quad \text{for a.e.} \quad t \in (0,T) \quad \text{with} \quad C > 0. \tag{4.7}$$

Proof. The existence for a.e. $t \in (0,T)$ of a unique weak solution $\psi = \psi(t,\cdot) \in V$ follows from Theorem 3.1. Then, using Theorem 3.2 we have for a.e. $t \in (0,T)$ that $\psi(t,\cdot) \in H^2(\Omega)$ and (4.7) holds. Note that ψ depends measurably on t, because of measurability of w_j on t, linearity of (4.1) and the estimate (4.7). Hence,

$$\int_0^T \|\psi(t)\|^2_{H^2(\Omega)}\, dt \le C\Big(\sum_{j=1}^{s} \int_0^T \|w_j(t)\|^2_H\, dt + T\Big) < \infty \tag{4.8}$$

and in consequence $\psi \in L^2(0,T;H^2(\Omega))$. □

Remark 4.1. Let $r \in (\frac{1}{2}, 1]$. It follows from [11, Theorems 2.81] that the trace $T : H^r(\Omega) \to H^{r-\frac{1}{2}}(\partial\Omega)$ is linear, continuous and

$$\|Tu\|_{H^{r-\frac{1}{2}}(\partial\Omega)} \le C\|u\|_{H^r(\Omega)}, \quad u \in H^r(\Omega).$$

Moreover, by [11, Theorems 2.80], the embedding $H^{r-\frac{1}{2}}(\partial\Omega) \subset L^2(\partial\Omega)$ is compact. In consequence $\tilde{T} : H^r(\Omega) \to L^2(\partial\Omega)$, $\tilde{T}u := i(Tu)$ for $u \in H^r(\Omega)$, where i means the identity on $H^{r-\frac{1}{2}}(\partial\Omega)$, is the linear continuous trace and

$$\|\tilde{T}u\|_{L^2(\partial\Omega)} \le C\|u\|_{H^r(\Omega)}, \quad u \in H^r(\Omega). \tag{4.9}$$

Lemma 4.2. *If assumption* (H_0) *holds, then Problem* **P** *has a unique solution.*

Proof. Note that $\nabla\psi \in L^2(0,T;L^6(\Omega))$, because $\nabla\psi \in L^2(0,T;V)$ and $V \subset L^6(\Omega)$ continuously. Moreover, it follows from Remark 4.1 that $w_i(\cdot) \in L^2(\partial\Omega)$. The proof of the existence and uniqueness of solutions to the linear problem is standard and can be done for example by the Galerkin method. The proof that uses the Galerkin method follows the steps of the proof of [13, Theorem 11.7] and [50, Theorem 23.A, Proposition 23.28]. $\square$

Lemma 4.3. *If Assumption* **H** *holds, then there exists* $T > 0$ *such that* $\Lambda : B(T, Q_0, Q_1, Q_2, Q_3) \to B(T, Q_0, Q_1, Q_2, Q_3)$ *for certain* $Q_0, Q_1, Q_2, Q_3 > 0$.

Proof. Let $w \in B(T, Q_0, Q_1, Q_2, Q_3)$, where Q_0, Q_3 are arbitrary, and the choice of T, Q_1, Q_2 will be specified later. Denote $\psi = \Lambda_E(w)$ and $c = \Lambda_P(w,\psi)$. We will derive *a priori* estimates for Problem **P**. Taking $\eta_i = c_i(t)$ in (4.2) we get

$$\frac{1}{2}\frac{d}{dt}\|c_i(t)\|_H^2 + D_i\|\nabla c_i(t)\|_H^2 + D_i\alpha z_i \int_\Omega c_i(t)\nabla\psi(t)\circ\nabla c_i(t)\,dx \\ = \int_{\partial_1\Omega}\big(a_{1i} - b_{1i}w_i(t)\big)c_i(t)\,d\sigma + \int_{\partial_2\Omega}\big(a_{2i} - b_{2i}w_i(t)\big)c_i(t)\,d\sigma, \tag{4.10}$$

for a.e. $t \in (0,T)$. We will prove, using the equation (4.10), the estimates $\|c_i\|^2_{L^2(0,T;H)} \le Q_0$, $\|\nabla c_i\|^2_{L^2(0,T;H)} \le Q_1$, $\|\partial_t c_i\|^2_{L^2(0,T;V^*)} \le Q_2$ and $\|c_i\|^4_{L^4(0,T;H)} \le Q_3$. The term $D_i\alpha z_i \cdot \int_\Omega |c_i(t)\nabla\psi(t)\circ\nabla c_i(t)|\,dx$ can be estimated using the continuous imbeddings $H^2(\Omega) \subset V \subset L^6(\Omega)$, the Hölder inequality, the Sobolev approximation inequality $\|u\|_{L^3(\Omega)} \le C\|u\|_V^{\frac{1}{2}}\|u\|_H^{\frac{1}{2}}$ for $u \in V$, the Young with ε inequality for $p = \frac{4}{3}$, $q = 4$ and Lemma 4.1, as below

$$\begin{aligned} D_i\alpha|z_i| \int_\Omega \big|c_i(t)\nabla\psi(t)\circ\nabla c_i(t)\big|\,dx &\le D_i\alpha|z_i|\|\nabla c_i(t)\|_H\|c_i(t)\|_{L^3(\Omega)}\|\nabla\psi(t)\|_{L^6(\Omega)} \\ &\le C\|\nabla c_i(t)\|_H\big(\|\nabla c_i(t)\|_H^2 + \|c_i(t)\|_H^2\big)^{\frac{1}{4}}\|c_i(t)\|_H^{\frac{1}{2}}\Big(\sum_{j=1}^s\|w_j(t)\|_H + 1\Big) \\ &\le C\big(\|\nabla c_i(t)\|_H^2 + \|c_i(t)\|_H^2\big)^{\frac{3}{4}}\|c_i(t)\|_H^{\frac{1}{2}}\Big(\sum_{j=1}^s\|w_j(t)\|_H + 1\Big) \\ &\le \frac{D_i}{4}\big(\|\nabla c_i(t)\|_H^2 + \|c_i(t)\|_H^2\big) + C\|c_i(t)\|_H^2\Big(\sum_{j=1}^s\|w_j(t)\|_H^4 + 1\Big) \\ &\le \frac{D_i}{4}\|\nabla c_i(t)\|_H^2 + C\Big(\sum_{j=1}^s\|w_j(t)\|_H^4 + 1\Big)\|c_i(t)\|_H^2. \end{aligned} \tag{4.11}$$

To estimate the boundary terms let $r \in (\frac{1}{2}, 1)$. Define $Z = H^r(\Omega)$. By Remark 4.1, the inequality (4.9) is true. Consider the triple of spaces $V \subset Z \subset H$. The embeddings $V \subset Z$ and $Z \subset H$ are compact (see [11, Theorem 2.80]). We can use the Ehrling lemma (see for example [41, Lemma 7.6]) to conclude that for any $\varepsilon > 0$ we can find $C(\varepsilon) > 0$ such that

$$\|u\|_{L^2(\partial\Omega)} \le \varepsilon \|\nabla u\|_H + C(\varepsilon)\|u\|_H, \quad u \in V.$$

Add that the norm $\|u\|_{L^2(\partial\Omega)}$ is understood in the trace sense. We estimate the boundary terms in (4.10). Using the Schwartz inequality, the Cauchy with ε inequality and the above inequality, we have

$$\begin{aligned} &\int_{\partial_1\Omega} \left|\big(a_{1i} - b_{1i} w_i(t)\big) c_i(t)\right| d\sigma + \int_{\partial_2\Omega} \left|\big(a_{2i} - b_{2i} w_i(t)\big) c_i(t)\right| d\sigma \\ &\le \|a_{1i} - b_{1i} w_i(t)\|_{L^2(\partial_1\Omega)} \|c_i(t)\|_{L^2(\partial_1\Omega)} + \|a_{2i} - b_{2i} w_i(t)\|_{L^2(\partial_2\Omega)} \|c_i(t)\|_{L^2(\partial_2\Omega)} \\ &\le C\big(1 + \|w_i(t)\|_{L^2(\partial\Omega)}\big) \|c_i(t)\|_{L^2(\partial\Omega)} \\ &\le \|c_i(t)\|^2_{L^2(\partial\Omega)} + C\|w_i(t)\|^2_{L^2(\partial\Omega)} + C \\ &\le \varepsilon_i \|\nabla c_i(t)\|^2_H + C(\varepsilon_i)\|c_i(t)\|^2_H + \varepsilon_{1i}\|\nabla w_i(t)\|^2_H + C(\varepsilon_{1i})\|w_i(t)\|^2_H + C, \end{aligned} \tag{4.12}$$

where the constants $\varepsilon_i, \varepsilon_{1i} > 0$ are at this point arbitrary and will be specified later. We take $\varepsilon_i = \frac{D_i}{4}$ in the last estimate, and using this estimate together with (4.11) in (4.10) we get

$$\begin{aligned} &\frac{d}{dt}\|c_i(t)\|^2_H + D_i\|\nabla c_i(t)\|^2_H \le \\ &\quad C\Big(\sum_{j=1}^{s} \|w_j(t)\|^4_H + 1\Big)\|c_i(t)\|^2_H + \varepsilon_{1i}\|\nabla w_i(t)\|^2_H + C(\varepsilon_{1i})\|w_i(t)\|^2_H + C. \end{aligned} \tag{4.13}$$

The Gronwall lemma implies that for all $t \in [0, T]$ we have

$$\begin{aligned} \|c_i(t)\|^2_H &\le e^{\int_0^t C(\sum_{j=1}^s \|w_j(\tau)\|^4_H + 1)\, d\tau} \times \\ &\quad \left[\|c_{0i}\|^2_H + \int_0^t \big(\varepsilon_{1i}\|\nabla w_i(\tau)\|^2_H + C(\varepsilon_{1i})\|w_i(\tau)\|^2_H + C\big)\, d\tau\right] \\ &\le e^{CT + C\sum_{j=1}^s \int_0^T \|w_j(\tau)\|^4_H\, d\tau} \times \\ &\quad \left[\|c_{0i}\|^2_H + CT + \int_0^T \big(\varepsilon_{1i}\|\nabla w_i(\tau)\|^2_H + C(\varepsilon_{1i})\|w_i(\tau)\|^2_H\big)\, d\tau\right] \\ &\le e^{C(T + sQ_3)}(\|c_{0i}\|^2_H + CT + C(\varepsilon_{1i})Q_0 + \varepsilon_{1i}Q_1). \end{aligned} \tag{4.14}$$

Integrating over the interval $(0, T)$ we obtain

$$\|c_i\|^2_{L^2(0,T;H)} \le CTe^{C(T+sQ_3)}(\|c_{0i}\|^2_H + T + C(\varepsilon_{1i})Q_0 + \varepsilon_{1i}Q_1). \tag{4.15}$$

Integrating (4.13) from 0 to T we get

$$D_i \int_0^T \|\nabla c_i(t)\|_H^2 \, dt \leq C \|c_i\|^2_{L^\infty(0,T;H)} \int_0^T \Big(\sum_{j=1}^s \|w_j(t)\|_H^4 + 1\Big) dt$$
$$+ CT + \int_0^T \big(\varepsilon_{1i}\|\nabla w_i(t)\|_H^2 + C(\varepsilon_{1i})\|w_i(t)\|_H^2\big)\, dt + \|c_{0i}\|_H^2.$$

Using (4.14) in the last inequality we get after cumbersome but straightforward computation

$$\begin{aligned}
&\|\nabla c_i\|^2_{L^2(0,T;H)} \\
&\leq C\big(\|c_{0i}\|_H^2 + T + C(\varepsilon_{1i})Q_0 + \varepsilon_{1i}Q_1\big)\big(1 + (sQ_3 + T)e^{C(T+sQ_3)}\big) \\
&\leq \varepsilon_{1i}Q_1 C\big(1 + (sQ_3+T)e^{C(T+sQ_3)}\big) \\
&\quad + C\big(\|c_{0i}\|_H^2 + T + C(\varepsilon_{1i})Q_0\big)\big(1 + (sQ_3+T)e^{C(T+sQ_3)}\big).
\end{aligned}$$

Without loss of generality we may assume that $T \leq 1$, thus

$$\begin{aligned}
\|\nabla c_i\|^2_{L^2(0,T;H)} &\leq \varepsilon_{1i}Q_1 C\big(1 + (sQ_3+1)e^{C(1+sQ_3)}\big) \\
&\quad + C\big(\|c_{i0}\|_H^2 + 1 + C(\varepsilon_{1i})Q_0\big)\big(1 + (sQ_3+1)e^{C(1+sQ_3)}\big).
\end{aligned} \tag{4.16}$$

Let $Q_0, Q_3 > 0$ be fixed. We put

$$\varepsilon_{1i} = \frac{1}{2C(1 + (sQ_3+1)e^{C(1+sQ_3)})}$$

and

$$Q_1 = 2C\big(\|c_{0i}\|_H^2 + 1 + C(\varepsilon_{1i})Q_0\big)\big(1 + (sQ_3+1)e^{C(1+sQ_3)}\big).$$

The inequality (4.16) yields $\|\nabla c_i\|^2_{L^2(0,T;H)} \leq Q_1$. From (4.15), assuming that $T \leq 1$, we get

$$\|c_i\|^2_{L^2(0,T;H)} \leq CTe^{C(1+sQ_3)}(\|c_{0i}\|_H^2 + 1 + C(\varepsilon_{1i})Q_0 + \varepsilon_{1i}Q_1) = TF(Q_0, Q_3),$$
$$\|c_i\|^4_{L^4(0,T;H)} \leq TF^2(Q_0, Q_3),$$

where $F(Q_0, Q_3)$ depends only on Q_0, Q_3, but not on T. Hence if we take $T = \min\{1, \frac{Q_0}{F(Q_0,Q_3)}, \frac{Q_3}{F^2(Q_0,Q_3)}\}$, we obtain $\|c_i\|^2_{L^2(0,T;H)} \leq Q_0$ and $\|c_i\|^4_{L^4(0,T;H)} \leq Q_3$. It remains to obtain the estimate for $\partial_t c_i$. For any $\eta_i \in L^2(0,T;V)$ using the continuous imbeddings $H^2(\Omega) \subset V \subset L^6(\Omega)$, $V \subset L^2(\partial\Omega)$, the Hölder inequality, the Sobolev approximation inequality $\|u\|_{L^3(\Omega)} \leq C\|u\|_V^{\frac{1}{2}}\|u\|_H^{\frac{1}{2}}$ for $u \in V$, the Cauchy inequality and Lemma 4.1, we have the following estimates

$$D_i \int_0^T \int_\Omega \left|\nabla c_i(t) \circ \nabla \eta_i(t)\right| dxdt \le D_i \int_0^T \|\nabla c_i(t)\|_H \|\nabla \eta_i(t)\|_H dt \tag{4.17}$$
$$\le D_i \|\nabla c_i\|_{L^2(0,T;H)} \|\eta_i\|_{L^2(0,T;V)},$$

$$D_i \alpha |z_i| \int_0^T \int_\Omega \left|c_i(t) \nabla \psi(t) \circ \nabla \eta_i(t)\right| dxdt \tag{4.18}$$
$$\le D_i \alpha |z_i| \int_0^T \|c_i(t)\|_{L^3(\Omega)} \|\nabla \psi(t)\|_{L^6(\Omega)} \|\nabla \eta_i(t)\|_H \, dt$$
$$\le C \int_0^T \left(\|\nabla c_i(t)\|_H^2 + \|c_i(t)\|_H^2\right)^{\frac{1}{4}} \|c_i(t)\|_H^{\frac{1}{2}} \Big(\sum_{j=1}^s \|w_j(t)\|_H + 1\Big) \|\nabla \eta_i(t)\|_H \, dt$$
$$\le C \|c_i\|_{L^\infty(0,T;H)}^{\frac{1}{2}} \int_0^T \Big(\|c_i(t)\|_V + \sum_{j=1}^s \|w_j(t)\|_H^2 + 1\Big) \|\nabla \eta_i(t)\|_H \, dt$$
$$\le C \|c_i\|_{L^\infty(0,T;H)}^{\frac{1}{2}} \Big(\|c_i\|_{L^2(0,T;V)} + \sum_{j=1}^s \|w_j\|_{L^4(0,T;H)}^2 + T^{\frac{1}{2}}\Big) \|\eta_i\|_{L^2(0,T;V)},$$

$$\int_0^T \int_{\partial_1\Omega} \left|\big(a_{1i} - b_{1i} w_i(t)\big)\eta_i(t)\right| d\sigma \, dt + \int_0^T \int_{\partial_2\Omega} \left|\big(a_{2i} - b_{2i} w_i(t)\big)\eta_i(t)\right| d\sigma \, dt \tag{4.19}$$
$$\le \int_0^T \|a_{1i} - b_{1i} w_i(t)\|_{L^2(\partial_1\Omega)} \|\eta_i(t)\|_{L^2(\partial_1\Omega)} \, dt$$
$$+ \int_0^T \|a_{2i} - b_{2i} w_i(t)\|_{L^2(\partial_2\Omega)} \|\eta_i(t)\|_{L^2(\partial_2\Omega)} \, dt$$
$$\le C \int_0^T \big(1 + \|w_i(t)\|_{L^2(\partial\Omega)}\big) \|\eta_i(t)\|_{L^2(\partial\Omega)} \, dt$$
$$\le C\big(T^{\frac{1}{2}} + \|w_i\|_{L^2(0,T;V)}\big) \|\eta_i\|_{L^2(0,T;V)}.$$

The inequalities (4.17)–(4.19) imply

$$\|\partial_t c_i\|_{L^2(0,T;V^*)} \le D_i \|\nabla c_i\|_{L^2(0,T;H)} + C \|c_i\|_{L^\infty(0,T;H)}^{\frac{1}{2}} \Big(\|c_i\|_{L^2(0,T;V)}$$
$$+ \sum_{j=1}^s \|w_j\|_{L^4(0,T;H)}^2 + T^{\frac{1}{2}}\Big) + C\big(T^{\frac{1}{2}} + \|w_i\|_{L^2(0,T;V)}\big).$$

Taking into account (4.14) to estimate the term $\|c_i\|_{L^\infty(0,T;H)}$, we can write

$$\|\partial_t c_i\|_{L^2(0,T;V^*)} \leq G(T, Q_0, Q_3, Q_1),$$

with the constant $G(T, Q_0, Q_3, Q_1)$. It is enough to take $Q_2 = G^2(T, Q_0, Q_3, Q_1)$. The proof is complete. □

Remark 4.2. Without the use of the Ehrling lemma, the proof would still be possible with additional not physical bounds on the constants present in the model.

We will denote $B = B(T, Q_0, Q_1, Q_2, Q_3)$ found in Lemma 4.3.

Lemma 4.4. *If Assumption* **H** *holds, then the mapping* $\Lambda : B \to B$ *is sequentially weakly continuous.*

Proof. Consider sequences $w_{in} \to w_i$ weakly in $L^2(0, T; V)$ with $(\partial_t w_{in}) \to \partial_t w_i$ weakly in $L^2(0, T; V^*)$ such that $w_n = (w_{1n}, ..., w_{sn})$, $w = (w_1, ..., w_s) \in B$. Let $c_n = \Lambda(w_n)$ and $\psi_n = \Lambda_E(w_n)$, $c_n = (c_{1n}, ..., c_{sn})$. We must prove that $c_{in} \to c_i$ weakly in $L^2(0, T; V)$ and $\partial_t c_{in} \to \partial_t c_i$ weakly in $L^2(0, T; V^*)$ for $c = \Lambda(w)$, $c = (c_1, ..., c_s)$. As $c_n \in B$, a bounded set in X_T, for a subsequence, not renumbered we must have $c_{in} \to c_i$ weakly in $L^2(0, T; V)$. Moreover $\partial_t c_{in} \to \bar{c_i}$ weakly in $L^2(0, T; V^*)$ where it must be $\bar{c_i} = \partial_t c_i$. If we are able to show that $c = \Lambda(w)$ then, by the uniqueness of the limit, the convergence will hold for the whole sequence and the proof will be complete. For any $\eta_i, \xi \in L^2(0, T; V)$ we have

$$\int_0^T \langle \partial_t c_{in}(t), \eta_i(t)\rangle_{V^*\times V}\, dt + \int_0^T \int_\Omega D_i\big(\nabla c_{in}(t) + \alpha z_i c_{in}(t)\nabla\psi_n(t)\big)\circ\nabla\eta_i(t)\, dxdt$$
$$= \int_0^T \int_{\partial_1\Omega} \big(a_{1i} - b_{1i} w_{in}(t)\big)\eta_i(t)\, d\sigma dt + \int_0^T \int_{\partial_2\Omega} \big(a_{2i} - b_{2i} w_{in}(t)\big)\eta_i(t)\, d\sigma dt, \tag{4.20}$$

$$\int_0^T \int_\Omega \nabla\psi_n(t) \circ \nabla\xi(t)\, dxdt + \int_0^T \int_{\partial\Omega} \frac{b(t)}{a(t)}\, \psi_n(t)\, \xi(t)\, d\sigma dt \tag{4.21}$$
$$= \lambda \sum_{j=1}^s \int_0^T \int_\Omega z_j w_{jn}(t)\xi(t)\, dxdt + \int_0^T \int_{\partial\Omega} \frac{h(t)}{a(t)}\, \xi(t)\, d\sigma dt.$$

From Lemma 4.1, we obtain the estimate

$$\|\psi_n\|_{L^2(0,T;H^2(\Omega))} \leq C\Big(\sum_{j=1}^s \|w_{jn}\|_{L^2(0,T;H)} + T^{\frac{1}{2}}\Big).$$

It follows that, for a subsequence, denoted again by n, we must have $\psi_n \to \psi$ weakly in $L^2(0, T; H^2(\Omega))$. We can pass to the limit in (4.21) which implies that

$$\int_0^T \int_\Omega \nabla\psi(t) \circ \nabla\xi(t)\,dxdt + \int_0^T \int_{\partial\Omega} \frac{b(t)}{a(t)}\,\psi(t)\,\xi(t)\,d\sigma dt$$
$$= \lambda \sum_{j=1}^{s} \int_0^T \int_\Omega z_j w_j(t)\xi(t)\,dxdt + \int_0^T \int_{\partial\Omega} \frac{h(t)}{a(t)}\,\xi(t)\,d\sigma dt,$$

whence $\psi = \Lambda_E(w)$.

We need to pass to the limit in (4.20). Note that the space $L^2(0,T;H^2(\Omega))$ is dense in $L^2(0,T;V)$. Hence there are sequences $\eta_{ik} \in L^2(0,T;H^2(\Omega))$, $\eta_{ik} \to \eta_i$ as $k \to \infty$ strongly in $L^2(0,T;V)$. We can write for any η_{ik}, by (4.20)

$$\int_0^T \langle \partial_t c_{in}(t), \eta_{ik}(t)\rangle_{V^*\times V}\,dt + \int_0^T \int_\Omega D_i\big(\nabla c_{in}(t) + \alpha z_i c_{in}(t)\nabla\psi_n(t)\big)\circ\nabla\eta_{ik}(t)\,dxdt$$
$$= \int_0^T \int_{\partial_1\Omega} \big(a_{1i} - b_{1i}w_{in}(t)\big)\eta_{ik}(t)\,d\sigma dt + \int_0^T \int_{\partial_2\Omega} \big(a_{2i} - b_{2i}w_{in}(t)\big)\eta_{ik}(t)\,d\sigma dt. \tag{4.22}$$

Passing to the limit with n and k in the terms with the time derivative and $D_i\nabla c_{in}(t) \circ\nabla\eta_{ik}(t)$ is clear and we omit it.

We will pass to the limit in the term with $\nabla\psi_n$ in (4.22). Let k be arbitrarily fixed. We have

$$\int_0^T \int_\Omega \big(c_{in}(t)\nabla\psi_n(t) - c_i(t)\nabla\psi(t)\big)\circ\nabla\eta_{ik}(t)\,dxdt \tag{4.23}$$
$$= \int_0^T \int_\Omega \big(c_{in}(t) - c_i(t)\big)\nabla\psi_n(t) \circ \nabla\eta_{ik}(t)\,dxdt$$
$$+ \int_0^T \int_\Omega c_i(t)\big(\nabla\psi_n(t) - \nabla\psi(t)\big)\circ\nabla\eta_{ik}(t)\,dxdt.$$

We use Lemma 4.1 once again which gives the estimate

$$\|\psi_n\|_{L^\infty(0,T;H^2(\Omega))} \le C\Big(\sum_{j=1}^{s} \|w_{jn}\|_{L^\infty(0,T;H)} + 1\Big). \tag{4.24}$$

Hence $\psi_n \to \psi$ weakly in $L^\infty(0,T;H^2(\Omega))$. Observe that

$$L^\infty(0,T;H^2(\Omega)) \ni \psi \to \int_0^T \int_\Omega c_i(t)\nabla\psi(t) \circ \nabla\eta_{ik}(t)\,dxdt \in \mathbb{R}$$

defines a linear and continuous functional. Indeed, using the continuous imbeddings $H^2(\Omega) \subset V \subset L^6(\Omega)$, $V \subset L^2(\partial\Omega)$, the Hölder inequality, the Sobolev approximation inequality $\|u\|_{L^3(\Omega)} \le C\|u\|_V^{\frac{1}{2}}\|u\|_H^{\frac{1}{2}}$ for $u \in V$ and the Cauchy inequality, we obtain

$$\begin{aligned}\int_0^T\int_\Omega \left|c_i(t)\nabla\psi(t)\circ\nabla\eta_{ik}(t)\right| dxdt &\le \int_0^T \|c_i(t)\|_{L^3(\Omega)}\|\nabla\psi(t)\|_{L^6(\Omega)}\|\nabla\eta_{ik}(t)\|_H\, dt\\
&\le C\int_0^T \left(\|\nabla c_i(t)\|_H^2 + \|c_i(t)\|_H^2\right)^{\frac{1}{4}}\|c_i(t)\|_H^{\frac{1}{2}}\|\nabla\psi(t)\|_V\|\nabla\eta_{ik}(t)\|_H\, dt\\
&\le C\|c_i\|^{\frac{1}{2}}_{L^\infty(0,T;H)}\int_0^T \left(1+\|c_i(t)\|_V\right)\|\eta_{ik}(t)\|_V\, dt\, \|\psi\|_{L^\infty(0,T;H^2(\Omega))}\\
&\le C\|c_i\|^{\frac{1}{2}}_{L^\infty(0,T;H)}\left(1+\|c_i\|_{L^2(0,T;V)}\right)\|\eta_{ik}\|_{L^2(0,T;V)}\|\psi\|_{L^\infty(0,T;H^2(\Omega))}. \qquad (4.25)\end{aligned}$$

It follows that

$$\int_0^T\int_\Omega c_i(t)\left(\nabla\psi_n(t) - \nabla\psi(t)\right)\circ\nabla\eta_{ik}(t)\, dxdt \to 0, \quad n\to\infty.$$

By the Aubin–Lions compactness theorem we have $c_{in} \to c_i$ strongly in $L^2(0,T;H)$. We remind that ψ_n is bounded in $L^\infty(0,T;H^2(\Omega))$, by (4.24). We have

$$\begin{aligned}&\int_0^T\int_\Omega \left|\left(c_{in}(t)-c_i(t)\right)\nabla\psi_n(t)\circ\nabla\eta_{ik}(t)\right| dxdt\\
&\quad\le \int_0^T \|c_{in}(t)-c_i(t)\|_H\|\nabla\psi_n(t)\circ\nabla\eta_{ik}(t)\|_H\, dt\\
&\quad\le \int_0^T \|c_{in}(t)-c_i(t)\|_H\|\nabla\psi_n(t)\|_{L^4(\Omega)}\|\nabla\eta_{ik}(t)\|_{L^4(\Omega)}\, dt\\
&\quad\le \|\psi_n\|_{L^\infty(0,T,H^2(\Omega))}\int_0^T \|c_{in}(t)-c_i(t)\|_H\|\eta_{ik}(t)\|_{H^2(\Omega)}\, dt\\
&\quad\le \|\psi_n\|_{L^\infty(0,T,H^2(\Omega))}\|c_{in}-c_i\|_{L^2(0,T;H)}\|\eta_{ik}\|_{L^2(0,T;H^2(\Omega))},\end{aligned}$$

whence

$$\int_0^T \int_\Omega \big(c_{in}(t) - c_i(t)\big)\nabla\psi_n(t) \circ \nabla\eta_{ik}(t)\,dxdt \to 0, \quad n \to \infty.$$

It follows that the integral on the left-hand side of (4.23) also converges to zero and in consequence

$$\int_0^T \int_\Omega D_i \alpha z_i c_{in}(t)\nabla\psi_n(t) \circ \nabla\eta_{ik}(t)\,dxdt \to \int_0^T \int_\Omega D_i \alpha z_i c_i(t)\nabla\psi(t) \circ \nabla\eta_{ik}(t)\,dxdt,$$

$n \to \infty$ in (4.22). To pass to the limit with k note that the functional

$$L^2(0,T;V) \ni \eta \to \int_0^T \int_\Omega c_i(t)\nabla\psi(t) \circ \nabla\eta(t)\,dxdt \in \mathbb{R} \tag{4.26}$$

is linear and continuous. It is implied immediately by the estimate (4.25) with η instead of η_{ik}. It is enough to take into account that $\psi \in L^\infty(0,T;H^2(\Omega))$.

Now we will pass to the limit in the boundary terms in (4.22). Consider the integral on $\partial_1\Omega$ only. Here passing to the limit follows from linearity of the functionals

$$L^2(0,T;V) \ni w \to \int_0^T \int_{\partial_1\Omega} b_{1i} w(t)\eta_{ik}(t)\,d\sigma dt \in \mathbb{R},$$

$$L^2(0,T;V) \ni \eta \to \int_0^T \int_{\partial_1\Omega} \big(a_{1i} - b_{1i} w_i(t)\big)\eta(t)\,d\sigma dt \in \mathbb{R}$$

and their continuity implied by the inequalities

$$\int_0^T \int_{\partial_1\Omega} \big|b_{1i} w(t)\eta_{ik}(t)\big|\,d\sigma dt \le C\|\eta_{ik}\|_{L^2(0,T;V)}\|w\|_{L^2(0,T;V)},$$

$$\int_0^T \int_{\partial_1\Omega} \big|\big(a_{1i} - b_{1i} w_i(t)\big)\eta(t)\big|\,d\sigma dt \le C\big(1 + \|w_i\|_{L^2(0,T;V)}\big)\|\eta\|_{L^2(0,T;V)}.$$

The proof is complete. □

Theorem 4.2. *If Assumption* **H** *holds, then there exists $T > 0$ such that Problem* **PE** *has a solution.*

Proof. The assertion follows immediately by Theorem 4.1 and Lemmas 4.1, 4.2, 4.3, 4.4. □

5. Uniqueness of weak solutions

In this section we prove that Problem **PE** cannot have more than one weak solution.

Theorem 5.1. *If Assumption* **H** *holds, then Problem* **PE** *has at most one solution on* $[0, T]$ *for an arbitrary* $T > 0$.

Proof. Suppose that Problem **PE** has two solutions (c_1, φ_1), (c_2, φ_2), $c_j = (c_{j1}, ..., c_{js})$, $j = 1, 2$ on $[0, T]$. We will show that they must be equal. By putting $\eta_i = c_{1i}(t) - c_{2i}(t)$ in (2.4), we get for a.e. $t \in (0, T)$

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|(c_{1i} - c_{2i})(t)\|_H^2 + D_i\|\nabla(c_{1i} - c_{2i})(t)\|_H^2 \\ &\quad + D_i\alpha z_i \int_\Omega c_{1i}(t)\nabla(\varphi_1 - \varphi_2)(t) \circ \nabla(c_{1i} - c_{2i})(t)\, dx \\ &\quad + D_i\alpha z_i \int_\Omega (c_{1i} - c_{2i})(t)\nabla\varphi_2(t) \circ \nabla(c_{1i} - c_{2i})(t)\, dx \\ &= -\int_{\partial_1\Omega} b_{1i}\big((c_{1i} - c_{2i})(t)\big)^2 d\sigma - \int_{\partial_2\Omega} b_{2i}\big((c_{1i} - c_{2i})(t)\big)^2 d\sigma. \end{aligned} \tag{5.1}$$

Note that the equation

$$\int_\Omega \nabla\varphi \circ \nabla\xi\, dx + \int_{\partial\Omega} \frac{b}{a}\varphi\xi\, d\sigma = \lambda \sum_{j=1}^{s} \int_\Omega z_j(c_{1j} - c_{2j})\xi\, dx \quad \text{for each} \quad \xi \in V \tag{5.2}$$

has the unique solution $\varphi = \varphi_1 - \varphi_2 \in L^2(0, T; H^2(\Omega))$ by the same arguments as in the proof of Lemma 4.1, and moreover

$$\|(\varphi_1 - \varphi_2)(t)\|_{H^2(\Omega)} \le C \sum_{j=1}^{s} \|(c_{1j} - c_{2j})(t)\|_H \quad \text{for a.e.} \quad t \in (0, T) \tag{5.3}$$

with $C > 0$. Analogously

$$\|\varphi_2(t)\|_{H^2(\Omega)} \le C \sum_{j=1}^{s} \|c_{2j}(t)\|_H \quad \text{for a.e.} \quad t \in (0, T) \quad \text{with} \quad C > 0. \tag{5.4}$$

By the similar argument as in the proof of Lemma 4.3, using (5.3) and (5.4) we obtain the following estimates of the integral terms in (5.1)

$$D_i\alpha|z_i| \int_\Omega \left|c_{1i}(t)\nabla(\varphi_1 - \varphi_2)(t) \circ \nabla(c_{1i} - c_{2i})(t)\right| dx$$

$$
\begin{aligned}
&\le C\|\nabla(c_{1i}-c_{2i})(t)\|_H\|c_{1i}(t)\|_{L^3(\Omega)}\|\nabla(\varphi_1-\varphi_2)(t)\|_{L^6(\Omega)}\\
&\le C\|\nabla(c_{1i}-c_{2i})(t)\|_H\|c_{1i}(t)\|_V\sum_{j=1}^{s}\|(c_{1j}-c_{2j})(t)\|_H\\
&\le \frac{D_i}{2}\|\nabla(c_{1i}-c_{2i})(t)\|_H^2+C\|c_{1i}(t)\|_V^2\sum_{j=1}^{s}\|(c_{1j}-c_{2j})(t)\|_H^2,
\end{aligned}
$$

$$
\begin{aligned}
&D_i\alpha|z_i|\int_\Omega\left|(c_{1i}-c_{2i})(t)\nabla\varphi_2(t)\circ\nabla(c_{1i}-c_{2i})(t)\right|dx\\
&\le C\|\nabla(c_{1i}-c_{2i})(t)\|_H\big(\|\nabla(c_{1i}-c_{2i})(t)\|_H^2+\|(c_{1i}-c_{2i})(t)\|_H^2\big)^{\frac14}\times\\
&\qquad\|(c_{1i}-c_{2i})(t)\|_H^{\frac12}\sum_{j=1}^{s}\|c_{2j}(t)\|_H\\
&\le C\big(\|\nabla(c_{1i}-c_{2i})(t)\|_H^2+\|(c_{1i}-c_{2i})(t)\|_H^2\big)^{\frac34}\|(c_{1i}-c_{2i})(t)\|_H^{\frac12}\sum_{j=1}^{s}\|c_{2j}(t)\|_H\\
&\le \frac{D_i}{2}\big(\|\nabla(c_{1i}-c_{2i})(t)\|_H^2+\|(c_{1i}-c_{2i})(t)\|_H^2\big)+C\|(c_{1i}-c_{2i})(t)\|_H^2\sum_{j=1}^{s}\|c_{2j}(t)\|_H^4\\
&\le \frac{D_i}{2}\|\nabla(c_{1i}-c_{2i})(t)\|_H^2+C\big(\sum_{j=1}^{s}\|c_{2j}(t)\|_H^4+1\big)\|(c_{1i}-c_{2i})(t)\|_H^2.
\end{aligned}
$$

In consequence

$$
\begin{aligned}
\frac{d}{dt}\|(c_{1i}-c_{2i})(t)\|_H^2 \le\, & C\Big(\|c_{1i}(t)\|_V^2\sum_{j=1}^{s}\|(c_{1j}-c_{2j})(t)\|_H^2\\
&+\big(\sum_{j=1}^{s}\|c_{2j}(t)\|_H^4+1\big)\|(c_{1i}-c_{2i})(t)\|_H^2\Big)
\end{aligned}
$$

and by adding, we get for a.e. $t\in(0,T)$

$$
\frac{d}{dt}\sum_{j=1}^{s}\|(c_{1j}-c_{2j})(t)\|_H^2 \le C\Big(\sum_{j=1}^{s}\big(\|c_{1j}(t)\|_V^2+\|c_{2j}(t)\|_H^4\big)+1\Big)\sum_{j=1}^{s}\|(c_{1j}-c_{2j})(t)\|_H^2. \tag{5.5}
$$

Hence the Gronwall lemma implies that for all $t\in[0,T]$

$$
\sum_{j=1}^{s}\|(c_{1j}-c_{2j})(t)\|_H^2 \le e^{\int_0^T C(\sum_{j=1}^{s}(\|c_{1j}(t)\|_V^2+\|c_{2j}(t)\|_H^4)+1)\,dt}\sum_{j=1}^{s}\|(c_{1j}-c_{2j})(0)\|_H^2. \tag{5.6}
$$

Therefore $c_1 = c_2$ because the right-hand side in (5.6) is equal to zero. It follows from the uniqueness of the solution to (5.2) that $\varphi_1 = \varphi_2$. □

6. Nonnegativity of weak solutions

In this section we prove that, provided the initial conditions c_{0i} are nonnegative for a.e. $x \in \Omega$, the concentrations $c_i(t)$ must also remain nonnegative.

Theorem 6.1. *Let Assumption* **H** *be true and* $c_{0i} \in H^+$. *Then for all* $t \in [0, T]$ *such that the solution of Problem* ***PE*** *exists on the interval* $[0, T]$ *we have* $c_i(t) \in H^+$.

Proof. The proof follows the lines of the corresponding part of the proof of [21, Lemma 4.1], [20, Theorem 6.1], see also [4, Proposition 1]. Consider the following auxiliary problem, which differs from Problem **PE** by replacement of c_i with c_i^+, where $c_i^+ = \max\{c_i, 0\}$ in the terms representing electrostatic forces in (2.4).

Problem PE$^+$. Find $c_i \in L^2(0, T; V)$ and $\varphi \in L^2(0, T; H^2(\Omega))$ such that $\partial_t c_i \in L^2(0, T; V^*)$ and for a.e. $t \in (0, T)$

$$\langle \partial_t c_i, \eta_i \rangle_{V^* \times V} + \int_\Omega D_i \big(\nabla c_i + \alpha z_i c_i^+ \nabla \varphi\big) \circ \nabla \eta_i \, dx \tag{6.1}$$
$$= \int_{\partial_1 \Omega} \big(a_{1i} - b_{1i} c_i\big) \eta_i \, d\sigma + \int_{\partial_2 \Omega} \big(a_{2i} - b_{2i} c_i\big) \eta_i \, d\sigma \quad \text{for each} \quad \eta_i \in V,$$

$$\int_\Omega \nabla \varphi \circ \nabla \xi \, dx + \int_{\partial\Omega} \frac{b}{a} \varphi \, \xi \, d\sigma = \lambda \sum_{j=1}^{s} \int_\Omega z_j c_j \xi \, dx + \int_{\partial\Omega} \frac{h}{a} \xi \, d\sigma \quad \text{for each} \quad \xi \in V, \tag{6.2}$$

and the initial condition (2.2) holds.

By the same argument as in the proof of Theorem 4.2 based on the Schauder–Tychonoff fixed point theorem it follows that Problem **PE$^+$** has a local in time weak solution $(\bar{c}, \bar{\varphi})$. We will prove that each (c, φ) solves Problem **PE$^+$** on a certain interval $[0, T]$ if and only if it solves Problem **PE** on this interval.

Indeed, assume that (c, φ) solves Problem **PE$^+$** on $[0, T]$. We will prove that $c_i(t) \in H^+$ for all $t \in [0, T]$. Taking $\eta_i = c_i^-(t) = -\min\{c_i(t), 0\}$ in (6.1) we get

$$-\frac{1}{2}\frac{d}{dt} \|c_i^-(t)\|_H^2 - D_i \|\nabla c_i^-(t)\|_H^2$$
$$= a_{1i} \|c_i^-(t)\|_{L^1(\partial_1\Omega)} + b_{1i} \|c_i^-(t)\|_{L^2(\partial_1\Omega)}^2 + a_{2i} \|c_i^-(t)\|_{L^1(\partial_2\Omega)} + b_{2i} \|c_i^-(t)\|_{L^2(\partial_2\Omega)}^2$$

for a.e. $t \in (0, T)$. Hence

$$\frac{1}{2}\frac{d}{dt} \|c_i^-(t)\|_H^2 + D_i \|\nabla c_i^-(t)\|_H^2 \le 0$$

for a.e. $t \in (0, T)$. After integration on the interval $(0, t)$ for $t \in [0, T]$ we get

$$\frac{1}{2}\|c_i^-(t)\|_H^2 + D_i \int_0^t \|\nabla c_i^-(s)\|_H^2\,ds \le \frac{1}{2}\|c_i^-(0)\|_H^2$$

for all $t \in [0, T]$, which, in view of $c_{0i} \in H^+$, yields that $c_i^-(t) = 0$ for all $t \in [0, T]$, whereas c satisfies (2.4). Note that we have also proved, that for every solution of Problem $\mathbf{PE}^+$ we must have $c_i(t) \in H^+$ for all t in the interval of solution existence.

Now assume that (c, φ) solves Problem **PE** on an interval $[0, T]$. We know that there exists a solution $(\bar{c}, \bar{\varphi})$ on a certain time interval $[0, T_0]$ of Problem $\mathbf{PE}^+$ and $(\bar{c}, \bar{\varphi})$ must also solve Problem **PE** on this time interval. If $T \le T_0$, then the uniqueness Theorem 5.1 implies that $(\bar{c}, \bar{\varphi}) = (c, \varphi)$ on $[0, T]$ and the assertion is proved. If $T_0 < T$ we will use the barrier method. Indeed, denote by $\bar{T}_0$ the supremum of all times T_0 such that Problem $\mathbf{PE}^+$ has a solution $(\bar{c}, \bar{\varphi})$ in $[0, T_0]$ (it may be that $\bar{T}_0 = +\infty$). If $\bar{T}_0 > T$ we arrive at the previous case $T \le T_0$. We will prove that the case $\bar{T}_0 \le T$ leads to a contradiction. Observe that $(\bar{c}, \bar{\varphi})$ must also solve Problem **PE** on each interval $[0, \bar{T}_0 - \varepsilon]$, which, by the uniqueness of solutions to Problem **PE** implies that $(\bar{c}, \bar{\varphi}) = (c, \varphi)$ on the interval $[0, \bar{T}_0)$. But, as $c_i \in C([0, T]; H)$, the values $\lim_{t \nearrow \bar{T}_0} \|\bar{c}_i(t)\|_H$ are well defined and finite, they are equal to $\|c_i(\bar{T}_0)\|_H$, and hence we can continue the solution $(\bar{c}, \bar{\varphi})$ of Problem $\mathbf{PE}^+$, starting from $\bar{T}_0$, which contradicts its maximality.

As both problems are equivalent, and for the solution of Problem **PE** we must have $c_i(t) \in H^+$ for all t, the assertion is proved. □

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