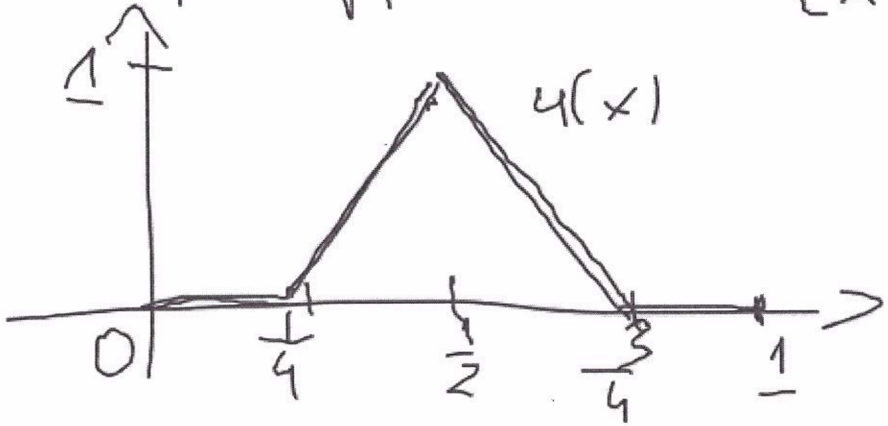
Metody elementów skończonych

$$\mathcal{U} = \{ u : \bar{\Omega} \to R \}, \quad \Omega \subset R^n$$

$$u \in \mathcal{U}, \quad supp(u) := \overline{\{ x \in \bar{\Omega} : u(x) \neq 0 \}}$$



$$\Omega = (0, 1)$$

$$supp(u) = \overline{\left(\frac{1}{4}, \frac{3}{4} \right)} = \left\langle \frac{1}{4}, \frac{3}{4} \right\rangle$$

$\mathcal{U}$, $a : U \times U \to R$ – forma dwuliniowa, symetryczna

U_N, $\{ u^1, \dots, u^N \}$

$$\mu\left(supp(u^i) \cap supp(u^j) \right) = 0 \Rightarrow a(u^i, u^j) = 0$$

$G_i = \{ j : \mu(\operatorname{supp}(u^i) \cap \operatorname{supp}(u^j)) \neq 0 \}$,
$i = 1, \dots, N$

$$\forall j \notin G_i \quad a(u^i, u^j) = 0$$

Def Metodę Galerkina ($U = V$) oraz metodę Ritza nazywamy MES, gdy

$$\exists M \in \mathbb{N} \ \forall N \in \mathbb{N} \ \forall i = 1, \dots, N \quad \# G_i \leq M.$$

$$\Omega = (0,1)$$

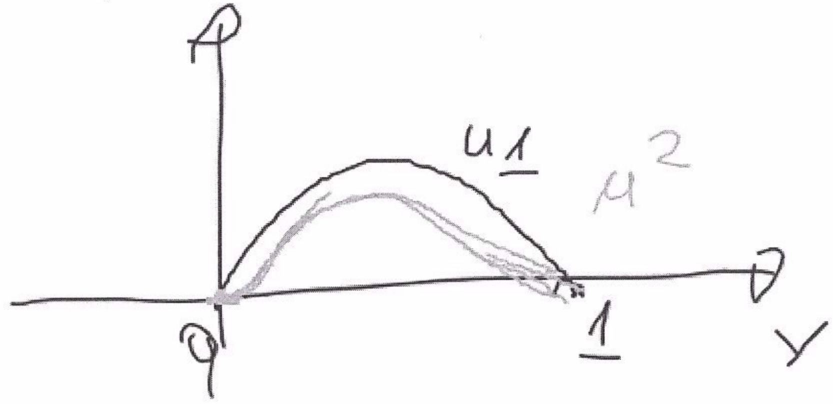
$$\text{I} \quad u^i(x) = (1-x)\,x^i, \quad i = 1, \dots, r$$

np. $r = 3$: $u^1(x) = (1-x)\,x$

$u^2(x) = (1-x)\,x^2$

$u^3(x) = (1-x)\,x^3$

$$u^i \in C^\infty \cap L$$



$U_r = \operatorname{span}(u^1, \dots, u^r)$

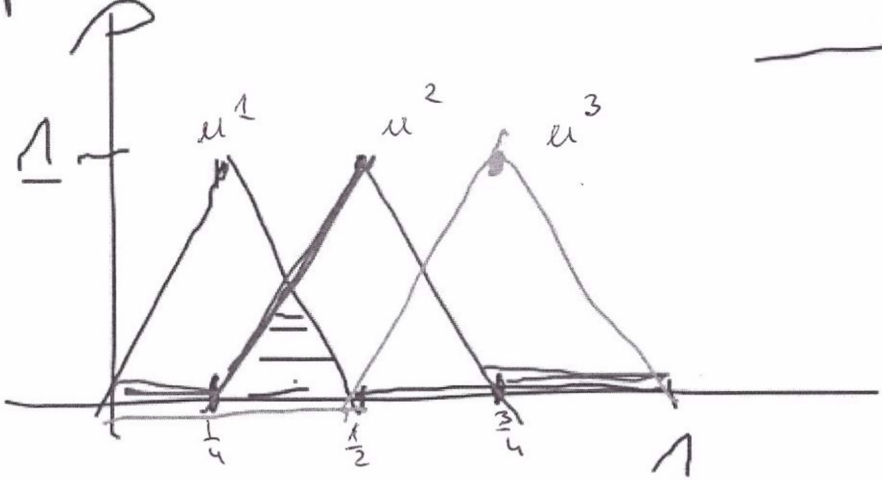
$$\overline{\bigcup_{r=1}^{\infty} U_r} = U$$

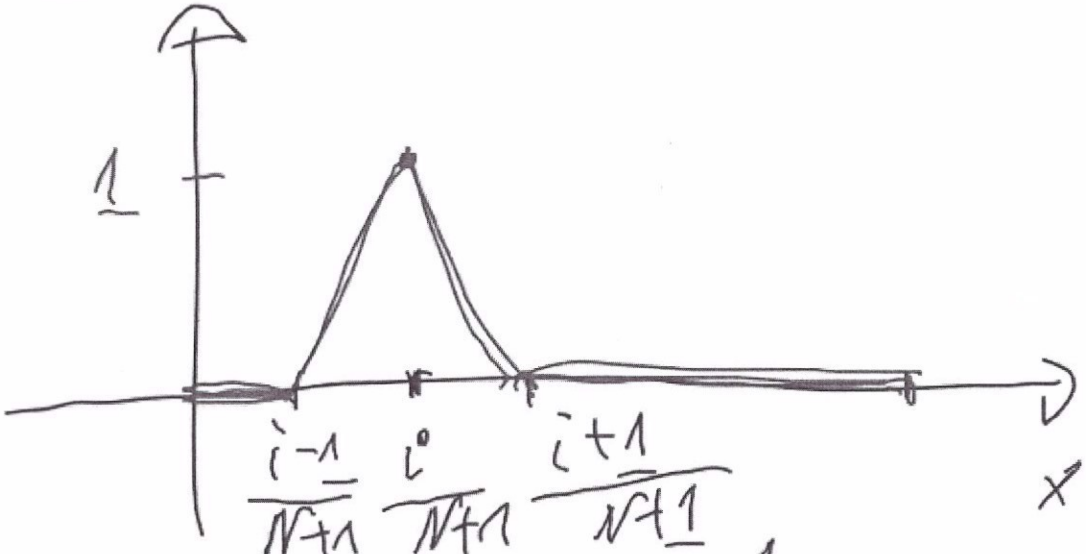
(II) $h = \dfrac{1}{N+1}$

$x_i = ih, \quad i = 0, \dots, N+1$

$u^i, \quad i = 1, \dots, N$

Np. $N = 3$





$u^i \in C^1_I \cap L$

$\mathcal{U}_N = \mathrm{span}(u^1, \dots, u^N)$

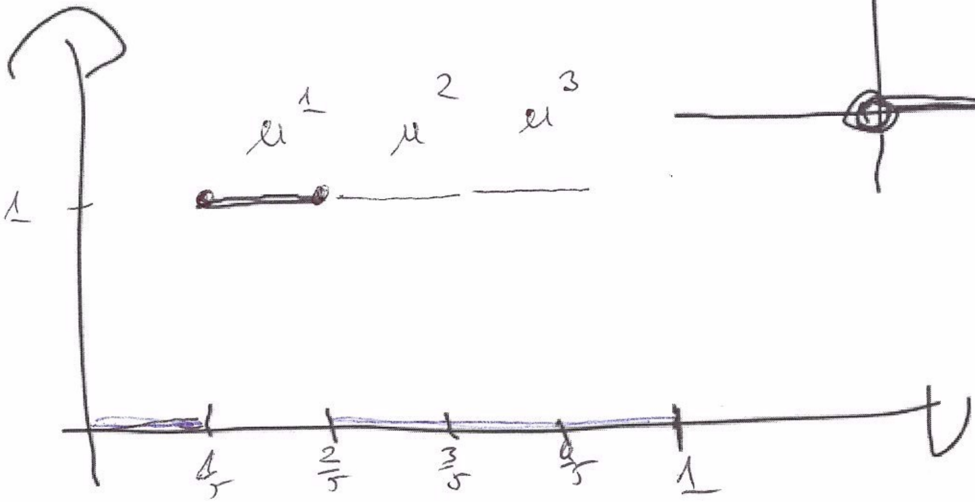
$\bigcup_{N=1}^{\infty} \mathcal{U}_N = \mathcal{U}$

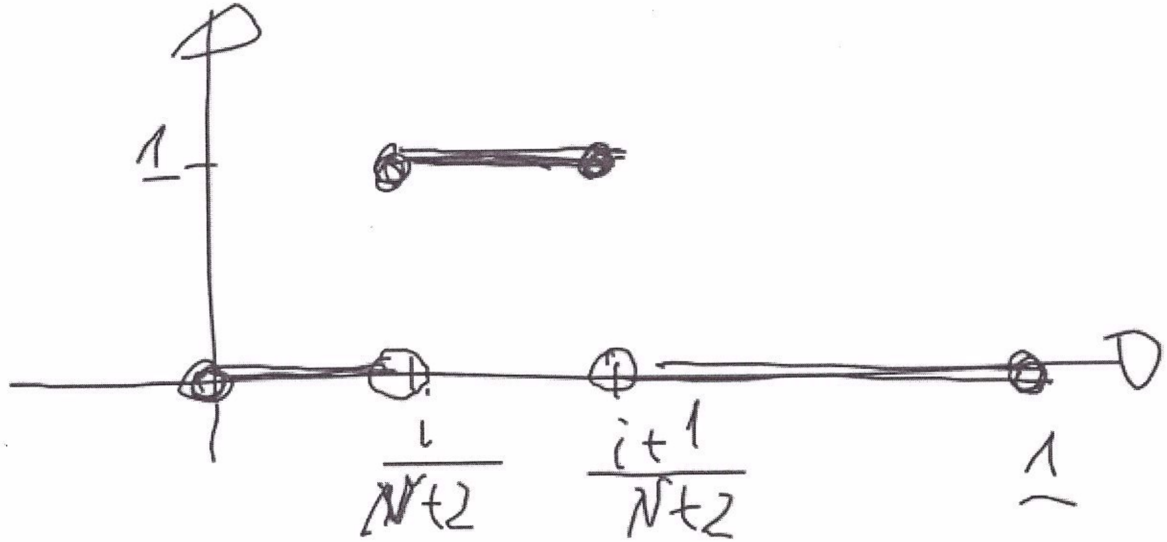
III $h = \frac{1}{N+2}$

$x_i = ih$, $i = 0, \dots, N+2$

$u^i(x)$, $i = 1, \dots, N$

Np. $N = 3$





$u^i \in C^0_I \cap L$

$\underline{U_N = \operatorname{span}(u^1, \dots, u^N)}$

$\bigcup_{N=1}^{\infty} U_N = U$

(*) $\begin{cases} -u'' = f(x) \\ u(0) = u(1) = 0 \end{cases}$, $f \in C^0_I$

$\mathcal{U} = C^1_I \cap L$

$a : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$

$a(u, v) = \int_0^1 u' v' \, dx$

$\varphi(v) = \int_0^1 f(x) v \, dx$

(**) $\boxed{u \in \mathcal{U} \quad \forall v \in \mathcal{U} \quad a(u, v) = \varphi(v)}$

(***) $\boxed{u_r \in \mathcal{U}_r \quad \forall v \in \mathcal{U}_r \quad a(u_r, v) = \varphi(v)}$

$\{u^1, \dots, u^N\}$ – $\frac{\text{II}}{}$

$\mathcal{U}_r = \operatorname{span}(u^1, \dots, u^N)$

$\sum_{j=1}^{r} a(u^j, u^i) c_j = \varphi(u^i)$

$i = 1, \dots, r$

$A = \left[a(u^j, u^i) \right]_{i,j=1}^{N}$

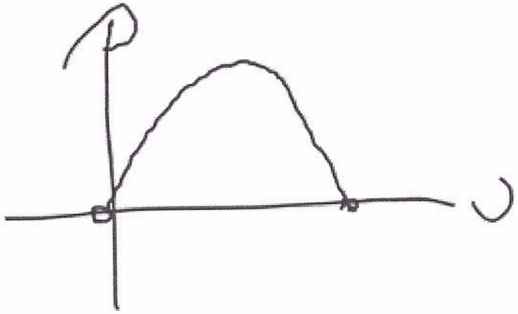
$b = [\varphi(u^1), \dots, \varphi(u^N)]^T$

$A c^T = b$

$u_r = \sum_{i=1}^{r} c_i u^i$

(*) $\begin{cases} -u'' = 1 \\ u(0) = u(1) = 0 \end{cases}$

$$u(x) = -\frac{1}{2}x^2 + \frac{1}{2}x$$



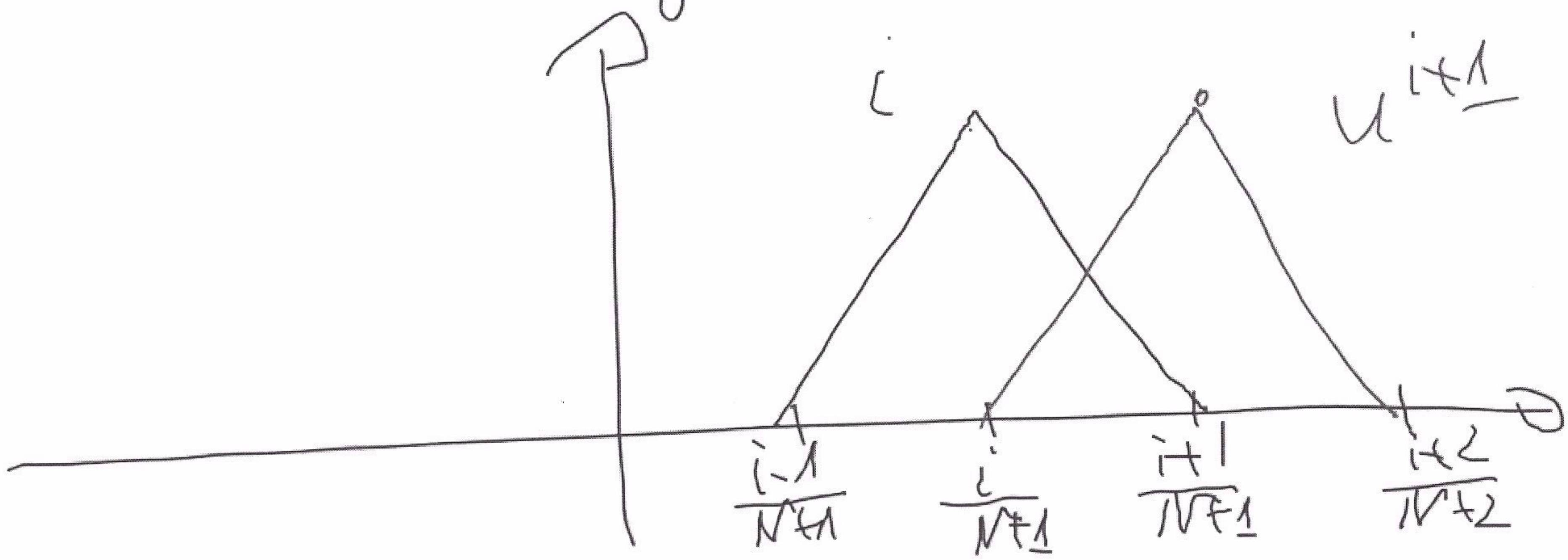
$$b_i = \int_0^1 \varphi^i(x)\,dx = \frac{1}{N+1}, \quad i = 1, \dots, N$$

$$A = \begin{bmatrix} 2(N+1) & -(N+1) & & & 0 \\ -(N+1) & 2(N+1) & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & -(N+1) \\ 0 & & & -(N+1) & 2(N+1) \end{bmatrix}, \quad b = \begin{bmatrix} \frac{1}{N+1} \\ \vdots \\ \vdots \\ \frac{1}{N+1} \end{bmatrix}$$

$$a(u^i, u^i) = \int_0^1 [(u^i)']^2 \, dx = 2(N+1)$$

$$a(u^j, u^i) = \int_0^1 (u^j)'(u^i)' \, dx = \, , \; i \neq j$$

$$= -(N+1)$$



$$\boxed{U := H_0^1(0,1)}$$ – patrz str. 11

$$(u,v) := \int_0^1 u'v'\,dx$$ – iloczyn skalarny

$$a : U \times U \to \mathbb{R}$$

$$a(u,v) = \int_0^1 u'v'\,dx$$

$$\geq \alpha \qquad \underline{\alpha}$$

$$a(v,v) = \int_0^1 (v')^2\,dx = \underline{1} \cdot \|v\|^2_{H_0^1(0,1)}$$

$$\forall v \in H_0^1(0,1)$$

1. U – rzeczywista ośrodkowa przestrzeń Hilberta
2. a – forma dwuliniowa, ograniczona, symetryczna, silnie dodatnio określona
3. φ – forma liniowa, ograniczona

A zatem (**) i (***) mają jedyne rozwiązanie w U. Ponadto metoda Galerkina (***) (i odpowiadająca jej metoda Ritza) jest zbieżna.

$\cancel{\varphi(v)}$ $\varphi : U \to \mathbb{R}$

$$\varphi(v) = \int_0^1 1 \cdot v\,dx$$

Def.

$$f: \mathbb{R} \to \mathbb{R}, \quad x_0 \in \mathbb{R}$$

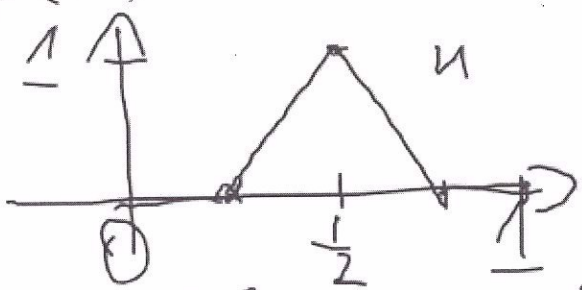
$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h} \in \mathbb{R}$$

Pochodna Sobolewa

$G \subset \mathbb{R}$ – otw.

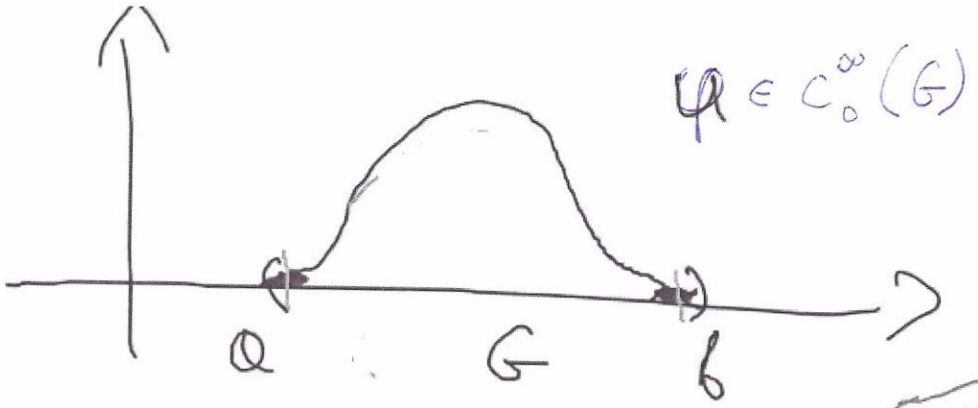
$u: G \to \mathbb{R}$

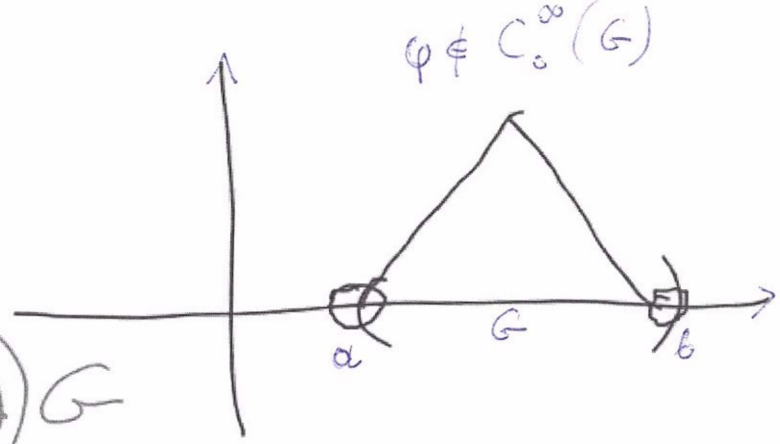
$\operatorname{supp}(u) := \overline{\{x \in G : u(x) \neq 0\}}$ – nośnik u



$$\operatorname{supp}(u) = \overline{\left(\tfrac{1}{4}, \tfrac{3}{4}\right)} = \left\langle \tfrac{1}{4}, \tfrac{3}{4} \right\rangle$$

$$\varphi \in C_0^\infty(G) := \{u: G \to \mathbb{R} : u \in C^\infty(G), \overset{\text{zw.}}{\operatorname{supp}(u)} \subset G\}$$





Def $u, v \in L^1_{loc}(G)$, $G \subset \mathbb{R}^n$

$$\int_G u\varphi'\,dx = -\int_G v\varphi\,dx \quad \forall \varphi \in C_0^\infty(G)$$

$\varphi^{(k)}$ $\quad (-1)^k$

$v := u'$ – słaba pochodna pierwszego rzędu funkcji u w G (poch. Sobolewa)

$u^{(k)}$

$$\cancel{u \in C^1(G) \Rightarrow} \int_G u\varphi'\,dx = -\int_G \underbrace{u'}_{v}\varphi\,dx + \cancel{u\varphi\big|_{\partial G}}$$

Def. $H^1(G) := \{u : u, u' \in L^2(G),\ u' - \text{pochodna Sobolewa}\}$

$H^1_0(G) := \{u \in H^1(G) : Tr(u) = 0\}$, jeśli $n = 1$, to $Tr(u) := u\big|_{\partial G}$.

↑ ślad