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1. Wyznaczyć charakterystykę równania:

   a) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + u = 0$ przechodzącą przez punkt $A_0(2,-3,1)$;

   b) $(x + u) \frac{\partial u}{\partial x} + (y + u) \frac{\partial u}{\partial y} = x + y + 2u$ przechodzącą przez punkt $A_0(1,1,-2)$;

c) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x + y$ przechodzącą przez punkt $A_0(-2,3,1)$;

d) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \sqrt{x^2 + y^2}$ przechodzącą przez punkt $A_0(-3,4,3)$;

e) $y \frac{\partial u}{\partial x} - x \frac{\partial u}{\partial y} - (1 + z^2) \frac{\partial u}{\partial z} = 2zu$ przechodzącą przez punkt $A_0(0,0,-1,1)$

2. Wyznaczyć charakterystyki równania:

a) $y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = 0$;

b) $(x + 2y) \frac{\partial u}{\partial x} - (3x + 4y) \frac{\partial u}{\partial y} = 0$;

c) $(x - 2y) \frac{\partial u}{\partial x} + (x - y) \frac{\partial u}{\partial y} = 0$;

d) $y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} + 2xyu^2 = 0$;

e) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} - \sqrt{x^2 + y^2} = 0$;

f) $y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} + u = 0$;

g) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x + y$;

h) $(x + u) \frac{\partial u}{\partial x} + (y + u) \frac{\partial u}{\partial y} = x + y + 2u$;

i) $(2x - y - u) \frac{\partial u}{\partial x} + (12x - 4y - 12u) \frac{\partial u}{\partial y} = 5u + y - 4x$;

j) $(3x + 12y - 4u) \frac{\partial u}{\partial x} + (u - x - 3y) \frac{\partial u}{\partial y} = 6u - x - 12y$.

3. Wyznaczyć całkę ogólną równania różniczkowego:

a) $xy^2 \frac{\partial u}{\partial x} + x^2 y \frac{\partial u}{\partial y} = (x^2 + y^2)u$;

b) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x^2 + y^2$;

c) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = u$;

d) $y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} + 2xyu^2 = 0$;

e) $(1 + xy)(x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y}) + (x^2 + y^2) u = 0$;

f) $yz \frac{\partial u}{\partial x} + zx \frac{\partial u}{\partial y} + xy \frac{\partial u}{\partial z} + xyz = 0$;

g) $x \frac{\partial u}{\partial y} - y \frac{\partial u}{\partial x} + (1 + z^2) \frac{\partial u}{\partial z} = 2zu$;

h) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = u$;

i) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2u$;

j) $\sqrt{x} \frac{\partial z}{\partial x} + \sqrt{y} \frac{\partial z}{\partial y} = \frac{1}{2}$;

k) $y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} + u = 0$;

l) $x \frac{\partial z}{\partial x} + (y + x^2) \frac{\partial z}{\partial y} = z$;

m) $(z - y) \frac{\partial z}{\partial x} + (x - z) \frac{\partial z}{\partial y} = y - x$;

n) $yz \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = 0$;

o) $x \frac{\partial z}{\partial x} + z \frac{\partial z}{\partial y} = 0$;

p) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{z}{2} \frac{\partial u}{\partial z} = 0$;

q) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = x + y$;

r) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$;

s) $y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = y$;

t) $x \frac{\partial z}{\partial x} - z \frac{\partial z}{\partial y} = 0 \ (x > 0)$;

u) $\frac{\partial u}{\partial x} + (2y - 3z) \frac{\partial u}{\partial y} + (3y + 2z) \frac{\partial u}{\partial z} = 0$;

v) $\frac{\partial u}{\partial x} - (y + 2z) \frac{\partial u}{\partial y} + (3y + 4z) \frac{\partial u}{\partial z} = 0$;

w) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + (z - \sqrt{x^2 + y^2 + z^2}) \frac{\partial u}{\partial z} = 0$;

x) $x (y^2 - z^2) \frac{\partial u}{\partial x} - y (x^2 + z^2) \frac{\partial u}{\partial y} + z (x^2 + y^2) \frac{\partial u}{\partial z} = 0$;

y) $(z - y)^2 \frac{\partial u}{\partial x} + z \frac{\partial u}{\partial y} + y \frac{\partial u}{\partial z} = 0$;

z) $(x^3 + 3xy^2) \frac{\partial u}{\partial x} + 2y^3 \frac{\partial u}{\partial y} + 2y^2 z \frac{\partial u}{\partial z} = 0$

4. Wyznaczyć powierzchnię całkową równania:

a) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = u$, jeżeli $u = \frac{1}{2} (y + z)$ dla $x = 2$.

b) $xy \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$, jeżeli $z = ye^{-2y}$ dla $x = 1$;

c) $x \frac{\partial z}{\partial x} + (y + x^2) \frac{\partial z}{\partial y} = z$, jeżeli $z = y - 4$ dla $x = 2$;

d) $y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = 2xyz$, jeżeli α) $z = \frac{1}{2} x^2 e^{x^2}$ dla $y = 0$;

β) $z = t$ dla $x = 1$, $y = t^2$;

e) $x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = 2z$, jeżeli $z = y$ dla $x = 1$;

f) $xy \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = yz$, jeżeli $x^2 + z^2 = 1$ dla $y = 0$;

g) $z \frac{\partial z}{\partial x} + yz \frac{\partial z}{\partial y} = z$, jeżeli $x^2 - z^2 = 1$ dla $y = 1$;

h) $x^2 \frac{\partial z}{\partial x} + y^2 \frac{\partial z}{\partial y} = z^2$, jeżeli $x^2 + y^2 = 1$ dla $z = 4$;

i) $xy \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} = z$, jeżeli $z = t^2$ dla $x = y = t$;

j) $(z - y) \frac{\partial z}{\partial x} + (x - z) \frac{\partial z}{\partial y} = y - x$, jeżeli $z = 2y$ dla $x = 0$;

k) $y \frac{\partial z}{\partial x} + z \frac{\partial z}{\partial y} = y$, jeżeli $x + z = 0$ dla $x^2 + y^2 = 1$;

l) $x \frac{\partial z}{\partial y} + z \frac{\partial z}{\partial y} = 0$, jeżeli $z = -y$ dla $x = 1$;

m) $y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = 0$, jeżeli $z = y^2$ dla $x = 0$;

n) $(1 + x^2) \frac{\partial z}{\partial x} + xy \frac{\partial z}{\partial y} = 0$, jeżeli $z = y^2$ dla $x = 0$;

o) $yz \frac{\partial z}{\partial x} + x \frac{\partial z}{\partial y} = 0$, jeżeli $z = x^2$ dla $y = 1$;

p) $(2x - y^2) \frac{\partial z}{\partial x} + 2y \frac{\partial z}{\partial y} = 0$, jeżeli $z = \frac{y}{2}$ dla $x = 0$;

q) $x \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$, jeżeli $u = x - y + 1$ dla $z = 1$;

r) $x \frac{\partial u}{\partial x} + yz \frac{\partial u}{\partial y} = 0$, jeżeli $u = x^z$ dla $y = 1$;

s) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + \frac{z}{2} \frac{\partial u}{\partial z} = 0$, jeżeli $u = y + z^2$ dla $x = 1$;

t) $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$, jeżeli $u = y + z$ dla $x = 2$;

u) $(z - y)^2 \frac{\partial u}{\partial x} + z \frac{\partial u}{\partial y} + y \frac{\partial u}{\partial z} = 0$, jeżeli $u = 2y(y - z)$ dla $x = 0$;

v) $xy \frac{\partial u}{\partial x} - y^2 \frac{\partial u}{\partial y} - x(1 + x^2) \frac{\partial u}{\partial z} = 0$, jeżeli $u = 2x^2 + x^4$ dla $y = 4$

Odpowiedzi

1. a) $x = 2e^t$, $y = -3e^t$, $u = e^{-t}$;

b) $x = \frac{4}{3} - \frac{1}{3} e^{3t}$, $y = \frac{4}{3} - \frac{1}{3} e^{3t}$, $u = -\frac{4}{3} - \frac{2}{3} e^{3t}$;

c) $x = -2 e^t$, $y = 3 e^t$, $u = e^t$;

d) $x = -3t$, $y = 4t$, $u = 5t - 2$ dla $t > 0$;

e) $x = 0$, $y = 0$, $z = -\mathrm{tg}(t + \frac{\pi}{4})$, $u = 2\cos^2(t + \frac{\pi}{4})$ dla $t \in (-\frac{3}{4}\pi, \frac{\pi}{4})$.

2. a) $x = C_1 e^t + C_2 e^{-t}$, $y = C_1 e^t - C_2 e^{-t}$, $u = C_3$;

b) $x = C_1 e^t + C_2 e^{2t}$, $y = -C_1 e^t - \frac{3}{2} C_2 e^{2t}$, $u = C_3$;

c) $x = C_1 \cos t + C_2 \sin t$, $y = \frac{1}{2}\left[(C_1 - C_2)\cos t + (C_1 + C_2)\sin t\right]$; $u = C_3$

d) $y^2 - x^2 = C_1$, $u = \frac{1}{x^2 + C_2}$;

e) $x = C_1 e^t$, $y = C_2 e^t$, $u = \sqrt{C_1^2 + C_2^2}\, e^t + C_3$;

f) $x = Ce^t$, $y = C_2 e^t$, $u = C_3 e^{-t}$;

g) $x = C_1 e^t$, $y = C_2 e^t$, $u = (C_1 + C_2) e^t + C_3$;

h) $x = C_1 + C_2 e^t + C_3 e^{3t}$, $y = C_1 - C_2 e^t + C_3 e^{3t}$, $u = -C_1 + 2C_3 e^{3t}$;

i) $x = 2 C_1 + C_2 e^t + C_3 e^{2t}$, $y = 3 C_1 - 2 C_3 e^{2t}$;

$u = C_1 + C_2 e^t + 2 C_3 e^{2t}$;

j) $x = -2\,C_1e^t - 8\,C_2e^{2t} - 3\,C_3e^{3t}, \quad y = C_1e^t + 3\,C_2e^{2t} + C_3e^{3t};$

$u = 2\,C_1e^t + 7C_2e^{2t} + 3\,C_3e^{3t}$

3. a) $u = xy\,F(x^2 - y^2)$

b) $u = \frac{1}{2}(x^2 + y^2) + F\,(\frac{y}{x})$ dla $x \neq 0;$

c) $F(\frac{y}{x}, \frac{z}{x}, \frac{u}{x}) = 0$ dla $x \neq 0$

d) $u = \dfrac{1}{x^2 + F(y^2 - x^2)};$

e) $\ln|u| = \frac{1}{2}\frac{y^2 - x^2}{1 + xy} + F(xy);$

f) $u = -\frac{1}{6}(x^2 + y^2 + z^2) + F\,(-2x^2 + y^2 + z^2, x^2 - 2y^2 + z^2)$ lub

$u = -\frac{1}{2}z^2 + F(x^2 - y^2, y^2 - z^2);$

g) $u = (1 + z^2)\,.\,F(x^2 + y^2, \frac{y - xz}{x + yz});$

h) $u = x\,F(\frac{y}{x})$ dla $x \neq 0;$

i) $F(\frac{y}{x}, \frac{\sqrt{|u|}}{x}) = 0$ dla $x \neq 0;$

j) $F(z - \sqrt{x}, \sqrt{x} - \sqrt{y}) = 0;$

k) $u = \dfrac{F(x^2 - y^2)}{x + y};$

l) $F\,(\frac{y - x^2}{x}, \frac{z}{x}) = 0$ dla $x \neq 0;$

m) $F(x + y + z, x^2 + y^2 + z^2) = 0;$

n) $F(z, x^2 - y^2z) = 0;$

o) $F(z, x, e^{-\frac{y}{z}}) = 0;$

p) $u = F\,(\frac{y}{x}, \frac{z^2}{x})$ dla $x \neq 0;$

r) $u = F(\frac{y}{x}, \frac{z}{x})$ dla $x \neq 0;$

q) $F(\frac{y}{x}, x + y - z) = 0$ dla $x \neq 0;$

s) $F(x^2 + y^2, z - x) = 0;$

t) $F(z, \ln x + \frac{y}{z}) = 0;$

u) $u = F\,(e^{-2x}(y\cos 3x + z\sin 3x), e^{-2x}(y\sin 3x - z\cos 3x));$

v) $u = F(e^{-2x}(y + z), e^{-x}(3y + 2z));$

w) $u = F(\frac{y}{x}, z + \sqrt{x^2 + y^2 + z^2});$

x) $u = F(x^2 + y^2 + z^2, \frac{yz}{x});$

y) $u = F(y^2 - z^2, 2x + (z - y)^2);$

z) $u = F(\frac{z}{y}, y + \frac{y^3}{x^2}).$

4) a) $u = \dfrac{(y + z)}{2}$

b) $z = x^2e^{-2y}\,y$

c) $z = y - x^2;$

d) α) $z = \dfrac{x^2 + y^2}{2}e^{x^2}$ $\quad\beta$) $z^4 = (x^2 + y^2 - 1)\,e^{4(x^2 - 1)};$

e) $z = xy;$

f) $(y^2 - 2x)^2(x^2 + z^2) = 4x^2;$

g) $(x - z)^2 - 2(x - z)(\ln y - z) = 1;$

h) $\left(\dfrac{4xz}{4z - 4x + xz}\right)^2 + \left(\dfrac{4yz}{4z - 4y + yz}\right)^2 = 1$

i) $z\,e^{-\frac{z}{y}} = yx\,e^{-y};$

j) $2(x^2 + y^2 + z^2) - 5(xy + yz + zx) = 0;$

k) $(x - z)^2 + 2(y^2 - z^2) = 2;$

l) $z = \frac{y}{\ln x - 1}$;

m) $z = x^2 + y^2$;

n) $z = \frac{y^2}{1 + x^2}$;

o) $z = \frac{x^2}{y^2}$;

p) $z = \frac{x}{y} + \frac{y}{2}$;

q) $u = xz - y + 1$;

r) $u = \frac{x^z}{y}$;

s) $u = \frac{y + z^2}{x}$;

t) $u = \frac{2(y + z)}{x}$;

u) $u = 2y(y - z) + 2x$;

v) $u = \frac{x^2y^2}{2^3} + \frac{x^4y^4}{4^4}$