Computer-aided proofs in dynamics I

Interval arithmetic and maps

Maciej Capiński

AGH University of Kraków

Plan of the lecture

- Computer-aided proofs - "philosophy"
- Brouwer theorem
- Interval Newton/Krawczyk method
- Wrapping effect
- Local coordinates

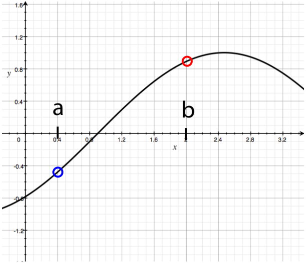
Literature

- W. Tucker, *Validated Numerics: A Short Introduction to Rigorous Computations*, Princeton University Press 2011
- J-B van den Berg *Introduction to rigorous numerics in dynamics: general functional analytic setup and an example that forces chaos* Proc. Sympos. Appl. Math. 2018
- T. Kapela, M. Mrozek, D. Wilczak, P. Zgliczyński, CAPD::DynSys: a flexible C++ toolbox for rigorous numerical analysis of dynamical systems, Commun. Nonlinear Sci. Numer. Simul. 2021

Computer-aided proofs - philosophy

Bolzano theorem

$$f : \mathbb{R} \to \mathbb{R} \qquad f(x) \stackrel{?}{=} 0$$



$$f(a) < 0 \qquad f(b) > 0$$

- There exists an x^* in (a, b) such that

$$f(x^*) = 0$$

Computer-aided proofs - philosophy

Bolzano theorem - no need to be too accurate

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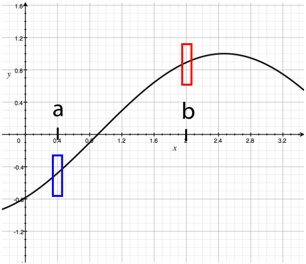

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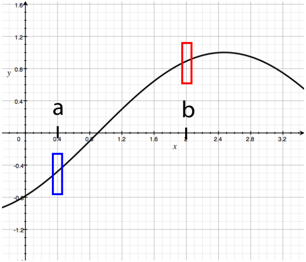

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CAP- the ingredients:

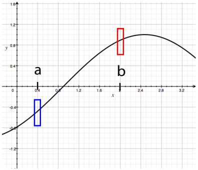
- Mathematical theorem
- Good understanding of the numerical properties of the problem
- Computer-aided validation of assumptions

- There exists an x^* in (a, b) such that

$$f(x^*) = 0$$

Kinds of tools that we shall use

Interval arithmetic



computations on intervals:

$$[1,2]+[3,4]=[4,6]$$
$$[1,2]-[3,4]=[-3,-1]$$
$$[1,2]*[3,4]=[3,8]$$
$$[1,2]/[3,4]=[\frac{1}{4},\frac{2}{3}]$$
$$[1,2]^{[3,4]}=[1^3,2^4]$$

...

extends to higher dimensions

$$[1,2]-[1,2]=[-1,1]$$

$$f:\mathbb{R}^n\to\mathbb{R}^n$$

What can be computed:

- $[f(U)]$, $[Df(U)]$
- higher order derivatives
- linear algebra; eg. $[A^{-1}]$

INTLAB[a]
RadiiPolynomial[b]
CAPD[c]

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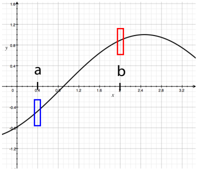

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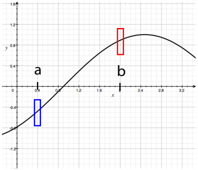

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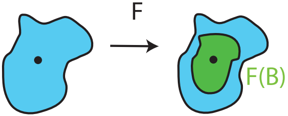
Examples of tools that we can use

Let $B \subset \mathbb{R}^n$ be homeomorphic to a closed ball

Theorem (Brouwer theorem)

If $F : B \to B$ is continuous, then there exists a $q \in B$

$$F(q) = q$$



Examples of tools that we can use

Theorem (Banach fixed pt thm)

Let $F : \mathbb{R}^n \to \mathbb{R}^n$. If for some $\lambda < 1$

$$\|F(x_1) - F(x_2)\| \leqslant \lambda \|x_1 - x_2\|$$

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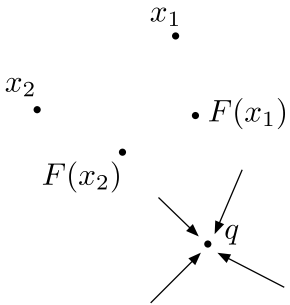
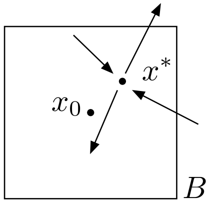


- The use of contraction can be exploited in various contexts

Interval Newton method

$$F : \mathbb{R}^n \to \mathbb{R}^n, \qquad C^1$$



$$B = \Pi_{i=1}^{n}[a_i, b_i]$$

$$x_0 \in B$$

Theorem (interval Newton)

If

$$x_0 - [DF(B)]^{-1}F(x_0) \subset B$$

Then $\exists! x^* \in B$ *such that*

$$F(x^*) = 0$$

(intuition)
Newton-Raphson:

$$F : \mathbb{R} \to \mathbb{R}$$

$$x_{n+1} = x_n - \frac{F(x_n)}{F'(x_n)}$$

Proof. chalk.

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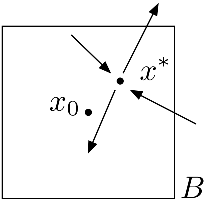

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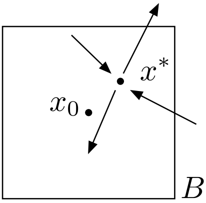

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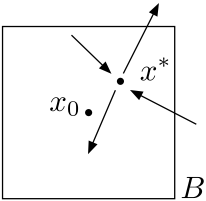

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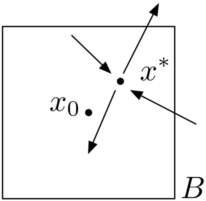

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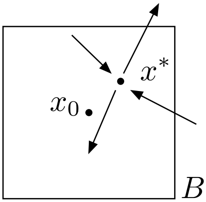

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Interval Newton method

Example - Hénon map

Theorem (interval Newton)

$$x_0 - [DF(B)]^{-1}F(x_0) \subset B.$$

$$\textit{Then } \exists! x^* \in B, \quad F(x^*) = 0.$$

$$h(x, y) = (1 - ax^2 + y, bx)$$

$a = 1.4, b = 0.3$

Fixed point:

$$h(x, y) = (x, y)$$

$$F(x, y) = (1 - ax^2 + y - x, bx - y) = 0$$

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if( subsetInterior(N,B) ) return 1;
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(show Newton method program)

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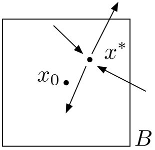
(show Newton method program)

Krawczyk Theorem

To get better results one gets creative



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$B = \Pi_{i=1}^n [a_i, b_i]$

$x_0 \in B$

$C \in \mathbb{R}^{n \times n}$

Theorem

Let

$$K = x_0 - CF(x_0) + (\mathrm{Id} - C[DF(B)])(B - x_0)$$

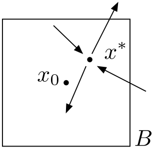
If $K \subset \mathrm{int}(B)$ then $\exists!\ x^* \in B$ such that

$$F(x^*) = 0$$

- $F(x_0) \approx 0$
- $C \approx DF(x_0)^{-1}$

(show Krawczyk method program)

Krawczyk vs Newton



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Newton method

$$N = x_0 - [DF(B)]^{-1} F(x_0)$$

$$N \subset B?$$

- solving a linear problem in **interval arithmetic**

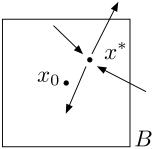
Krawczyk method

$$K = x_0 - CF(x_0) + (\mathrm{Id} - C[DF(B)])(B - x_0)$$

$$K \subset \mathrm{int}(B)?$$

- guessing/computing C **non-rigorously**

Krawczyk vs Newton



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Fork in the road

Theorem (Banach fixed pt thm)

Let $F : X \to X$. If for some $\lambda < 1$

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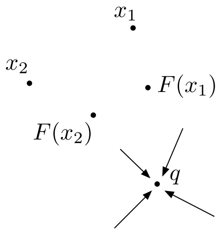


Analytic approach

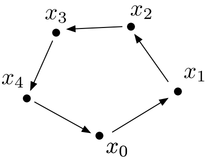
- Solving ODEs
- Boundary value problems
- Invariant manifolds
- DDEs, PDEs

geometric & topological approach

- Chaotic dynamics for ODEs
- Invariant manifolds
- **Blenders**
- DDEs, PDEs

J-B van den Berg "Introduction to rigorous numerics in dynamics: general functional analytic setup" https://www.math.vu.nl/ janbouwe/pub/introrignumdyn.pdf

Periodic orbits



$f : \mathbb{R}^n \to \mathbb{R}^n$

$f^k(x) = x$ when
$F(x) = 0$

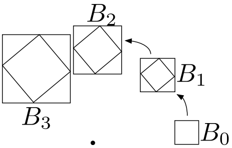
for $F(x) = f^k(x) - x$

Problem with Newton method

$N = x_0 - [DF(B)]^{-1} F(x_0)$

$[Df^k(B)] = [Df(f^{k-1}(B))] \cdot [Df^{k-1}(B)]$

$f(x) = Rx + g(x)$
R is rotation



$B_0 = B \qquad D_0 = \mathrm{Id}$

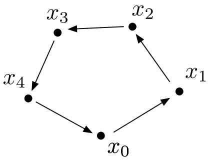
For $i = 1, \dots, k$

$B_i = [f(B_{i-1})] \qquad D_i = [Df(B_{i-1})] \cdot D_{i-1}$

Then

$[Df^k(B)] = D_k$

Periodic orbits



$f : \mathbb{R}^n \to \mathbb{R}^n$

$f^k(x) = x$ when
$F(x) = 0$

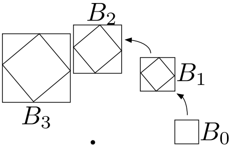
for $F(x) = f^k(x) - x$

Problem with Newton method

$N = x_0 - [DF(B)]^{-1} F(x_0)$

$[Df^k(B)] = [Df(f^{k-1}(B))] \cdot [Df^{k-1}(B)]$

$f(x) = Rx + g(x)$
R is rotation



$B_0 = B \qquad D_0 = \mathrm{Id}$

For $i = 1, \ldots, k$

$B_i = [f(B_{i-1})] \qquad D_i = [Df(B_{i-1})] \cdot D_{i-1}$

Then

$[Df^k(B)] = D_k$

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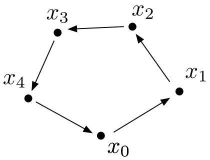

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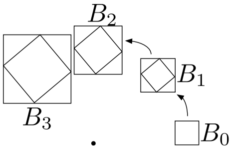

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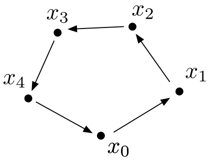

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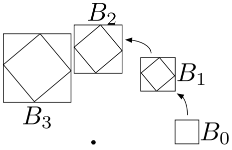

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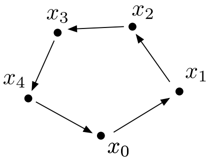

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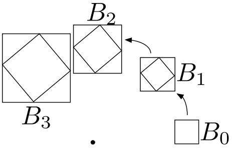

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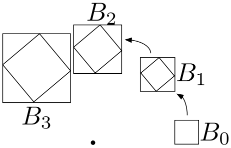
Wrapping effect

Remedy 1 - parallel shooting

Problem with Newton method 

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Remedy:
$f^k(x_1) = x_1$ when $F(x) = 0$ for

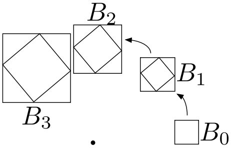
$$F(x_1, \ldots, x_k) = \begin{pmatrix} f(x_1) - x_2 \\ f(x_2) - x_3 \\ \vdots \\ f(x_k) - x_1 \end{pmatrix}$$

There is no free lunch:

- $F : \mathbb{R}^{nk} \to \mathbb{R}^{nk}$ instead of $\mathbb{R}^n$
- The matrix DF is cumbersome to compute
- Use Krawczyk method!

Wrapping effect

Remedy 2 - Take care of coordinates



$$F(x+y) = F(x) + \int_0^1 \frac{d}{ds} F(x+sy)\,ds$$
$$= F(x) + \int_0^1 DF(x+sy)\,ds\, y$$
$$\in F(x) + [DF(X)]y$$

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

Let us represent a set as:

$$X = x + AY$$

where $x, Y \subset \mathbb{R}^n$, $A \subset \mathbb{R}^{n \times n}$

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so $F(X) \subset \bar{X}$ for

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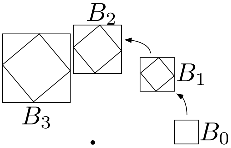

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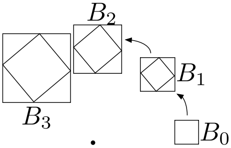

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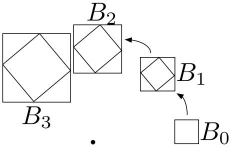

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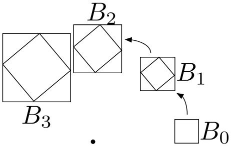

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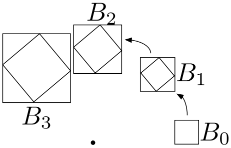

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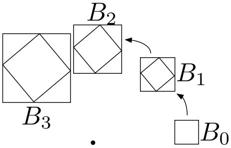

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$$f(x) = Rx + \varepsilon g(x)$$

where R is a rotation matrix

(show code)

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Initial set:
$X_0 = x_0 + Y_0$
choose some local coordinates:
$A_k = R^k$
$x_k = f^k(x_0)$
Compute inductively:
$Y_{k+1} = (A_{k+1}^{-1} Df(X_k) A_k) Y_k$
$X_{k+1} = x_k + A_k Y_k$
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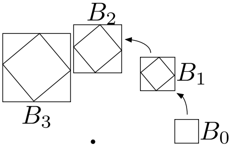

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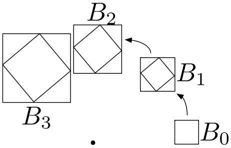

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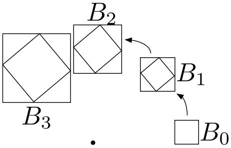

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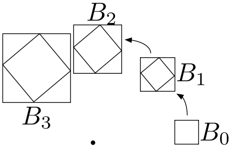

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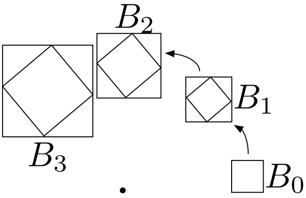
Key points

- Good numerical understanding first
- Good theorem (eg Newton vs Krawczyk)
- (Most) often being aware of coordinates helps



Tomorrow: CAP – topological shadowing