Computer-aided proofs in dynamics II

Covering relations and cone conditions

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AGH University of Kraków

Plan of the lecture

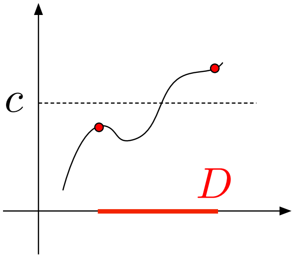
- Brouwer degree
- Covering relations
- Topological shadowing
- Cone conditions
- Horizontal/vertical discs

Brouwer degree

$D \subset \mathbb{R}^n$ open set, $c \in \mathbb{R}^n$

$\overline{D}$ compact

$F : D \to \mathbb{R}^n$ continuous



Definition

$\deg(F, D, c)$ is a number in $\mathbb{Z}$ s.t.

- $\deg(F, D, c) \neq 0$ then $\exists x \in D$ such that
$$F(x) = c$$
- $H : [0, 1] \times D \to \mathbb{R}^n$ homotopy s.t. $c \notin H([0, 1], \partial D)$ then
$$\deg(H_\lambda, D, c) = \deg(H_0, D, c)$$
- $F(x) = A(x - x_0) + c$ and $x_0 \in D$ then
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- $E \subset D$ and $F^{-1}(c) \cap D \subset E$ then
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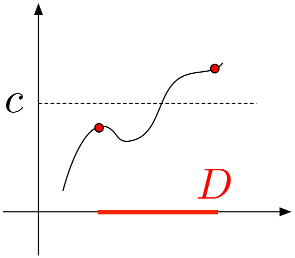

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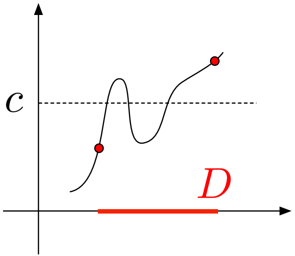

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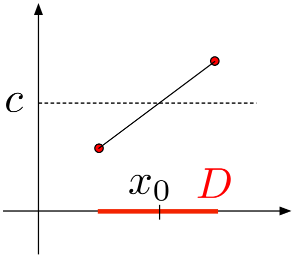

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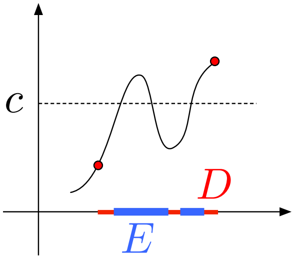

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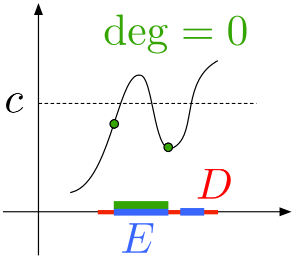

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h-sets

$B_u \subset \mathbb{R}^u$, $B_s \subset \mathbb{R}^s$ closed unit balls

$$n = u + s$$

Definition (h-set)

$N := (|N|, \eta)$ where $|N| \subset \mathbb{R}^n$,
$\eta : \mathbb{R}^n \to \mathbb{R}^n$ homeomorphism s.t.

$$\eta\left(B_u \times B_s\right) = |N|.$$

$N^- = \eta(\partial B_u \times B_s)$
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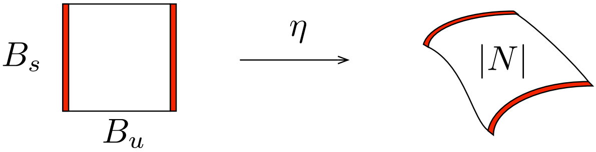

- In computer-aided proofs, in local coordinates, h-sets are cubes; recall: $X = x + A\,B$.

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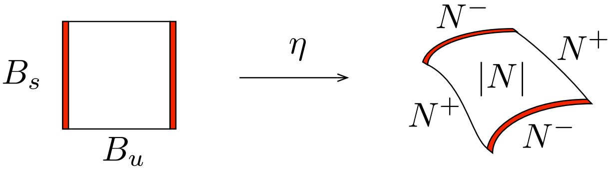

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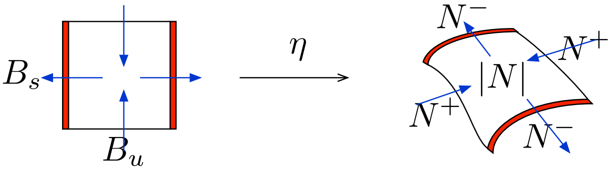

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Covering relation

$$N_\eta^- = \partial B_u \times B_s$$
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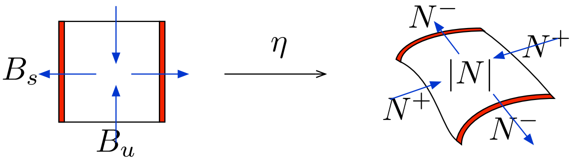

We have $F : \mathbb{R}^n \to \mathbb{R}^n$.

$N = (|N|, \eta)$, $M = (|M|, \xi)$ are h-sets. Define

$$f := \xi^{-1} \circ F \circ \eta$$

Definition (N F-covers M)

We say that $N \overset{F}{\Longrightarrow} M$ iff.
$\exists$ homotopy $H : [0,1] \times B_u \times B_s \to \mathbb{R}^2$ s.t.

$$H(0, \cdot) = f$$
$$H\Big([0,1], N_\eta^-\Big) \cap M_\xi = \emptyset$$
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where $A(\partial B_u) \cap B_u = \emptyset$

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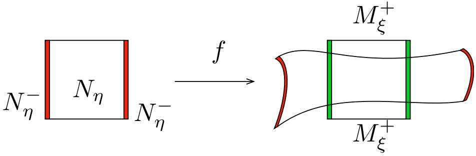

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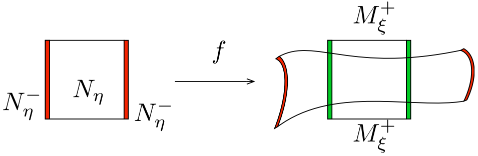

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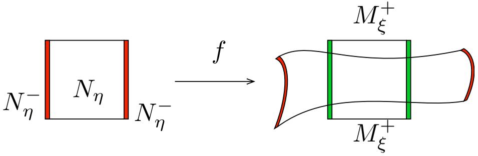

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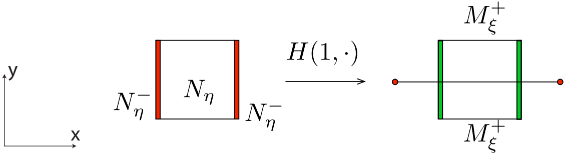

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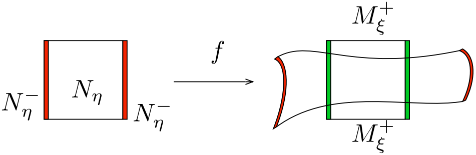
Covering relation

Validation

$f = \xi^{-1} \circ F \circ \eta$

$$m(A) := \frac{1}{\|A^{-1}\|}$$

$$\|Ax\| \geqslant m(A)\|x\|$$



Assume that

$$[Df(N)] = \begin{pmatrix} A & C_1 \\ C_2 & B \end{pmatrix}$$

Lemma

If $\|f(0)\| \leqslant r$ *and*

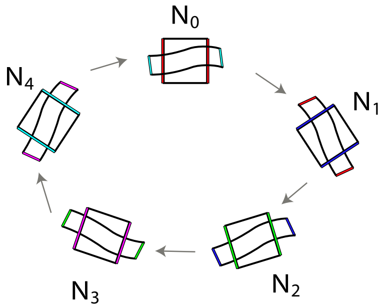
$$\begin{aligned} m(A) - \|C_1\| &> 1 + r \\ \|B\| + \|C_2\| &< 1 - r \end{aligned}$$

then $N \Longrightarrow M$.

Proof. chalk.
(Show code)

Covering relation

Shadowing theorem



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Theorem

$$N_0 \overset{F}{\Rightarrow} N_1 \overset{F}{\Rightarrow} \ldots \overset{F}{\Rightarrow} N_0$$

Then there exists a periodic orbit passing through the sets.

Proof. chalk.

[ZG] Zgliczyński, Gidea. "Covering relations for multidimensional dynamical systems". JDE 2004

Covering relation

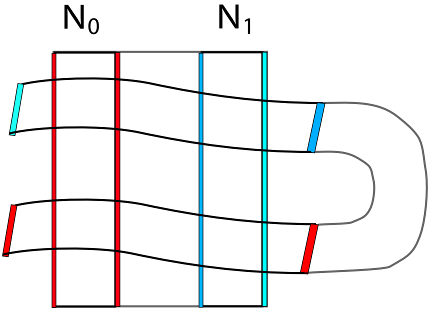
Horseshoe

$$N_0 \overset{F}{\Rightarrow} N_1$$

$$N_0 \overset{F}{\Rightarrow} N_0$$

$$N_1 \overset{F}{\Rightarrow} N_0$$

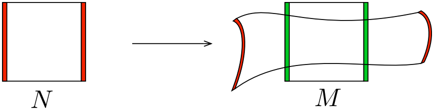
$$N_1 \overset{F}{\Rightarrow} N_1$$



- Orbits of any prescribed sequences of zeros and ones.
- Periodic orbits of any period. For example:

$$N_0 \overset{F}{\Rightarrow} N_1 \overset{F}{\Rightarrow} N_1 \overset{F}{\Rightarrow} N_1 \overset{F}{\Rightarrow} N_0$$

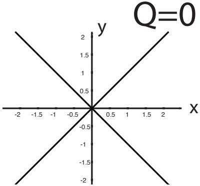
Cones



$$N \overset{F}{\Rightarrow} M$$

$$Q : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$$

$$Q(x, y) = \|x\|^2 - \|y\|^2$$



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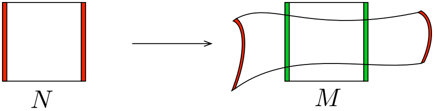

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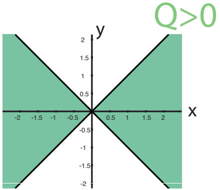

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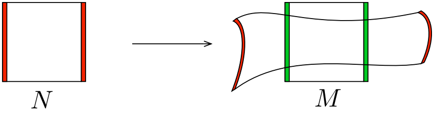

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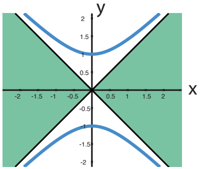

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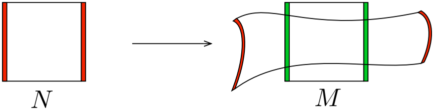

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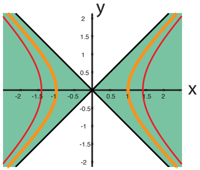
$$Q = a, Q = b, \quad a > b > 0$$



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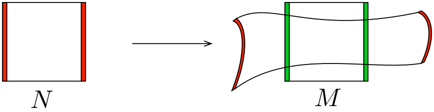

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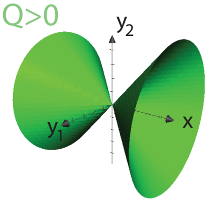
$$u = 1, \quad s = 2$$



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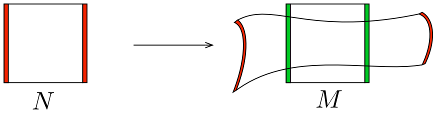

$$N \overset{F}{\Rightarrow} M$$

$$Q : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$$
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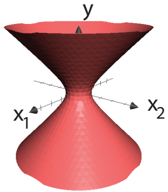
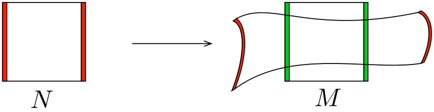
$$u = 2, s = 1 \quad Q = a > 0$$



Cone conditions



$$N \overset{F}{\Rightarrow} M$$

$$Q : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$$

$$Q(x, y) = \|x\|^2 - \|y\|^2$$

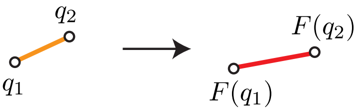
$$F : \mathbb{R}^n \to \mathbb{R}^n$$

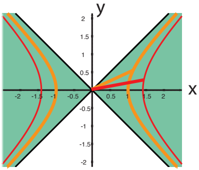
Definition (cone cond.)

If $Q(q_1 - q_2) \geqslant 0$ then

$$Q(F(q_1) - F(q_2)) \geqslant 0$$

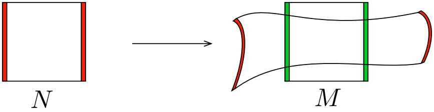
$$Q = a, Q = b, \quad a > b > 0$$





Cone conditions

$$Q : \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R}$$
$$Q(x, y) = \|x\|^2 - \|y\|^2$$



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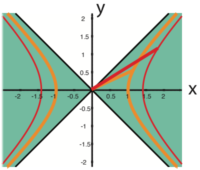
Example

$$(x, y) \mapsto (3x, 2y)$$



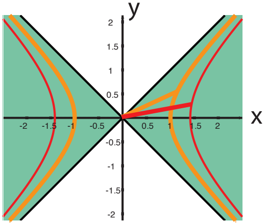
Cone conditions

Validation

$$Q(F(q)-F(p))-Q(q-p)\geqslant 0$$

$$Q(v)=v^TCv$$

$$Q(x,y)=$$
$$=\|x\|^2-\|y\|^2$$
$$=x^Tx-y^Ty$$
$$=\begin{pmatrix}x^T & y^T\end{pmatrix}\begin{pmatrix}\mathrm{Id} & 0\\ 0 & -\mathrm{Id}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$



$$F(q)-F(p)\in[DF(N)](q-p)$$

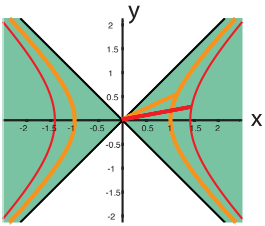
$$Q(F(q)-F(p))-Q(q-p)\in(q-p)^T([DF(N)]^TC[DF(N)]-C)(q-p)$$

Cone conditions
Validation

$$Q(F(q) - F(p)) - Q(q-p) \geqslant 0$$

$$Q(v) = v^T C v$$

$$Q(x,y) =$$
$$= \|x\|^2 - \|y\|^2$$
$$= x^T x - y^T y$$
$$= \begin{pmatrix} x^T & y^T \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$



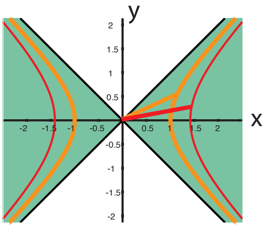
$$F(q) - F(p) \in [DF(N)](q-p)$$

$$Q(F(q) - F(p)) - Q(q-p) \in (q-p)^T([DF(N)]^T C [DF(N)] - C)(q-p)$$

Cone conditions
Validation

$$Q(F(q) - F(p)) - Q(q - p) \geqslant 0$$

$$Q(v) = v^T C v$$



$$\begin{aligned} &Q(x, y) = \\ &= \|x\|^2 - \|y\|^2 \\ &= x^T x - y^T y \\ &= \left(x^T\, y^T\right) \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \end{aligned}$$

$$F(q) - F(p) \in [DF(N)](q - p)$$

$$Q(F(q) - F(p)) - Q(q - p) \in (q - p)^T ([DF(N)]^T C [DF(N)] - C)(q - p)$$

Cone conditions

Validation

$$Q(F(q) - F(p)) - Q(q - p) \geqslant 0$$

$$Q(v) = v^T C v$$

$$Q(x, y) =$$
$$= \|x\|^2 - \|y\|^2$$
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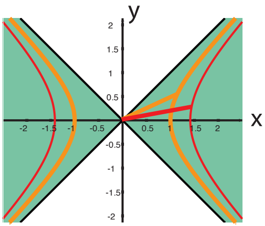

$$F(q) - F(p) \in [DF(N)](q - p)$$

$$Q(F(q) - F(p)) - Q(q - p) \in (q - p)^T \left(\underline{[DF(N)]^T C [DF(N)] - C}\right)(q - p)$$

Cone conditions

Validation

$$Q(F(q) - F(p)) - mQ(q-p) \geqslant 0$$

$$Q(v) = v^T C v$$

$$Q(x,y) =$$
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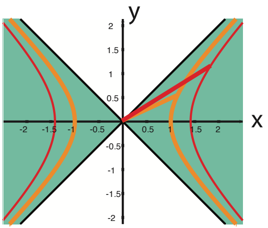
$$= x^T x - y^T y$$
$$= \begin{pmatrix} x^T & y^T \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -\mathrm{Id} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

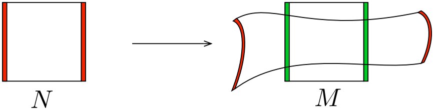


$$F(q) - F(p) \in [DF(N)](q-p) \qquad m > 0$$

$$\underline{[DF(N)]^T C [DF(N)] - mC}$$

code example

Horizontal discs



$$N \stackrel{F}{\Rightarrow} M$$

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

$$N = \overline{B_u} \times \overline{B_s}$$

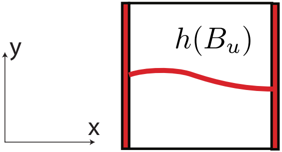
Definition (horizontal disc)

$$h : B_u \to N$$

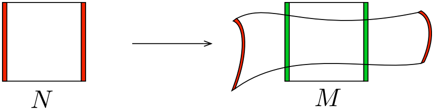
- $\pi_x h(x) = x$

cone conditions:

- $x_1 \neq x_2$ then $Q(h(x_1) - h(x_2)) \geqslant 0$



Horizontal discs



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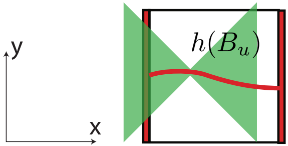


Horizontal discs

$$F : \mathbb{R}^n \to \mathbb{R}^n$$

$$N = B_u \times B_s$$

Lemma

If h is a disc in N and $N \overset{F}{\Longrightarrow} M$, then there exists $h^* : B_u \to M$

$$F(h(B_u)) \cap M = h^*(B_u)$$

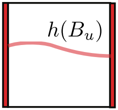
Proof. chalk.

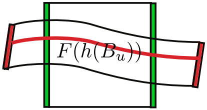
Definition (N F-covers M)

We say that $N \overset{F}{\Longrightarrow} M$ iff.
$\exists$ homotopy $H : [0,1] \times B_u \times B_s \to \mathbb{R}^2$ s.t.

$$H(0,\cdot) = f$$
$$H\Big([0,1], N_\eta^-\Big) \cap M_\xi = \emptyset$$
$$H\Big([0,1], N_\eta\Big) \cap M_\xi^+ = \emptyset$$
$$H(1,x,y) = (Ax, 0)$$

where $A(\partial B_u) \cap B_u = \emptyset$





[Z] Zgliczyński, Covering relations, cone conditions and the stable manifold theorem, JDE 2009

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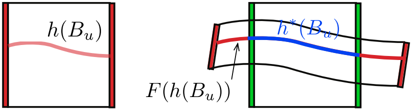

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Why does this make us happy?

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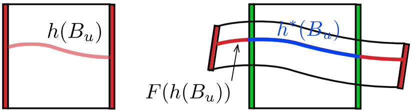
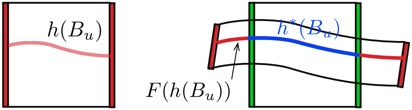


- blenders
- stable/unstable manifolds of normally hyperbolic fixed points
- normally hyperbolic manifolds
- Arnold diffusion

Why does this make us happy?

Some references



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- P. Zgliczyński "Covering relations, cone conditions and the stable manifold theorem" JDE 2009
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Tomorrow: Blenders.