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COVERING RELATIONS AND THE EXISTENCE OF TOPOLOGICALLY NORMALLY HYPERBOLIC INVARIANT SETS

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Abstract. We present a topological method for the detection of normally hyperbolic type invariant sets for maps. The invariant set covers the sub-manifold without a boundary in $\mathbb{R}^k$. For the method to hold we only need to assume that the movement of the system transversal to the manifold has directions of topological expansion and contraction. The movement in the direction of the manifold can be arbitrary. The result is based on the method of covering relations and local Brouwer degree theory.

1. **Introduction.** Covering relations are topological tools used for proofs of non-trivial symbolic dynamics of dynamical systems. The method is based on the Brouwer fixed point index, and the setting is such that it allows for rigorous numerical verification. The method has been applied in computer assisted proofs for the Hénon map, Rössler equations [15], [4], Lorenz equations [6], Chua circuit [5] or Kuramoto-Shivashinsky ODE [14], amongst others. The method is based on singling out a number of regions, called h-sets, which have hyperbolic type properties. Using these properties one can find orbits of the system, which shadow the h-sets along their trajectories. So far the method has always relied on the fact that the systems had a strong expanding and contracting local coordinates. The aim of this paper is to develop a method, which would also allow for a third central coordinate, where the dynamics is not as distinctive. The method will be used for finding invariant sets in the setting of (topological) normal hyperbolicity.

We consider a dynamical system in a small neighborhood of a compact sub-manifold in $\mathbb{R}^k$. We suspect that in this neighborhood we have an invariant set (manifold). The reason for the existence of such a set, is that in the investigated region the system has normally-hyperbolic type properties. The properties considered are of purely topological nature. For each point in the region, locally three (possibly multidimensional) directions can be singled out. The first two are the directions of topological contraction and expansion. The third is a direction associated with the coordinate of our sub-manifold. In this direction we need not say much about the dynamics, and refer to it as the central direction. We assume that if we start on a

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section with a fixed central coordinate, then locally we have topological contraction and expansion of our map in the first two directions. This is expressed in terms of covering relations. The movement in the central direction can be arbitrary, as long as the projections onto the first two coordinates preserve their topological properties. It turns out that the expansion and contraction are enough for us to establish the existence of an invariant set, which covers the sub-manifold.

Since the method is topological, we do not obtain any regularity properties for our invariant set. We also do not obtain its uniqueness. These are the main limitations of the result. The topological nature of the argument though does give us also a number of advantages. The setting is such, that it allows for rigorous computer assisted verification of the conditions. It is not necessary to consider any C^1 conditions. Also the required assumptions are local in nature. Another advantage is that one does not need to investigate the dynamics in the central direction, which from the point of view of rigorous numerics is very awkward to handle.

The paper is organized as follows. In the second section we give brief preliminaries on covering relations and on the properties of the Brouwer degree, which are used for the proof of the main result. Section three contains the main result. In section four we show how the assumptions of the main theorem may be verified using rigorous computer assisted methods. We also show how the result compares with the method of Haro and de la Llave [8], which is one of the most recent results on the detection of normally hyperbolic invariant manifolds, designed for rigorous computer assisted implementation (for related work see also [9], [10], [1], [2] and [3]). Section five contains examples of applications of the method.

2. **Preliminaries.**

In this section we introduce the basic background on h-sets, covering relations and Brouwer degree theory.

2.1. **Covering relations.**

Let $\overline{B_n}(0,1)$ denote a closed ball of radius one centered at zero in $\mathbb{R}^n$.

Definition 2.1. [7] An h-set, is an object consisting of the following data

1. N - a compact subset of $\mathbb{R}^k$
2. $u(N), s(N) \in \{0,1,2,3,\ldots\}$, such that $u(N)+s(N)=k$
3. a homeomorphism $\eta_N : \mathbb{R}^k \to \mathbb{R}^k = \mathbb{R}^{u(N)} \times \mathbb{R}^{s(N)}$ such that

$$\eta_N(N) = \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1)$$

We set

$$\begin{aligned}
N_\eta &= \overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1),\\
N_\eta^- &= \partial\overline{B_{u(N)}}(0,1) \times \overline{B_{s(N)}}(0,1),\\
N_\eta^+ &= \overline{B_{u(N)}}(0,1) \times \partial\overline{B_{s(N)}}(0,1),\\
N^- &= \eta_N^{-1}(N_\eta^-), \quad N^+ = \eta_N^{-1}(N_\eta^+).
\end{aligned}$$

Definition 2.2. [7] Assume N, M are h-sets, such that $u(N)=u(M)=u$ and $s(N)=s(M)=s$. Let $f : N \to \mathbb{R}^k$ be a continuous map. Let $f_\eta = \eta_M \circ f \circ \eta_N^{-1} :$ $N_\eta \to \mathbb{R}^u \times \mathbb{R}^s$. We say that

$$N \overset{f}{\Longrightarrow} M$$

(N f-covers M) if the following conditions are satisfied

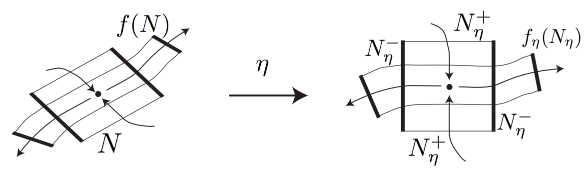

FIGURE 1. An h-set N, and a covering relation $N \stackrel{f}{\Longrightarrow} N$, in the setting of a hyperbolic fixed point.

1. There exists a continuous homotopy $h : [0,1] \times N_\eta \to \mathbb{R}^u \times \mathbb{R}^s$ such that the following conditions hold true

$$h_0 = f_\eta,$$

$$h([0,1], N_\eta^-) \cap M_\eta = \emptyset, \tag{1}$$

$$h([0,1], N_\eta) \cap M_\eta^+ = \emptyset. \tag{2}$$

2.1. If $u > 0$, then there exists a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that

$$h_1(x,y) = (Ax, 0), \quad \text{where } x \in \mathbb{R}^u \text{ and } y \in \mathbb{R}^s, \tag{3}$$

$$A(\partial B_u(0,1)) \subset \mathbb{R}^u \backslash \overline{B_u}(0,1). \tag{4}$$

2.2. If $u = 0$, then

$$h_1(x) = 0, \quad \text{for } \ x \in N_\eta.$$

The idea behind Definition 2.2 is that the coordinate $x \in \mathbb{R}^u$ is the direction of topological expansion and $y \in \mathbb{R}^s$ is the coordinate of topological contraction (the notations u, s stand for "unstable" and "stable" respectively).

To provide some more intuition for the Definitions 2.1 and 2.2 let us illustrate the setting in the case of a hyperbolic fixed point (see Figure 1). I such a case we can take $N_\eta = M_\eta$ to be a small box surrounding the fixed point, chosen in the linearized coordinates of hyperbolic expansion and contraction. The homotopy h corresponds to a projection onto the unstable coordinate, and the homeomorphism η is the local change of coordinates around the fixed point (a more detailed discussion on how to choose η, N, M, and in particular on how to construct the homotopy h will be given in the proof of Proposition 1 and in Section 5).

Let us note that the class of functions satisfying Definition 2.2 is broader than those having a hyperbolic invariant set. In particular, Definition 2.2 does not require the function to be differentiable.

2.2. **Properties of the local Brouwer degree.** For a bounded open set $D \subset \mathbb{R}^n$, a continuous function $f : D \to \mathbb{R}^n$, and $c \in \mathbb{R}^n$ such that $c \in \mathbb{R}^n \setminus f(\partial D)$, we denote by $\deg(f, D, c)$ the Brouwer degree of f with respect to the set D at c [12].

2.2.1. *Solution property.* [12] If $\deg(f, D, c) \neq 0$ then there exists an $x \in D$ with $f(x) = c$.

2.2.2. *Homotopy property.* [12] Let $H : [0,1] \times D \to \mathbb{R}^n$ be continuous. Suppose that

$$\bigcup_{\lambda \in [0,1]} H_\lambda^{-1}(c) \cap D \quad \text{is compact,} \tag{5}$$

then

$$\forall \lambda \in [0,1] \quad \deg(H_\lambda, D, c) = \deg(H_0, D, c).$$

If $[0,1] \times \overline{D} \subset \mathrm{dom}(H)$ and $\overline{D}$ is compact, then (5) follows from the condition

$$c \notin H([0,1], \partial D).$$

2.2.3. *Degree property for affine maps.* [12] Suppose that $f(x) = B(x - x_0) + c$, where B is a linear map and $x_0 \in R^n$. If the equation $B(x) = 0$ has no nontrivial solutions (i.e if $Bx = 0$, then $x = 0$) and $x_0 \in D$, then

$$\deg(f, D, c) = \mathrm{sgn}(\det B). \tag{6}$$

2.2.4. *Excision property.* [12] Suppose that we have an open set E such that $E \subset D$ and

$$f^{-1}(c) \cap D \subset E,$$

then

$$\deg(f, D, c) = \deg(f, E, c).$$

3. **Main Result.** Let D be a compact set in $\mathbb{R}^k$. Let us assume that there exists a neighborhood U of D and a homeomorphism $\phi : U \to \mathbb{R}^k$ such that

$$\phi(D) = \Lambda \times N,$$

where $N = \overline{B_u}(0,1) \times \overline{B_s}(0,1)$ and Λ is a compact $c = k - u - s$ dimensional sub-manifold, without a boundary, in $\mathbb{R}^k$.

We consider a homeomorphism

$$f : U \to U.$$

We will look for an invariant set in the interior of D, which covers the manifold $\phi^{-1}(\Lambda, 0, 0)$. The exact meaning of this statement will be made clear in the formulation of the result.

A point p in $\Lambda \times N$ will be represented as $p = (\theta, x, y)$, where θ, x, and y correspond to Λ, $\overline{B_u}(0,1)$ and $\overline{B_s}(0,1)$ coordinates respectively. For a given point $\theta \in \Lambda$ we will use the notations f_θ and, f_θ^{-1} for functions

$$f_\theta, f_\theta^{-1} : N \to \mathbb{R}^{u+s},$$

$$\begin{aligned} f_\theta(x,y) &:= \pi_{us} \circ \phi \circ f \circ \phi^{-1}(\theta, x, y), \\ f_\theta^{-1}(x,y) &:= \pi_{us} \circ \phi \circ f^{-1} \circ \phi^{-1}(\theta, x, y), \end{aligned} \tag{7}$$

where π_{us} is the projection onto the x, y coordinates. In line with Definition 2.2 we will adapt a notation in which x will be the unstable and y will be the stable coordinate (in the topological sense of covering relations) for the maps f_θ. For the coordinate θ we will not assume any expansion or contraction properties (θ can be thought of as the central direction). We assume that for any $\theta \in \Lambda$

$$N \overset{f_\theta}{\Longrightarrow} N, \tag{8}$$

and

$$N \overset{f_\theta^{-1}}{\Longrightarrow} N. \tag{9}$$

For (9) we make a natural assumption that the roles of the stable and unstable directions are reversed with respect to (8). The coordinates x become the stable coordinates and y the unstable coordinates for the maps f_θ^{-1}.

The following Theorem is the main result of the paper. It gives a tool for the detection of an invariant set for the map f, which covers the manifold Λ.

Theorem 3.1. *If $f : U \to U$ is a homeomorphism, and for every point $\theta \in \Lambda$ the set N is covered by the maps f_θ and f_θ^{-1}*

$$N \overset{f_\theta}{\Longrightarrow} N, \qquad N \overset{f_\theta^{-1}}{\Longrightarrow} N, \tag{10}$$

then for any $\theta \in \Lambda$ the set

$$\begin{aligned} K_\theta := \{p \in D | f^m(p) \in D \text{ for all } m \in \mathbb{Z}, \text{ and} \\ p = \phi^{-1}(\theta, x, y) \text{ for some } x \in \overline{B_u}(0,1), y \in \overline{B_s}(0,1)\} \end{aligned}$$

is nonempty and lies in the interior of D.

Proof. Without loss of generality we can assume that $D = \Lambda \times N$. We therefore assume that we have our map

$$f = (f_c, f_u, f_s) : \Lambda \times \overline{B_u}(0,1) \times \overline{B_s}(0,1) \to \mathbb{R}^k.$$

(the indexes c, u, s standing for "central", "unstable" and "stable" respectively). We will use the notations f_c^{-1}, f_u^{-1}, f_s^{-1} for the functions

$$f_c^{-1}(\cdot) = (f^{-1}(\cdot))_c, \qquad f_u^{-1}(\cdot) = (f^{-1}(\cdot))_u, \qquad f_s^{-1}(\cdot) = (f^{-1}(\cdot))_s.$$

We note that from (7) follows that

$$\begin{aligned} f_\theta(x, y) &= (f_u(\theta, x, y), f_s(\theta, x, y)), \\ f_\theta^{-1}(x, y) &= (f_u^{-1}(\theta, x, y), f_s^{-1}(\theta, x, y)). \end{aligned} \tag{11}$$

Let us fix a point

$$\rho_0 \in \Lambda.$$

We will show that $K_{\rho_0} \neq \emptyset$ and that $K_{\rho_0} \subset \operatorname{int} D$.

Let us take an $m \in \mathbb{N}$, and two arbitrary linear functions $g : \mathbb{R}^u \to \mathbb{R}^u$, $h : \mathbb{R}^s \to \mathbb{R}^s$ such that

$$g(\partial \overline{B_u}(0,1)) \subset \mathbb{R}^u \backslash \overline{B_u}(0,1), \tag{12}$$

$$h(\partial \overline{B_s}(0,1)) \subset \mathbb{R}^s \backslash \overline{B_s}(0,1). \tag{13}$$

We define a compact set

$$XD := \underbrace{D \times \ldots \times D}_{2m+1} \subset \mathbb{R}^{(2m+1)k},$$

and a map

$$F : XD \to \mathbb{R}^{(2m+1)k},$$

$$
\begin{aligned}
F(\mathbf{x}) =& F(\theta_{-m}, x_{-m}, y_{-m}, \ldots, \theta_0, x_0, y_0, \ldots, \theta_m, x_m, y_m) \\
:=& (\theta_{-m} - f_c^{-1}(\theta_{-m+1}, x_{-m+1}, y_{-m+1}), \\
& \qquad x_{-m} - f_u^{-1}(\theta_{-m+1}, x_{-m+1}, y_{-m+1}), \\
& \qquad\qquad y_{-m} - f_s^{-1}(\theta_{-m+1}, x_{-m+1}, y_{-m+1}), \\
& \ldots \\
& \theta_{-1} - f_c^{-1}(\theta_0, x_0, y_0), x_{-1} - f_u^{-1}(\theta_0, x_0, y_0), y_{-1} - f_s^{-1}(\theta_0, x_0, y_0), \\
& \theta_0 - \rho_0, x_0 - g(x_m), y_0 - h(y_{-m}), \\
& \theta_1 - f_c(\theta_0, x_0, y_0), x_1 - f_u(\theta_0, x_0, y_0), y_1 - f_s(\theta_0, x_0, y_0), \\
& \ldots \\
& \theta_m - f_c(\theta_{m-1}, x_{m-1}, y_{m-1}), x_m - f_u(\theta_{m-1}, x_{m-1}, y_{m-1}), \\
& \qquad y_m - f_s(\theta_{m-1}, x_{m-1}, y_{m-1})).
\end{aligned}
$$

(The functions g and h are inserted into F for technical reasons. These will become apparent during the proof). Throughout the course of the proof we will show that $F(\mathbf{x}) = 0$ for some point $\mathbf{x} \in \text{int } XD$. This is the main and most important part of the argument, which will take up the majority of the proof. Let us note that by showing this we will obtain a pair $(x_0, y_0) \in \text{int } N$, such that

$$f^i(\rho_0, x_0, y_0) \in \text{int } D \quad \text{for } i = -m, \ldots, m. \tag{14}$$

The proof that $F(\mathbf{x}) = 0$ for some $\mathbf{x} \in \text{int } XD$ will be done in three stages. For each stage we will consider a different homotopy. In the final stage of the argument we will combine all three of them together, and obtain the result by the use of properties of the local Brouwer degree.

Let us first consider a homotopy $H : [0,1] \times XD \to R^{(2m+1)k}$ as follows. On each coordinate θ_i we define H_λ as

$$
\begin{aligned}
(H_\lambda(\mathbf{x}))_{i,c} &:= \theta_i - f_c^{-1}(\theta_{i+1}, (1-\lambda)x_{i+1}, (1-\lambda)y_{i+1}) \quad \text{for } i < 0, \\
(H_\lambda(\mathbf{x}))_{i,c} &:= \theta_i - f_c(\theta_{i-1}, (1-\lambda)x_{i-1}, (1-\lambda)y_{i-1}) \quad \text{for } i > 0,
\end{aligned}
$$

leaving the function on the other coordinates identical to F. Clearly we have

$$F = H_0. \tag{15}$$

We will show that

$$\deg(H_\lambda, \text{int} XD, 0) \text{ is independent from } \lambda. \tag{16}$$

From the Homotopy Property of the Brouwer degree (See Section 2.2), to do so it is sufficient to show that

$$H_\lambda(\mathbf{x}) \neq 0 \quad \text{for each } \mathbf{x} \in \partial XD,\ \lambda \in [0,1].$$

Since Λ is without boundary, if $\mathbf{x} \in \partial XD$ we must have either $x_i \in \partial \overline{B_u}(0,1)$ or $y_i \in \partial \overline{B_s}(0,1)$ for some $i \in \{-m, \ldots, m\}$. We will consider the cases with $x_i \in \partial \overline{B_u}(0,1)$ for $i \geq 0$ first. For $0 \leq i \leq m-1$, in order to have $H_\lambda(\mathbf{x}) = 0$ we would need to have

$$(x_{i+1}, y_{i+1}) - (f_u, f_s)(\theta_i, x_i, y_i) = 0,$$

for some $y_i \in \overline{B_s}(0,1)$, $(x_{i+1}, y_{i+1}) \in N$. This is impossible since from the fact that

$$N \overset{f_{\theta_i}}{\Longrightarrow} N,$$

follows that $(f_u, f_s)(\theta_i, x_i, y_i) \notin N$ (see (11), Definition 2.2 and (1) in particular. In our case $N_\eta^- = \partial\overline{B_u}(0,1) \times \overline{B_s}(0,1)$ and $M_\eta = N$). For $i = m$, the fact that $x_0 - g(x_m) \neq 0$ follows from (12). Let us now consider that $y_i \in \partial\overline{B_s}(0,1)$ for $i > 0$. If $y_i \in \partial\overline{B_s}(0,1)$ then in order to have $H_\lambda(\mathbf{x}) = 0$ we would need

$$(x_i, y_i) - (f_u, f_s)(\theta_{i-1}, x_{i-1}, y_{i-1}) = 0,$$

for some $x_i \in \overline{B_u}(0,1)$, $(x_{i-1}, y_{i-1}) \in N$. This is impossible since from the fact that

$$N \overset{f_{\theta_{i-1}}}{\Longrightarrow} N,$$

we have $(f_u, f_s)(\theta_{i-1}, N) \cap \overline{B_u}(0,1) \times \partial\overline{B_s}(0,1) = \emptyset$ (see (11), Definition 2.2 and (2) in particular. In our case $N_\eta = N$ and $M_\eta^+ = \overline{B_u}(0,1) \times \partial\overline{B_s}(0,1)$). The fact that for $x_i \in \partial\overline{B_u}(0,1)$ with $i < 0$ and $y_i \in \partial\overline{B_s}(0,1)$ with $i \leq 0$ we cannot have $H_\lambda(\mathbf{x}) = 0$ follows by a mirror argument, using the fact that f_θ^{-1} covers N. Let us just note that for the inverse map the role of the coordinates is reversed: the x coordinates become stable and y unstable. We also use h instead of g in the argument for $y_{-m} \in \partial\overline{B_s}(0,1)$. This finishes establishing (16).

In this part of the proof, before we introduce the second homotopy, we will restrict the set XD to some smaller subset, by the use of the Excision Property of the Brouwer degree. Let us first define by induction a sequence of points starting with ρ_0

$$\begin{aligned} \rho_i &:= f_c^{-1}(\rho_{i+1}, 0, 0) \quad \text{for } -m \leq i \leq 1, \\ \rho_i &:= f_c(\rho_{i-1}, 0, 0) \quad \text{for } 1 \leq i \leq m. \end{aligned}$$

From the fact that f and f^{-1} are continuous we can choose a sequence of sets $U_i \subset \Lambda$, $i = -m, \ldots, m$, which are small neighborhoods of the points ρ_i such that

$$\begin{aligned} \left(f^{-1}\left(\overline{U_i}, 0, 0\right)\right)_c &\subset U_{i-1} \quad \text{for } -m+1 \leq i \leq 0. \\ \left(f\left(\overline{U_i}, 0, 0\right)\right)_c &\subset U_{i+1} \quad \text{for } 0 \leq i \leq m-1. \end{aligned} \tag{17}$$

Let us define the following subset of XD

$$XU = \overline{U_{-m}} \times N \times \ldots \times \overline{U_m} \times N.$$

Since on the coordinates θ_i the function H_1 is equal to

$$\begin{aligned} (H_1(\mathbf{x}))_{0,c} &= \theta_0 - \rho_0 \quad \text{for } i = 0, \\ (H_1(\mathbf{x}))_{i,c} &= \theta_i - f_c^{-1}(\theta_{i+1}, 0, 0) \quad \text{for } i < 0, \\ (H_1(\mathbf{x}))_{i,c} &= \theta_i - f_c(\theta_{i-1}, 0, 0) \quad \text{for } i > 0, \end{aligned} \tag{18}$$

from (17) and (18) we have that $0 \notin H_1(XD \backslash \mathrm{int} XU)$. This means that from the Excision Property (See Section 2.2) we have

$$\deg(H_1, \mathrm{int} XD, 0) = \deg(H_1, \mathrm{int} XU, 0). \tag{19}$$

Since the sets U_i can be chosen to be arbitrarily small, we can assume without loss of generality that using local coordinates we have

$$U_i = B_c(\rho_i, 1), \quad \text{for } i = -m, \ldots, m. \tag{20}$$

Let us note that this assumption will be important for us from the point of view that now we can assume that the sets U_i are convex, which we could not apriori assume about the whole manifold Λ.

Let us now consider the following homotopy $G : [0,1] \times \overline{XU} \to \mathbb{R}^{(2m+1)k}$ as follows

$$(G_\lambda(\mathbf{x}))_0 = (F(\mathbf{x}))_0 \,,$$

$$\left.\begin{array}{l} (G_\lambda(\mathbf{x}))_{i,c} = \theta_i - \lambda\rho_i - (1-\lambda) f_c^{-1}(\theta_{i+1},0,0) \\ (G_\lambda(\mathbf{x}))_{i,u} = x_i - f_u^{-1}(\lambda\rho_{i+1} + (1-\lambda)\theta_{i+1}, x_{i+1}, y_{i+1}) \\ (G_\lambda(\mathbf{x}))_{i,s} = y_i - f_s^{-1}(\lambda\rho_{i+1} + (1-\lambda)\theta_{i+1}, x_{i+1}, y_{i+1}) \end{array}\right\} \quad \text{for } i < 0,$$

$$\left.\begin{array}{l} (G_\lambda(\mathbf{x}))_{i,c} = \theta_i - \lambda\rho_i - (1-\lambda) f_c(\theta_{i-1},0,0) \\ (G_\lambda(\mathbf{x}))_{i,u} = x_i - f_u(\lambda\rho_{i-1} + (1-\lambda)\theta_{i-1}, x_{i-1}, y_{i-1}) \\ (G_\lambda(\mathbf{x}))_{i,s} = y_i - f_s(\lambda\rho_{i-1} + (1-\lambda)\theta_{i-1}, x_{i-1}, y_{i-1}) \end{array}\right\} \quad \text{for } i > 0.$$

Let us note that from the definitions of G and H (see (18)) we have

$$G_0(\mathbf{x}) = H_1(\mathbf{x}). \tag{21}$$

Using a similar argument to the one used for (16) we will show that

$$\deg(G_\lambda, \mathrm{int}XU, 0) \text{ is independent from } \lambda. \tag{22}$$

Let us introduce a notation $\theta_{i,\lambda}$

$$\theta_{i,\lambda} := \lambda\rho_i + (1-\lambda)\theta_i.$$

To show (22) it is enough to prove that for $\mathbf{x} \in \partial XU$ we have $G_\lambda(\mathbf{x}) \neq 0$. If $\mathbf{x} \in \partial XU$ then either $\theta_i \in \partial\overline{B_c}(\rho_i,1)$, $x_i \in \partial\overline{B_u}(0,1)$ or $y_i \in \partial\overline{B_s}(0,1)$ for some $i \in \{-m,\ldots,m\}$. If $\theta_i \in \partial\overline{B_c}(\rho_i,1)$ then for $i=0$ we have $G_{\lambda\,0,c}(\mathbf{x}) = \theta_0 - \rho_0 \neq 0$. For $i \neq 0$, if $\theta_i \in \partial\overline{B_c}(\rho_i,1)$ from (17) we know that $f_c^{-1}(\theta_{i+1},0,0) \in \mathrm{int}\ \overline{B_c}(\rho_i,1)$ for $i<0$, and $f_c(\theta_{i-1},0,0) \in \mathrm{int}\ \overline{B_c}(\rho_i,1)$ for $i>0$, which from the definition of G_λ implies that we cannot have $G_\lambda(\mathbf{x}) = 0$. If $x_i \in \partial\overline{B_u}(0,1)$ with $0 \leq i \leq m-1$ then since for any $\theta_{i,\lambda}$ we have

$$N \overset{f_{\theta_{i,\lambda}}}{\Longrightarrow} N,$$

we know that $(f_u,f_s)\left(\theta_{i,\lambda}, \partial\overline{B_u}(0,1) \times \overline{B_s}(0,1)\right) \cap N = \emptyset$. This means that we cannot have

$$(x_{i+1},y_{i+1}) - (f_u,f_s)(\theta_{i,\lambda},x_i,y_i) = 0,$$

for any $y_i \in \overline{B_s}(0,1)$, $(x_{i+1},y_{i+1}) \in N$, therefore $G_\lambda(\mathbf{x}) \neq 0$. If $x_m \in \partial\overline{B_u}(0,1)$ then from (12) we have $x_0 - g(x_m) \neq 0$, hence $G_\lambda(\mathbf{x}) \neq 0$. For $y_i \in \partial\overline{B_s}(0,1)$ for $1 \leq i \leq m$ from the fact that

$$N \overset{f_{\theta_{i-1,\lambda}}}{\Longrightarrow} N,$$

we have $(f_u,f_s)(\theta_{i-1,\lambda},N) \cap \overline{B_u}(0,1) \times \partial\overline{B_s}(0,1) = \emptyset$. This means that we will not have

$$(x_i,y_i) - (f_u,f_s)(\theta_{i-1,\lambda},x_{i-1},y_{i-1}) = 0,$$

for all $x_i \in \overline{B_u}(0,1)$, $(x_{i-1},y_{i-1}) \in N$, hence $G_\lambda(\mathbf{x}) \neq 0$. The fact that for $x_i \in \partial\overline{B_u}(0,1)$ with $i<0$ and $y_i \in \partial\overline{B_s}(0,1)$ with $i \leq 0$ we cannot have $G_\lambda(\mathbf{x}) = 0$ follows by a mirror argument, using the fact that for any $\lambda \in [0,1]$ the map $f_{\theta_{i,\lambda}}^{-1}$ covers N (We once again note that for the inverse map the role of the coordinates is reversed: the x coordinates become stable and y unstable. We also use h instead of g in the argument for $y_{-m} \in \partial\overline{B_s}(0,1)$). This finishes establishing (22).

Let us consider now a last homotopy $K : [0,1] \times XU \to \mathbb{R}^{mk}$,

$$\begin{aligned} K_\lambda(\mathbf{x}) =(& \theta_{-m} - \rho_{-m}, (x_{-m}, y_{-m}) - h_\lambda^{-m}(x_{-m+1}, y_{-m+1}), \\ & \dots \\ & \theta_{-1} - \rho_{-1}, (x_{-1}, y_{-1}) - h_\lambda^{-1}(x_0, y_0), \\ & \theta_0 - \rho_0, x_0 - g(x_m), y_0 - h(y_{-m}), \\ & \theta_1 - \rho_1, (x_1, y_1) - h_\lambda^{1}(x_0, y_0), \\ & \dots \\ & \theta_m - \rho_m, (x_m, y_m) - h_\lambda^{m}(x_m, y_{m-1})), \end{aligned} \tag{23}$$

where for $i = -m+1, \dots, m$ the function h_λ^i is the homotopy from the definition of the covering

$$N \overset{f_{\rho_{i+1}}^{-1}}{\Longrightarrow} N, \quad \text{for } i < 0,$$

$$N \overset{f_{\rho_{i-1}}}{\Longrightarrow} N, \quad \text{for } i > 0.$$

Let us observe that from the definition of G and K we have

$$K_0(\mathbf{x}) = G_1(\mathbf{x}). \tag{24}$$

We will now show that

$$\deg(K_\lambda, XU, 0) \quad \text{is independent from } \lambda. \tag{25}$$

From the Homotopy Property of the Brouwer degree once again, it is sufficient to show that

$$K_\lambda(\mathbf{x}) \neq 0 \quad \text{for each } \mathbf{x} \in \partial XU,\ \lambda \in [0,1].$$

Let $\mathbf{x} \in \partial XU$, then we must either have $\theta_i \in \partial U_i$, $x_i \in \partial \overline{B_u}(0,1)$ or $y_i \in \partial \overline{B_s}(0,1)$. If $\theta_i \in \partial U_i$ then from (20) and (23) we can clearly see that we cannot have $K_\lambda(\mathbf{x})$ equal to zero on the θ_i coordinate. If $x_i \in \partial \overline{B_u}(0,1)$ then for $i = 0, \dots, m-1$, from the fact that

$$N \overset{f_{\rho_i}}{\Longrightarrow} N,$$

we have that $h^{i+1}([0,1], \partial \overline{B_u}(0,1), \overline{B_s}(0,1)) \cap N = \emptyset$ and therefore for any $y_i \in \overline{B_s}(0,1)$, $(x_{i+1}, y_{i+1}) \in \overline{B_u}(0,1) \times \overline{B_s}(0,1)$ we have

$$(x_{i+1}, y_{i+1}) - h_\lambda^{i+1}(x_i, y_i) \neq 0,$$

hence $K_\lambda(\mathbf{x}) \neq 0$. If $x_m \in \partial \overline{B_u}(0,1)$ from (12) we have $x_0 - g(x_m) \neq 0$ which means that $K_\lambda(\mathbf{x}) \neq 0$. If $y_i \in \partial \overline{B_s}(0,1)$ then for $i > 0$, from the fact that

$$N \overset{f_{\rho_{i-1}}}{\Longrightarrow} N,$$

we have $h^i([0,1], N) \cap \overline{B_u}(0,1) \times \partial \overline{B_s}(0,1) = \emptyset$ and therefore

$$(x_i, y_i) - h_\lambda^i(x_{i-1}, y_{i-1}) \neq 0,$$

for any $x_i \in \overline{B_u}(0,1)$ and $(x_{i-1}, y_{i-1}) \in N$, hence $K_\lambda(\mathbf{x}) \neq 0$. The cases $x_i \in \partial \overline{B_u}(0,1)$ for $i < 0$ and $y_i \in \partial \overline{B_s}(0,1)$ for $i \leq 0$ can be shown by a mirror argument using $f_{\rho_{i+1}}^{-1}$ instead of $f_{\rho_{i-1}}$ (One has to remember that for the inverse map the role of the coordinates is reversed: the x coordinates become stable and y unstable. Otherwise the argument is identical). This finishes establishing (25).

So far, taking (15), (16), (19), (21), (22), (24) and (25) into account we have shown that

$$\deg(F, \mathrm{int} XD, 0) = \deg(K_1, \mathrm{int} XU, 0). \tag{26}$$

If we knew that

$$\deg(K_1, \mathrm{int} XU, 0) \neq 0, \tag{27}$$

then by (26) and the Solution Property of the Brouwer degree (See Section 2.2), we would have that there exists an $\mathbf{x} \in \mathrm{int}\ XD$ such that $F(\mathbf{x}) = 0$, which would finish establishing (14).

To show (27) let us write out the function K_1,

$$\begin{aligned} K_1(\mathbf{x}) =(&\ \theta_{-m} - \rho_{-m}, x_{-m}, y_{-m} - A_{-m} y_{-m+1}, \\ &\ldots \\ &\theta_{-1} - \rho_{-1}, x_{-1}, y_{-1} - A_{-1} y_0, \\ &\theta_0 - \rho_0, x_0 - g(x_m), y_0 - h(y_{-m}), \\ &\theta_1 - \rho_1, x_1 - A_1 x_0, y_1, \\ &\ldots \\ &\theta_m - \rho_m, x_m - A_m x_{m-1}, y_m), \end{aligned} \tag{28}$$

where $A_i = h_1^i$ are linear maps. The map $K_1(\mathbf{x})$ is clearly affine and of the form

$$K_1(\mathbf{x}) = (Id - A)(\mathbf{x} - c),$$

where

$$\begin{aligned} A\mathbf{x} = &(0, 0, A_{-m} y_{-m+1}, \ldots, 0, 0, A_{-1} y_0, \\ &0, g(x_m), h(y_{-m}), 0, A_1 x_0, 0, \ldots, 0, A_m x_{m-1}, 0), \\ c = &(\rho_{-m}, 0, 0, \ldots, \rho_m, 0, 0), \end{aligned}$$

hence by the Degree Property of Affine Maps (see Section 2.2) we have

$$\deg(K_1, XU, 0) = sgn \det(Id - A).$$

We will show that $(Id - A)$ is non-degenerate. Let us assume that $(Id - A)\mathbf{x} = 0$. From (28) we have $x_i = y_{-i} = 0$ for $i = -m, \ldots, -1$. Also

$$\begin{aligned} x_0 &= g \circ A_m \circ A_{m-1} \circ \ldots \circ A_1 x_0, \\ y_0 &= h \circ A_{-m} \circ A_{-m+1} \circ \ldots \circ A_{-1} y_0, \end{aligned}$$

which by the fact that

$$\begin{aligned} A_i \left(\partial \overline{B_u}(0,1) \right) &\subset \mathbb{R}^u \setminus \overline{B_u}(0,1), \\ g \left(\partial \overline{B_u}(0,1) \right) &\subset \mathbb{R}^u \setminus \overline{B_u}(0,1), \\ h \left(\partial \overline{B_u}(0,1) \right) &\subset \mathbb{R}^u \setminus \overline{B_u}(0,1), \end{aligned}$$

means that $x_0 = y_0 = 0$. From the fact that $x_0 = y_0 = 0$ and the fact that $(Id - A)\mathbf{x} = 0$ follows that for $i \geq 1$

$$\begin{aligned} x_i &= A_i \circ \ldots \circ A_1 x_0 = 0, \\ y_{-i} &= A_{-i} \circ \ldots \circ A_{-1} y_0 = 0. \end{aligned}$$

This means that $\det(Id - A) \neq 0$, which finishes the proof of (14).

From the above argument we know that for any $m \in \mathbb{N}$ we have a point $(x_0(m), y_0(m)) \in N$ such that

$$f^i(\rho_0, x_0(m), y_0(m)) \in \ \mathrm{int}\ D \qquad \text{for } i \in \{-m, \ldots m\}.$$

Since the set N is compact we can pass to a convergent subsequence tending to $(x_0, y_0) \in N$ for which $f^i(\rho_0, x_0, y_0) \in D$ for all $i \in \mathbb{Z}$. Our construction implies that $f^i(\rho_0, x_0, y_0)$ has to be in the interior of D for all $i \in \mathbb{Z}$. We therefore know that our set K_{ρ_0} is non-empty and lies in the interior of D, which finishes the proof. □

Remark 1. In Theorem 3.1 the main emphasis should be put on the fact that we have a trajectory in the interior of D starting from an arbitrary $\theta \in \Lambda$. Showing only an existence of an invariant set in the interior of D can be easily done by the use of standard Conley index type arguments. Such arguments though will not give us an invariant set which covers the whole manifold Λ.

Remark 2. Let us note that we have made no strong assumptions on the topology of Λ, or on the dynamics on it. This is the underlying reason which made the proof awkward to handle. For convex Λ, or for $\Lambda = \mathbb{T}^c$, the proof can be considerably simplified, since for these cases it is easy to define homotopies on Λ.

Remark 3. If $f : D \to \mathbb{R}^k$ is a continuous map, instead of a homeomorphism, then from the assumption that f_θ cover N, using an analogous argument (without backward covering), one can establish the existence of a (forward) invariant set for the map f.

Remark 4. The main conditions (10) are local in nature. For (rigorous) computer assisted applications this gives us an opportunity of tailoring the coordinates in D around a given θ and verifying (10) in local coordinates.

Remark 5. The invariant set does not need to be unique in the sense that for a given $\theta \in \Lambda$ we may have more than one x, y for which $f^n(\theta, x, y) \in \mathrm{int} D$ for all $n \in \mathbb{Z}$ (see the example in Section 5.2). We also do not obtain any regularity results for the set $\bigcup_{\theta\in\Lambda} K_\theta$. We do not even know if it is a manifold or not.

4. **Rigorous numerical verification and application of the result.** In this section we will show how Theorem 3.1 may be applied using rigorous computer assisted methods. We will also compare the result with the method of Haro and de la Llave [8]. The method of Haro and de la Llave relies strongly on the fact that the invariant manifold is a normally hyperbolic torus and that the movement on the torus is a rotation. In our method the manifold can be an arbitrary compact sub-manifold in $\mathbb{R}^n$ without a boundary, the movement on it can be arbitrary, and we do not require normal hyperbolicity (As mentioned above, due to the fact that our assumptions are much weaker we lose the uniqueness and regularity results).

Let us start by presenting the result of Haro and de la Llave.

Theorem 4.1. [8] *Let $U \subset \mathbb{R}^n$ be an open set. Let $F : \mathbb{T}^d \times U \subset \mathbb{T}^d \times \mathbb{R}^n \to \mathbb{R}^n$ be a map of class C^{r+1}, with $r \geq 1$ such that for all $\theta \in \mathbb{T}^d$ the map $F(\theta, \cdot) : U \to \mathbb{R}^n$ is a local diffeomorphism. Let $\omega \in \mathbb{R}^d$ be a rotation.*

We consider a skew-product

$$\bar{x} = F(\theta, x), \qquad \bar{\theta} = \theta + \omega,$$

that is, a bundle map on the bundle $E = \mathbb{T}^d \times \mathbb{R}^n$.

Let $K : \mathbb{T}^d \to U \subset \mathbb{R}^n$ be a C^r map such that:

a. K is an approximate invariant torus, that is

$$\|F(\theta, K(\theta)) - K(\theta + \omega)\|_{C^r} \leq \varepsilon.$$

b. The transfer operator $\mathcal{L}$ over the rotation ω, acting on complex sections $\Delta : \mathbb{T}^d \to \mathbb{C}^n$ by

$$\mathcal{L}\Delta(\theta) = DF(\theta - \omega, K(\theta - \omega))\Delta(\theta - \omega),$$

is hyperbolic as an operator on C^0.

Then

1. *If ε is small enough, there exists a C^r map $K_F : \mathbb{T}^d \to U \subset \mathbb{R}^n$ such that*

$$F(\theta, K_F(\theta)) = K_F(\theta + \omega), \tag{29}$$

and $||K_F - K||_{C^r} = O(\varepsilon)$.

2. *The solution K_F above is the only C^0 solution of (29) in a C^0 neighborhood of K.*
3. *The torus K_F is normally hyperbolic.*

Moreover, the map $F \to K_F$ is C^1 when F is given the C^{r+1} topology and K_F the C^r topology.

Let us make some remarks on the above result.

Remark 6. The result depends heavily on the fact that the motion on the torus is a rotation.

Remark 7. The condition b. of Theorem 4.1 is on an infinite dimensional functional space. Its verification most often is not straightforward.

Now we will show how the conditions of Theorem 3.1 can be verified in practice and make a couple remarks to compare the result with Theorem 4.1. First let us introduce the following definition.

Definition 4.2. Let $U \subset \mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be a C^1 function. We define the interval enclosure of df on the set U as

$$[df(U)] = \left\{ A \in \mathbb{R}^{n\times n} | A_{ij} \in \left[\inf_{x\in U} \frac{df_i}{dx_j}(x), \sup_{x\in U} \frac{df_i}{dx_j}(x) \right] \text{ for all } i,j = 1,\ldots,n \right\}.$$

Proposition 1. *Let Λ be a c-dimensional sub-manifold without a boundary (in particular $\Lambda = \mathbb{T}^c$). Let $N = B_u(0,1) \times B_s(0,1) \subset U \subset \mathbb{R}^{u+s}$ and $F : \Lambda \times U \to \Lambda \times U$, be a homeomorphism of the form*

$$F(\theta, p) = f(\theta, p) + g^+(\theta, p),$$
$$F^{-1}(\theta, p) = f^{-1}(\theta, p)) + g^-(\theta, p),$$

where $f = (\mathrm{id}, f_u, f_s) : \Lambda \times U \to \Lambda \times U$ is a diffeomorphism and the functions $g^\pm = (g_c^\pm, g_u^\pm, g_s^\pm)$ satisfy

$$|g_u^\pm(\theta, p)| \le \varepsilon_u^\pm, \tag{30}$$
$$|g_s^\pm(\theta, p)| \le \varepsilon_s^\pm, \tag{31}$$

for all $\theta \in \Lambda$ and $p \in U$. If there exist $\delta_u^+, \delta_s^+, \delta_u^-, \delta_s^- > 0$, such that for any $\theta \in \Lambda$

$$f(\theta, 0) \in \Lambda \times B_u(0, \delta_u^+) \times B_s(0, \delta_s^+), \tag{32}$$
$$f^{-1}(\theta, 0) \in \Lambda \times B_u(0, \delta_u^-) \times B_s(0, \delta_s^-), \tag{33}$$

and for any $\theta \in \Lambda$, $A \in \left[\frac{d(f_u,f_s)}{dp}(\theta,N)\right]$ *and* $C \in \left[\frac{d(f_u,f_s)}{dp}(\theta,N)\right]$ *we have*

$$\inf\{|A_u\,(v_u,v_s)| : |v_u| = 1, |v_s| \le 1\} > 1 + \varepsilon_u^+ + \delta_u^+, \tag{34}$$

$$\sup\{|A_s\,(v_u,v_s)| : |v_s| \le 1, |v_u| \le 1\} < 1 - \varepsilon_s^+ - \delta_s^+, \tag{35}$$

$$\sup\{|C_u\,(v_u,v_s)| : |v_s| \le 1, |v_u| \le 1\} < 1 - \varepsilon_u^- - \delta_u^-, \tag{36}$$

$$\inf\{|C_s\,(v_u,v_s)| : |v_s| = 1, |v_u| \le 1\} > 1 + \varepsilon_s^- + \delta_s^-, \tag{37}$$

then

$$N \overset{F_\theta}{\Longrightarrow} N, \tag{38}$$

$$N \overset{F_\theta^{-1}}{\Longrightarrow} N. \tag{39}$$

In particular there exists a function $K : \Lambda \to 2^{\mathbb{R}^n}\backslash\{\emptyset\}$, *such that for any* $\theta \in \Lambda$ *and* $x \in K(\theta)$

$$\begin{aligned} K(\theta) &\subset \mathrm{int}N, \\ F(\theta,x) &\in K(\theta + g_c^+(\theta,x)), \\ F^{-1}(\theta,x) &\in K(\theta + g_c^-(\theta,x)). \end{aligned} \tag{40}$$

Proof. To show (38), for any $\theta \in \Lambda$ we need to define the homotopy h from Definition 2.2. For a given θ and for any $p = (p_u,p_s) \in N$, $\lambda \in [0,1]$ we define h as

$$\begin{aligned} h(\lambda,p) = (1-\lambda)f_\theta(0) + \Bigg(\int_0^1 \left(\frac{d\,(f_u,f_s)}{dp}(\theta,(1-\lambda)tp)\right)_u dt \cdot (p_u,(1-\lambda)p_s) \\ , (1-\lambda)\int_0^1 \left(\frac{d\,(f_u,f_s)}{dp}(\theta,tp)\right)_s dt \cdot p\Bigg) + (1-\lambda)g^+(\theta,p). \end{aligned}$$

Since for $i = u,s$

$$(f_\theta(p) - f_\theta(0))_i = \int_0^1 \left(\frac{d\,(f_u,f_s)}{dp}(\theta,tp)\right)_i dt \cdot p,$$

we have $h(0,p) = f_\theta(p) + g^+(\theta,p) = F_\theta(p)$. For $\lambda = 1$ we have

$$h(1,p) = (Ap_u, 0) \quad \text{with} \quad Ap_u := \left(\frac{d\,(f_u,f_s)}{dp}(\theta,0)\right)_u (p_u,0)\,.$$

For any λ from $[0,1]$, from the fact that

$$A^\lambda := \int_0^1 \frac{d\,(f_u^+,f_s^+)}{dp}(\theta,(1-\lambda)tp)dt \in \left[\frac{d\,(f_u^+,f_s^+)}{dp}(\theta,N)\right],$$

for any $p \in N^- = \partial\overline{B_u}(0,1) \times \overline{B_s}(0,1)$ using (30), (32) and (34) we have

$$\begin{aligned} |\,(h(\lambda,p))_u\,| &\ge |\int_0^1 \left(\frac{d\,(f_u^+,f_s^+)}{dp}(\theta,(1-\lambda)tp)\right)_u dt \cdot (p_u,(1-\lambda)p_s)\,| \\ &\quad - \delta_u^+ - \varepsilon_u^+ \\ &= |A_u^\lambda\,(p_u,(1-\lambda)p_s)\,| - \delta_u^+ - \varepsilon_u^+ \\ &> 1. \end{aligned} \tag{41}$$

This proves that for any $\lambda \in [0,1]$ we have $h(\lambda,N^-) \cap N = \emptyset$. Also for $\lambda = 1$ from (41), since $A(x) = A^1(x,0)$, we have $A\,(\partial B_u(0,1)) \cap \overline{B_u}(0,1) = \emptyset$.

For $p \in N$, using (31), (33) and (35) we have

$$\begin{aligned} |\left(h(\lambda,p)\right)_s| &\leq |(1-\lambda)\int_0^1 \left(\frac{d\left(f_u,f_s\right)}{dp}(\theta,tp)\right)_s dt\cdot p| + \delta_s^+ + \varepsilon_s^+ \\ &= (1-\lambda)|A_s^0 p| + \delta_s^+ + \varepsilon_s^+ \\ &< 1, \end{aligned}$$

which means that $h([0,1],N)\cap N^+ = \emptyset$. This finishes establishing (38).

Using a mirror argument (keeping in mind that the role of the stable and unstable directions is reversed for the inverse map) we can show that for any $\theta \in \Lambda$ we have (39). The result (40) follows directly from Theorem 3.1. □

Remark 8. For Proposition 1 to hold we do not need to assume that the manifold Λ is a torus. What is more we do not need any assumptions on the dynamics on Λ. In particular the conditions (34),...,(37) do not imply normal hyperbolicity, since F need not be differentiable, and the movement on Λ can be arbitrary.

Remark 9. The conditions (34),...,(37) involve only standard derivatives on compact sets. They can be verified using rigorous numerics. The result can be used for (rigorous) computer assisted proofs. What is more, both Theorem 3.1 and Proposition 1 can give us an explicit bound on the size of the perturbation under which the invariant set persists (see examples in Section 5).

Remark 10. If we have a normally hyperbolic invariant manifold in $\mathbb{R}^{c+n}$ then Proposition 1 can be applied. To do so one has to consider the coordinates in line with the directions of hyperbolic contraction and expansion and focus on a small neighborhood of the manifold.

Remark 11. Since the result is established through topological tools only, we loose all of the regularity results of our invariant set. In comparison with Theorem 4.1 the result is quite weak. What is more, as it stands, it does not give a possibility of detecting the stable and unstable manifolds as is done in [8]. The advantage of the method lies in the simplicity of the required conditions.

It should be possible though to obtain some regularity results by adding appropriate cone conditions to the assumptions. The cone conditions should also give us results for the foliations of the stable and unstable manifolds. The results which will include the cone conditions in the spirit of [16] are under preparation and will appear in forthcoming work.

5. **Application of the results.**

In this Section we will present two examples. The first is the rotating Hénon map. The map has been studied in [8], [11] and [13]. The existence of the invariant set will be established by the use of Proposition 1. In the second example we will deal with a "toy" problem, which has a degenerate derivative on Λ. For such cases Proposition 1 cannot be applied. The existence of the invariant set is proved by applying Theorem 3.1 directly. This example demonstrates also that for a $\theta \in \Lambda$ we can have more than just one point in K_θ.

5.1. **The rotating Hénon map.**

5.1.1. *Statement of the problem.* We will consider the rotating Hénon map

$$F_\varepsilon : \Lambda \times \mathbb{R}^2 \to \Lambda \times \mathbb{R}^2,$$
$$F_\varepsilon(\theta, x, y) = (\theta + g_c^+(\theta, x, y), 1 + y - ax^2 + \varepsilon \cos(2\pi\theta), bx), \tag{42}$$

where Λ is a c-dimensional submanifold without a boundary (say $\Lambda = \mathbb{T}^c$), and $g_c : \Lambda \times \mathbb{R}^2 \to \Lambda$ is such that the map (42) is a diffeomorphism.

The dynamics of (42) with $a = 0.68$, $b = 0.1$, $\Lambda = \mathbb{T}^1$ and

$$g_c^+(\theta, x, y) = \theta + \omega \pmod 1, \tag{43}$$

$\omega \in \mathbb{R}$, has been investigated by Haro and De la Llave in [8], for a demonstration of a numerical algorithm for finding invariant manifolds and their whiskers in quasi periodically forced systems.

In this section we will not use the assumption (43) that g_c^+ is a rotation. What is more we do not assume that Λ is a one dimensional torus. In this more general setting we will prove that for the parameters $a = 0.68$ and $b = 0.1$, for all $\varepsilon \leq \frac{1}{2}$, there exists an invariant set of (42) which covers the manifold Λ and is contained in a set

$$U_\varepsilon = \Lambda \times [x_- - 1.1\varepsilon, x_- + 1.1\varepsilon] \times [y_- - 0.12\varepsilon, y_- + 0.12\varepsilon],$$

where (x_-, y_-) is a fixed point for the (standard) Hénon map,

$$x_- = \frac{-(1-b) - \sqrt{(1-b)^2 + 4a}}{2a} \approx -2.043\,3,$$
$$y_- = bx_- \approx -0.204\,33.$$

Remark 12. Since we do not have any strong assumptions on g_c^+ this example is not normally hyperbolic. What is more, since Proposition 1 is our main tool, using the same method as is presented below we could obtain similar results for a non-differentiable perturbation on all three coordinates. This is not done to keep the example relatively simple and in line with [8].

Remark 13. If we include (43) into our assumptions, then the example becomes normally hyperbolic. This though does not make it less interesting, since we not only obtain the persistence of the manifold (which is automatic in such setting), but also obtain explicit bounds on the size of the perturbation and the size of the region in which the invariant set is contained. Such bounds are not easily obtainable from the normal hyperbolicity theory.

5.1.2. *The unperturbed map.* We start by investigating the case of $\varepsilon = 0$. We will ignore the coordinate θ and concentrate on a map

$$F(x, y) = (1 + y - ax^2, bx).$$

The point (x_-, y_-) is one of the two fixed points $(x_\pm, y_\pm)$ of the map F

$$x_\pm = \frac{-(1-b) \pm \sqrt{(1-b)^2 + 4a}}{2a}, \quad y_\pm = bx_\pm.$$

We have

$$DF(x, y) = \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix},$$

with two eigenvalues $\lambda_1 = -ax + \sqrt{b + a^2x^2}$, $\lambda_2 = -ax - \sqrt{b + a^2x^2}$. For (x_-, y_-) the eigenvalues are

$$\lambda_1 \approx 2.\,814\,4, \quad \lambda_2 \approx -3.\,553\,1 \times 10^{-2}.$$

We will consider the following Jordan forms of the matrix $DF(x_-,y_-)$

$$DF(x_-,y_-)=\Phi_\varepsilon J\Phi_\varepsilon^{-1},$$

$$\Phi_\varepsilon=\varepsilon\kappa\begin{pmatrix}\tau & \eta\\ -\tau\lambda_2 & -\lambda_1\eta\end{pmatrix},\quad J=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix},\quad \Phi_\varepsilon^{-1}=\frac{1}{\varepsilon}\begin{pmatrix}-\frac{1}{\tau}\lambda_1 & -\frac{1}{\tau}\\ \frac{1}{\eta}\lambda_2 & \frac{1}{\eta}\end{pmatrix},$$

where $\kappa=1/(\lambda_2-\lambda_1)$. The constants τ,η serve the purpose of an appropriate rescaling of the stable and unstable directions in the local coordinates, and will be chosen later on. When we will consider the perturbed Hénon map in Section 5.1.3, for a given $\varepsilon>0$ we will use the maps Φ_ε and Φ_ε^{-1}.

We introduce local coordinates of hyperbolic expansion and contraction around the point (x_-,y_-) as

$$(\tilde{x},\tilde{y})=\Phi_\varepsilon^{-1}\left(x-x_-,y-y_-\right). \tag{44}$$

The map F in the local coordinates is

$$\tilde{F}(\tilde{x},\tilde{y})=\Phi_\varepsilon^{-1}\left(F\left(\Phi_\varepsilon(\tilde{x},\tilde{y})+(x_-,y_-)\right)-(x_-,y_-)\right),$$

and its derivative $d\tilde{F}$ is equal to

$$\begin{aligned}
d\tilde{F}(\tilde{x},\tilde{y}) &= \Phi_\varepsilon^{-1}\circ dF(\Phi_\varepsilon(\tilde{x},\tilde{y})+(x_-,y_-))\circ\Phi_\varepsilon\\
&= \Phi_\varepsilon^{-1}\circ dF\begin{pmatrix}\varepsilon\kappa\left(\tau\tilde{x}+\eta\tilde{y}\right)+x_-\\ -\varepsilon\kappa\left(\tau\lambda_2\tilde{x}+\eta\lambda_1\tilde{y}\right)+y_-\end{pmatrix}\circ\Phi_\varepsilon\\
&= \Phi_\varepsilon^{-1}\circ\begin{pmatrix}-2a\left(\varepsilon\kappa\left(\tau\tilde{x}+\eta\tilde{y}\right)+x_-\right) & 1\\ b & 0\end{pmatrix}\circ\Phi_\varepsilon\\
&= \Phi_\varepsilon^{-1}\circ\left(\begin{pmatrix}-2ax_- & 1\\ b & 0\end{pmatrix}+\begin{pmatrix}-2a\varepsilon\kappa\left(\tau\tilde{x}+\eta\tilde{y}\right) & 0\\ 0 & 0\end{pmatrix}\right)\circ\Phi_\varepsilon\\
&= J+R_\varepsilon,
\end{aligned}$$

where

$$R_\varepsilon=-2a\varepsilon\kappa^2\left(\tau\tilde{x}+\eta\tilde{y}\right)\begin{pmatrix}-\lambda_1 & -\frac{\eta}{\tau}\lambda_1\\ \frac{\tau}{\eta}\lambda_2 & \lambda_2\end{pmatrix}.$$

For any $\tilde{x},\tilde{y}\in[-1,1]$ we have the following estimates, which will be used later on for the verification of the covering conditions

$$\begin{aligned}
&[d\tilde{F}(B(0,1)\times B(0,1))]\\
&\subset\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix}+\varepsilon\left(\tau+\eta\right)\begin{pmatrix}[-\frac{1}{2},\frac{1}{2}] & [-\frac{1}{2}\frac{\eta}{\tau},\frac{1}{2}\frac{\eta}{\tau}]\\ [-\frac{6}{1000}\frac{\tau}{\eta},\frac{6}{1000}\frac{\tau}{\eta}] & [-\frac{6}{1000},\frac{6}{1000}]\end{pmatrix}.
\end{aligned} \tag{45}$$

Now we turn to the inverse map. The inverse map to F is

$$F^{-1}(x,y)=\left(\frac{1}{b}y,-1+x+\frac{a}{b^2}y^2\right),$$

and has a derivarive

$$dF^{-1}(x,y)=\begin{pmatrix}0 & \frac{1}{b}\\ 1 & \frac{2a}{b^2}y\end{pmatrix}.$$

In the local coordinates (44) the inverse map is

$$\tilde{F}^{-1}(\tilde{x},\tilde{y})=\Phi_\varepsilon^{-1}\left(F^{-1}\left(\Phi_\varepsilon(\tilde{x},\tilde{y})+(x_-,y_-)\right)-(x_-,y_-)\right),$$

and its derivative $d\tilde{F}^{-1}$ is equal to

$$
\begin{aligned}
d\tilde{F}^{-1}(\tilde{x},\tilde{y}) &= \Phi_{\varepsilon}^{-1}\circ dF^{-1}(\Phi_{\varepsilon}(\tilde{x},\tilde{y})+(x_-,y_-))\circ\Phi_{\varepsilon}\\
&= \Phi_{\varepsilon}^{-1}\circ dF^{-1}\begin{pmatrix} \varepsilon\kappa\left(\tau\tilde{x}+\eta\tilde{y}\right)+x_- \\ -\varepsilon\kappa\left(\tau\lambda_2\tilde{x}+\eta\lambda_1\tilde{y}\right)+y_-\end{pmatrix}\circ\Phi_{\varepsilon}\\
&= \Phi_{\varepsilon}^{-1}\circ\begin{pmatrix} 0 & \frac{1}{b} \\ 1 & \frac{2a}{b^2}\left(-\varepsilon\kappa\left(\tau\lambda_2\tilde{x}+\eta\lambda_1\tilde{y}\right)+y_-\right)\end{pmatrix}\circ\Phi_{\varepsilon}\\
&= \Phi_{\varepsilon}^{-1}\circ\left(\begin{pmatrix} 0 & \frac{1}{b} \\ 1 & \frac{2a}{b^2}y_-\end{pmatrix}+\begin{pmatrix} 0 & 0 \\ 0 & -\frac{2a}{b^2}\varepsilon\kappa\left(\tau\lambda_2\tilde{x}+\eta\lambda_1\tilde{y}\right)\end{pmatrix}\right)\circ\Phi_{\varepsilon}\\
&= J^{-1}+R'_{\varepsilon},
\end{aligned}
$$

where

$$
R'_{\varepsilon}=\frac{2a}{b^2}\varepsilon\kappa^2\left(\tau\lambda_2\tilde{x}+\eta\lambda_1\tilde{y}\right)\begin{pmatrix} -\lambda_2 & -\frac{\eta}{\tau}\lambda_1 \\ \frac{\tau}{\eta}\lambda_2 & \lambda_1\end{pmatrix}.
$$

For $\tilde{x},\tilde{y}\in[-1,1]$ this gives us the following estimates

$$
\begin{aligned}
&[d\tilde{F}^{-1}(B(0,1)\times B(0,1))] \qquad (46)\\
&\subset\begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2}\end{pmatrix}+\varepsilon\left(\tau\left|\lambda_2\right|+\eta|\lambda_1|\right)\begin{pmatrix} [-\frac{6}{10},\frac{6}{10}] & [-50\frac{\eta}{\tau},50\frac{\eta}{\tau}] \\ [-\frac{\tau}{\eta}\frac{6}{10},\frac{\tau}{\eta}\frac{6}{10}] & [-50,50]\end{pmatrix}.
\end{aligned}
$$

5.1.3. *Verification of the covering conditions.* Let $N=\overline{B}(0,1)\times\overline{B}(0,1)$. We define $\phi:\Lambda\times\mathbb{R}^2\to\Lambda\times\mathbb{R}^2$ as

$$
\begin{aligned}
\phi(\theta,x,y) &= (\theta,\Phi_{\varepsilon}^{-1}\left(x-x_-,y-y_-\right)),\\
\phi^{-1}(\theta,x,y) &= (\theta,\Phi_{\varepsilon}\left(x,y\right)+(x_-,y_-)).
\end{aligned}
$$

We will show that for any $\theta\in\Lambda$

$$
N\overset{(F_{\varepsilon})_{\theta}}{\Longrightarrow}N, \qquad (47)
$$

$$
N\overset{(F_{\varepsilon}^{-1})_{\theta}}{\Longrightarrow}N. \qquad (48)
$$

We will now apply Proposition 1 to establish (47). From (7) and the fact that $F_0(\theta,x_-,y_-)=\left(\theta+g_c^{+}\left(\theta,x,y\right),x_-,y_-\right)$ we have

$$
\begin{aligned}
\left(F_{\varepsilon}\right)_{\theta}(0,0) &= \pi_{us}\circ\phi\circ F_{\varepsilon}\circ\phi^{-1}\left(\theta,0,0\right)\\
&= \pi_{us}\circ\phi\circ F_{\varepsilon}(\theta,x_-,y_-)\\
&= \pi_{us}\circ\phi\left(F_0(\theta,x_-,y_-)+(0,\varepsilon\cos2\pi\theta,0)\right)\\
&= \pi_{us}\circ\left(\theta+g_c^{+}\left(\theta,x,y\right),\Phi_{\varepsilon}^{-1}\left((0,0)+(\varepsilon\cos2\pi\theta,0)\right)\right)\\
&= (-\frac{1}{\tau}\lambda_1\cos2\pi\theta,\frac{1}{\eta}\lambda_2\cos2\pi\theta),
\end{aligned}
$$

which gives us

$$
\left(f_{\varepsilon}\right)_{\phi}\left(\Lambda,0,0\right)\subset\Lambda\times\overline{B_u}(0,\frac{1}{\tau}\left|\lambda_1\right|)\times\overline{B_s}(0,\frac{1}{\eta}\left|\lambda_2\right|). \qquad (49)
$$

From (45) we have that for any $\theta\in\Lambda$ and $A\in[\frac{d(F_{\varepsilon})_{\theta}}{d(x,y)}(N)]$

$$
\begin{aligned}
&\inf\left\{\left|A_u(0,x,y)\right|:|x|=1,|y|\leq1\right\}\geq\left|\lambda_1\right|-\varepsilon\left(\tau+\eta\right)\frac{1}{2}\left(1+\frac{\eta}{\tau}\right), \qquad (50)\\
&\sup\left\{\left|A_s(0,x,y):|x|\leq1,|y|\leq1\right|\right\}\leq\left|\lambda_2\right|+\varepsilon\left(\tau+\eta\right)\frac{6}{1000}\left(1+\frac{\tau}{\eta}\right).
\end{aligned}
$$

From (49) and (50), by Proposition 1 (in our case since F_ε is a diffeomorphism $\varepsilon_u^{\pm} = \varepsilon_s^{\pm} = 0$. Also $\delta_u^+ = |\lambda_1|/\tau$, $\delta_s^+ = |\lambda_2|/\eta$), if we have

$$|\lambda_1| - \varepsilon(\tau+\eta)\frac{1}{2}\left(1+\frac{\eta}{\tau}\right) > 1 + \frac{1}{\tau}|\lambda_1|, \tag{51}$$

$$|\lambda_2| + \varepsilon(\tau+\eta)\frac{6}{1000}\left(1+\frac{\tau}{\eta}\right) < 1 - \frac{1}{\eta}|\lambda_2|, \tag{52}$$

then we have established (47). The conditions (51) and (52) hold for all $\varepsilon \leq \frac{1}{2}$ with $\tau = 3$, $\eta = \frac{3}{40}$.

To establish (48) we first compute

$$F_\varepsilon^{-1}(\theta,x,y) = \left(\left(F_\varepsilon^{-1}(\theta,x,y)\right)_c, \frac{1}{b}y, x' - 1 + \frac{a}{b^2}y^2 - \varepsilon\cos\left(2\pi F_\varepsilon^{-1}(\theta,x,y)\right)\right),$$

which gives

$$\begin{aligned}
\left(F_\varepsilon^{-1}\right)_\theta(0,0) &= \pi_{us}\circ\phi\circ F_\varepsilon^{-1}\circ\phi^{-1}(\theta,0,0) \\
&= \pi_{us}\circ\phi\circ F_\varepsilon^{-1}(\theta,x_-,y_-) \\
&= \pi_{us}\circ\phi\left(F_0^{-1}(\theta,x_-,y_-) + (0,0,\varepsilon\cos\left(2\pi\left(F_\varepsilon^{-1}(\theta,x,y)\right)_c\right))\right) \\
&= \pi_{us}\circ\left(\left(F_0^{-1}(\theta,x_-,y_-)\right)_c,\right. \\
&\qquad \left.\Phi_\varepsilon^{-1}\left((0,0) + (0,\varepsilon\cos\left(2\pi\left(F_\varepsilon^{-1}(\theta,x,y)\right)_c\right))\right)\right) \\
&= \left(-\frac{1}{\tau}\cos\left(2\pi\left(F_\varepsilon^{-1}(\theta,x,y)\right)_c\right), \frac{1}{\eta}\cos\left(2\pi\left(F_\varepsilon^{-1}(\theta,x,y)\right)_c\right)\right),
\end{aligned}$$

hence

$$(F_\varepsilon)_\phi^{-1}(\tilde{\theta},0,0) \subset B(0,\frac{1}{\tau})\times B(0,\frac{1}{\eta}). \tag{53}$$

From (46) we know that for any $\theta\in\Lambda$ and $B \in [\frac{d\left(F_\varepsilon^{-1}\right)_\theta}{d(x,y)}(N)]$ we have (let us note that the roles of the stable and unstable coordinates have been exchanged with respect to the forward map)

$$\begin{aligned}
\inf\{|B_u(0,x,y)| : |x|=1, |y|\leq 1\} &\geq \left|\frac{1}{\lambda_2}\right| - \varepsilon(\tau|\lambda_2| + \eta|\lambda_1|)\left(\frac{\tau}{\eta}\frac{6}{10} + 50\right), \\
\sup\{|B_s(0,x,y)| : |x|\leq 1, |y|\leq 1\} &\leq \frac{1}{\lambda_1} + \varepsilon(\tau|\lambda_2| + \eta|\lambda_1|)\left(\frac{6}{10} + 50\frac{\eta}{\tau}\right).
\end{aligned}$$

Hence from (53), by Proposition 1 (in our case $\varepsilon_u^{\pm} = \varepsilon_s^{\pm} = 0$, $\delta_u^+ = 1/\tau$ and $\delta_s^+ = 1/\eta$), if we have

$$\left|\frac{1}{\lambda_2}\right| - \varepsilon(\tau|\lambda_2| + \eta\lambda_1)\left(\frac{\tau}{\eta}\frac{6}{10} + 50\right) > 1 + \frac{1}{\eta}, \tag{54}$$

$$\frac{1}{\lambda_1} + \varepsilon(\tau|\lambda_2| + \eta\lambda_1)\left(\frac{6}{10} + 50\frac{\eta}{\tau}\right) < 1 - \frac{1}{\tau}, \tag{55}$$

then we have established (48). The conditions (54) and (55) hold for $\varepsilon \leq \frac{1}{2}$ with $\tau = 3$, $\eta = \frac{3}{40}$.

5.1.4. *The estimate of the region in which the invariant set is contained.* So far we have shown that for $\varepsilon \leq \frac{1}{2}$ we have the covering relations (47), (48) for $N = \overline{B}(0,1)\times\overline{B}(0,1)$. This means that we have an invariant set which covers the manifold inside a set

$$D = \phi^{-1}\left(\Lambda \times N\right).$$

This gives us the following bounds

$$\begin{aligned} D &= \phi^{-1}\left(\Lambda \times N\right) \\ &= \Lambda \times \{(x_-, y_-) + \Phi_\varepsilon\left(\overline{B}(0,1) \times \overline{B}(0,1)\right)\} \\ &\subset \Lambda \times \{(x_-, y_-) + [-\varepsilon|\kappa|(\tau+\eta), \varepsilon|\kappa|(\tau+\eta)] \\ &\quad \times [-\varepsilon|\kappa|\left(\tau|\lambda_2| + \eta|\lambda_1|\right), \varepsilon|\kappa|\left(\tau|\lambda_2| + \eta|\lambda_1|\right)]\}. \end{aligned}$$

With $\tau = 3$ and $\eta = \frac{3}{40}$ this gives us $D \subset U_\varepsilon$.

5.2. **An example with a degenerate derivative on Λ.** The conditions (34), ...,(37) might suggest that for the application of Theorem 3.1 one should need hyperbolic contraction and expansion. This is not the case. The result is purely topological and does not rely on hyperbolicity. To demonstrate this let us consider the following example for which we apply Theorem 3.1 directly, without using Proposition 1.

5.2.1. *Statement of the problem.* Let $h : \mathbb{R} \to \mathbb{R}$,

$$h(x) = \begin{cases} 3x - 2 & \text{for } x \geq 1 \\ x^3 & \text{for } |x| < 1 \\ 3x + 2 & \text{for } x \leq -1. \end{cases}$$

Let us consider the following ODE

$$\begin{aligned} \dot{\theta} &= g(\theta, x, y) + \varepsilon_1(\theta, x, y) \\ \dot{x} &= h(x) + \varepsilon_2(\theta, x, y) \\ \dot{y} &= -h(y) + \varepsilon_3(\theta, x, y), \end{aligned} \tag{56}$$

where $\theta \in \mathbb{T}^n$ and $g, \varepsilon_1, \varepsilon_2, \varepsilon_3$ are locally Lipschitz.

Let

$$M := \frac{3}{e^3 - 1}\left(\frac{1}{\sqrt{5}} - \frac{1}{2}\right).$$

Using Theorem 3.1 we will show that if

$$\sup\{|\varepsilon_i(\theta, x, y)|, \theta \in \mathbb{T}^n, x, y \in B(0,1)\} < M \qquad \text{for } i = 2, 3, \tag{57}$$

then for any $\theta_0 \in \mathbb{T}^n$ there exists a trajectory of (56), starting at (θ_0, x_0, y_0) for some

$$(x_0, y_0) \in B(0, \frac{1}{2}) \times B(0, \frac{1}{2}),$$

which stays in the interior of the set $\mathbb{T}^n \times B(0, \frac{1}{2}) \times B(0, \frac{1}{2})$.

5.2.2. *Verification of the covering conditions.* Let $\varepsilon := (\varepsilon_1, \varepsilon_2, \varepsilon_3)$ and $f^\varepsilon : \mathbb{T}^n \times \mathbb{R}^2 \to \mathbb{T}^n \times \mathbb{R}^2$, $f^\varepsilon = (f_c^\varepsilon, f_u^\varepsilon, f_s^\varepsilon)$ be a time $t = 1/2$ shift along the trajectory of (56). First we consider $\varepsilon = 0$. For $x, y \in B(0, \frac{1}{2})$ we have

$$f_u^0(\theta, x, y) = \begin{cases} \frac{1}{\sqrt{\frac{1}{x^2}-1}} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -\frac{1}{\sqrt{\frac{1}{x^2}-1}} & \text{for } x < 0 \end{cases} \qquad f_s^0(\theta, x, y) = \begin{cases} \frac{1}{\sqrt{\frac{1}{y^2}+1}} & \text{for } y > 0 \\ 0 & \text{for } y = 0 \\ -\frac{1}{\sqrt{\frac{1}{y^2}+1}} & \text{for } y < 0. \end{cases}$$

Taking $u = s = 1$ and $N := B_u(0, \frac{1}{2}) \times B_s(0, \frac{1}{2})$ we have that for any $\theta \in \mathbb{T}^n$

$$\left(f_u^0, f_s^0\right)(\theta, N) = B_u(0, \frac{1}{\sqrt{3}}) \times B_u(0, \frac{1}{\sqrt{5}}), \tag{58}$$

which means that

$$N \overset{f_\theta^0}{\Longrightarrow} N.$$

Let $\phi^\varepsilon(t) = (\theta^\varepsilon(t), x^\varepsilon(t), y^\varepsilon(t))$ be a solution of (56) with an initial condition

$$(\theta_0, x_0, y_0) \in \mathbb{T}^n \times \overline{B}(0, \frac{1}{2}) \times \overline{B}(0, \frac{1}{2}).$$

Using (57) and the fact that f_u^0 and f_s^0 are Lipschitz with a constant $L = 3$ we have

$$|x^\varepsilon(1) - x^0(1)| < \frac{M}{L}(e^L - 1) = \frac{1}{\sqrt{5}} - \frac{1}{2}, \tag{59}$$

$$|y^\varepsilon(1) - y^0(1)| < \frac{M}{L}(e^L - 1) = \frac{1}{\sqrt{5}} - \frac{1}{2}. \tag{60}$$

This means that for ε satisfying (57), from (58), (59) and (60) we have $N \overset{f_\theta^\varepsilon}{\Longrightarrow} N$. A mirror argument can be applied to show that $N \overset{(f^\varepsilon)_\theta^{-1}}{\Longrightarrow} N$, which by Theorem 3.1 establishes our result.

Remark 14. The invariant set obtained in Example 5.2 does not need to be a single torus. We can for example take

$$\varepsilon_2(\theta, x, y) = -\varepsilon x,$$
$$\varepsilon_3(\theta, x, y) = \varepsilon y,$$

with small ε. Both the flows on x and y are now independent. Looking at the x coordinate, for $\varepsilon \leq 0$ we have a single fixed point at zero. For $\varepsilon > 0$ we have three fixed points: $-\sqrt{\varepsilon}$, 0 and $\sqrt{\varepsilon}$. An analogous discussion can be made for fixed points on the y coordinate.

Looking now at the full system, for $\varepsilon \leq 0$ we have a single invariant torus $\mathbb{T}^n \times \{0\}$, but for $\varepsilon > 0$ the invariant set is $\mathbb{T}^n \times [-\sqrt{\varepsilon}, \sqrt{\varepsilon}]^2$ (see Figire 2).

Remark 15. We can see that in Example 5.2, Theorem 3.1 allowed us to obtain an explicit bound on the size of the perturbation under which the invariant set persists. Let us also note that we have made no strong assumptions about the functions g and ε_1.

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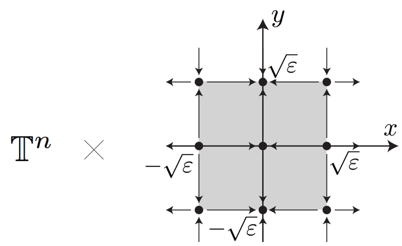

FIGURE 2. The invariant set for the system in Example 5.2 with $\varepsilon_2(\theta, x, y) = -\varepsilon x$, $\varepsilon_3(\theta, x, y) = \varepsilon y$ and $\varepsilon > 0$.

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