

Transition Tori in the Planar Restricted Elliptic Three Body Problem

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Abstract. We consider the elliptic three body problem as a perturbation of the circular problem. We show that for sufficiently small eccentricities of the elliptic problem, and for energies sufficiently close to the energy of the libration point L_2 , a Cantor set of Lyapounov orbits survives the perturbation. The orbits are perturbed to quasi-periodic invariant tori. We show that for a certain family of masses of the primaries, for such tori we have transversal intersections of stable and unstable manifolds, which lead to chaotic dynamics involving diffusion over a short range of energy levels. Some parts of our argument are nonrigorous, but are strongly backed by numerical computations.

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1. Introduction

In the planar restricted circular three body problem (PRC3BP) two large masses μ and $1 - \mu$ rotate on planar circular Keplerian orbits. For convenience we will call the larger body - the Sun and the smaller massive body - the planet. The problem deals with the motion of a third massless particle (the comet or the spacecraft), which moves on the same plane as the two larger bodies under their gravitational pull. This problem was considered by Llibre, Martinez and Simo in [20] for energies of solutions close to the energy of the libration point L_2^μ . There it has been shown that there exists a family of parameters $\{\mu_k\}_{k=2}^\infty$ for which we have a homoclinic orbit to the libration point $L_2^{\mu_k}$. Moreover, it has been shown that for μ close to any of the values μ_k , for a Lyapounov orbit around L_2^μ with energy sufficiently close to the energy of L_2^μ , its stable and unstable manifolds intersect transversally. This dynamics is restricted to a constant energy manifold and leads to a homoclinic tangle and symbolic dynamics. Later a similar problem has been numerically investigated by Koon, Lo, Marsden and Ross [17], where smaller energies were considered. In such a case the chaotic dynamics is extended to include homoclinic and heteroclinic tangles along stable and unstable manifolds of Lyapounov orbits around both the libration point L_1^μ and L_2^μ . This has been later proven by Wilczak and Zgliczyński using a method of covering relations and rigorous computer assisted computations in [28, 29], for the case of the Sun-Jupiter system and the energy of the comet Oterma.

All of the above mentioned results have a common feature: since the problem follows from an autonomous Hamiltonian, the transversality of the intersections and the chaotic dynamics of the system are always restricted to a constant energy manifold. In this paper we are going to consider the planar restricted elliptic three body problem (PRE3BP), where the equations are no longer autonomous, which means that a change of energy of solutions is possible. We will consider the circular problem considered by Llibre, Martinez and Simo in [20] and generalize it to allow the orbits of the planet and the Sun to be elliptic with small eccentricities e . We will treat this as a perturbation of the circular case. We will show that most of the Lyapounov orbits around L_2^μ persist under such perturbation as KAM-tori. Moreover, we will show that the symbolic dynamics associated with these orbits also survives. This kind of the 'structural stability' of symbolic dynamics constitute the main result of this paper. It will turn out that we also have chaotic diffusion along the energy level. In effect the dynamics of the elliptic problem is by one dimension richer than the dynamics of the circular problem, where all solutions are restricted to a constant energy manifold.

The diffusion between energies discussed in this paper follows from a mechanism similar to the 1964 Arnold's example [2]. Arnold conjectured that this phenomenon appears in the three body problem. The result of this paper is a small step towards a proof of this conjecture, but the described dynamics does not fulfill all requirements. First of all, prior to perturbation we do not have a fully integrable system. We start with the circular problem with a setting in which we already have a transversal homoclinic

connection between Lyapounov orbits, which in our setting play the role of lower dimensional normally hyperbolic invariant tori. Such systems are referred to as "a priory unstable", or even "a priory chaotic". Secondly, and most importantly, our diffusion is between energies with distance of order $(e\mu^{1/3})^{1/2}$ and not of order one.

Throughout some of the so far explored examples a certain pattern can be observed in the methods used for problems involving diffusion in priory unstable systems (see for example [22] for a result of Moeckel on detection of transition tori in the case of the planar five body problem; or the work of Delshams, Llave and Seara [8] on diffusion of energy for perturbations of geodesic flows on a two dimensional torus; also Wiggins [25], [26] discusses this mechanism in the case of perturbations of completely integrable systems). First a normally hyperbolic invariant manifold foliated by invariant tori is found. The tori are required to have hyperbolic stable and unstable manifolds and a transversal intersection of these manifolds. Secondly a perturbation of the system is considered. By perturbation theory ([15], [27]) of normally hyperbolic manifolds, the normally hyperbolic invariant manifold and its stable and unstable manifolds persist under the perturbation. Next step is to show that on the perturbed invariant manifold most of the invariant tori survive. This under appropriate nondegeneracy conditions is a result of the Kolmogorov Arnold Moser Theory (KAM) [3],[16]. Using KAM-technics (for example [14], [15] or [31]) it can be shown that most of the invariant tori persist and form a Cantor set having a positive measure in the invariant manifold. The last step is to show that the stable and unstable manifolds of the surviving tori intersect transversally. This can be done by the use of a Melnikov type method along a homoclinic orbit of the unperturbed problem. The transversal intersections between the invariant manifolds of the perturbed tori lead to homoclinic tangles for each of the surviving tori. In addition to this we also have a chaotic diffusion along the Cantor set of homoclinic tangles between the tori. In this paper we will follow this procedure.

When applying the method to prove the existence of transition chains for a given physical problem the steps of the above described procedure, which present the biggest obstacles are usually the verification of the assumptions of the KAM theorem and computation of the Melnikov integral.

The fundamental role for our investigation is played by the Hill's problem. In the neighborhood of the libration point the PCR3BP and PER3BP, when written in a suitably rescaled Hill's coordinates, are perturbations of the Hill's problem depending on two small parameters μ and e . This gives us a 'local' picture around L_2^μ , the existence of normally hyperbolic invariant manifold and KAM-tori. In our case the twist property needed for the KAM theorem follows from the Lyapounov-Moser Theorem [23]. For sufficiently small μ we will prove the twist property for the family of periodic orbits around L_2^μ , by approximating the PRC3BP with the Hill's problem and thus obtain the following theorem (for a detailed formulation see Theorem 31).

Theorem 1 *There exist positive constants $R_{Hill}, \kappa, \mu^* \in \mathbb{R}$ such that for all mass parameters $\mu < \mu^*$ and any perturbation e such that $e\mu^{-2/3} < \kappa$ most Lyapounov orbits with radii in Hill's coordinates not exceeding R_{Hill} are perturbed to quasi periodic orbits*

(in other words, to invariant two dimensional invariant tori in extended phase space). The set of radiuses for which the tori survive forms a Cantor set with complement measure smaller than $O((e\mu^{-1/3})^{1/2})$.

Our second result concerns the study of the stable and unstable leaves of the Cantor set of surviving KAM tori. We do so by applying a modification of a Melnikov method to obtain the following result (for a detailed formulation see Theorem 46).

Theorem 2 *Assume that for the sequence of masses $\{\mu_k\}_{k=2}^\infty$ (the sequence is specified in Theorem 3) the twist condition holds and that a derivative of a Melnikov integral (93) at zero is nonzero, then for any given μ_k there exists a radius $R(\mu_k)$ such that for perturbations e with $e\mu_k^{-2/3} < \kappa$ and sufficiently small $e\mu_k^{-1/3}$ there exists a homoclinic and a heteroclinic tangle between the surviving tori or radii smaller than $R(\mu_k)$. The tangle implies existence of symbolic dynamics and diffusion in energy. The diffusion occurs between surviving tori on an interval of energies of order $(e\mu_k^{-1/3})^{1/2}$.*

To apply Theorem 2 we need to back the argument by numerical verification of its assumptions. We have verified the twist condition numerically for large masses μ_k and provided a rigorous argument that for sufficiently small masses the condition holds true. For the Melnikov integral (93) we can rigorously prove that it is convergent and that it is zero at zero. It needs to be stressed though that the assumption that the derivative of the Melnikov integral at zero is nonzero has only been verified numerically.

We believe that the above mentioned numerical computations can be performed using an rigorous-computer-assisted approach in the spirit of [28]. Such arguments require careful estimates, use of topological tools, and are the subject of ongoing work.

In our work we have been unable to obtain uniform bounds for the size of the radii $R(\mu_k)$ from Theorem 2 with respect to μ_k . From our proof it only follows that these need to decrease together with μ_k so that at least $\mu_k^{-1/3}R(\mu_k)$ is smaller than some constant. It is possible though that in a number of needed estimates $R(\mu_k)$ has to be chosen even smaller. We remark also that the symbolic dynamics proved in Theorem 2 will hold not only for the family of parameters $\{\mu_k\}$, but also for other masses μ for which $|\mu_k - \mu| < \epsilon_k$ with sufficiently small ϵ_k .

The paper is organized as follows. Section 2 contains preliminaries, where we recall the earlier results on the planar restricted three body problem of [20], and introduce basic facts about the Hill's problem and the PRE3BP. In Section 3 we present the Lyapounov–Moser theorem [23] and show how to apply it to obtain the twist property. In Section 4 we show that we have a twist on the family of Lyapounov orbits around L_2^μ . In Section 5 we apply the normally hyperbolic invariant manifold theorem together with the KAM Theorem to show that most of the Lyapounov orbits around L_2^μ persist under perturbation from the circular problem to the elliptic problem and prove the first of our two main theorems. In Section 6 we use a Melnikov type argument to detect the transversal intersections between the stable and unstable manifolds of the perturbed Lyapounov orbits. In Section 7 we compute the Melnikov integral. In Section 8 we gather together our results and prove Theorem 2.

2. Preliminaries

2.1. The Planar Restricted Circular Three Body Problem

In the planar restricted circular three body problem (PRC3BP) we consider the motion of a small massless particle (a comet or a spacecraft), under the gravitational pull of two larger bodies of mass μ and $1 - \mu$ (called the planet and the Sun, respectively) which move around the origin on circular orbits of period 2π on the same plane as the massless body. The Hamiltonian of the problem is given by [1]

$$H(\mu, q, p, t) = \frac{p_1^2 + p_2^2}{2} - \frac{1 - \mu}{r_1(t)} - \frac{\mu}{r_2(t)}, \quad (1)$$

where $(p, q) = (q_1, q_2, p_1, p_2)$ are the coordinates of the massless particle and $r_1(t)$ and $r_2(t)$ are the distances from the masses $1 - \mu$ and μ respectively. After introducing a new coordinates system (x, y, p_x, p_y)

$$\begin{aligned} x &= q_1 \cos t + q_2 \sin t, & p_x &= p_1 \cos t + p_2 \sin t, \\ y &= -q_1 \sin t + q_2 \cos t, & p_y &= -p_1 \sin t + p_2 \cos t, \end{aligned} \quad (2)$$

which rotates together with the two larger masses, the larger masses become motionless and one obtains [1] an autonomous Hamiltonian

$$H(\mu, x, y, p_x, p_y) = \frac{(p_x + y)^2 + (p_y - x)^2}{2} - \Omega(x, y), \quad (3)$$

where

$$\begin{aligned} \Omega(x, y) &= \frac{x^2 + y^2}{2} + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}, \\ r_1 &= \sqrt{(x - \mu)^2 + y^2}, \quad r_2 = \sqrt{(x + 1 - \mu)^2 + y^2}. \end{aligned} \quad (4)$$

The motion of the particle is given by the equation

$$\dot{\mathbf{x}} = J \nabla H(\mu, \mathbf{x}), \quad (5)$$

where $\mathbf{x} = (x, y, p_x, p_y) \in \mathbb{R}^4$, $J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$ and Id is a two dimensional identity matrix.

The movement of the flow (5) is restricted to the hypersurfaces determined by the energy level C ,

$$M(\mu, C) = \{(x, y, p_x, p_y) \in \mathbb{R}^4 | H(\mu, x, y, p_x, p_y) = C\}. \quad (6)$$

In the x, y coordinates this means that the movement is restricted to the so called Hill's region defined by

$$R(\mu, C) = \{(x, y) \in \mathbb{R}^2 | \Omega(x, y) \geq -C\}.$$

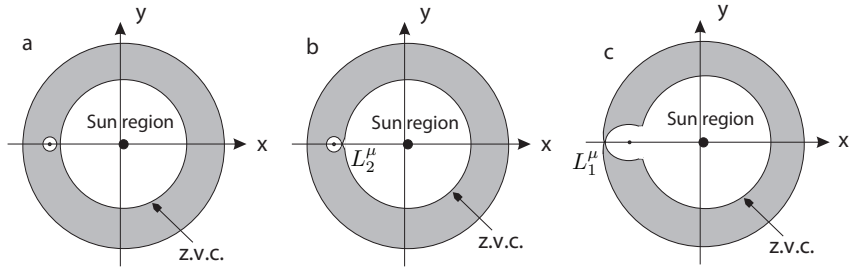


Figure 1. The Hill's region for various energy levels: when the energy C is smaller than C_2^μ (a), when $C = C_2^\mu$ (b) and when $C = C_1^\mu > C_2^\mu$ (c).

The shape of the Hill's region $R(\mu, C)$ will differ with C (see Figure 1). The focus of our attention in this paper will be on the case when the energy C is equal to or slightly larger than C_2^μ . For the energy C equal to C_2^μ we have the libration point L_2^μ which is of the form $(-k, 0, 0, -k)$ with $k > 0$. We shall investigate the dynamics inside of the inner part of the Hill's region to the right of the point L_2^μ . We shall refer to it as the "Sun region" (see Figure 1). The boundary of this region (see Figures 1 and 2) is a zero velocity curve (z.v.c.). The linearized vector field at the point L_2^μ has two real and two purely imaginary eigenvalues, thus it follows [20] from the Lyapounov theorem that for energies C larger and sufficiently close to C_2^μ there exists a family of periodic orbits $l_\mu(C)$ emanating from the equilibrium point L_2^μ .

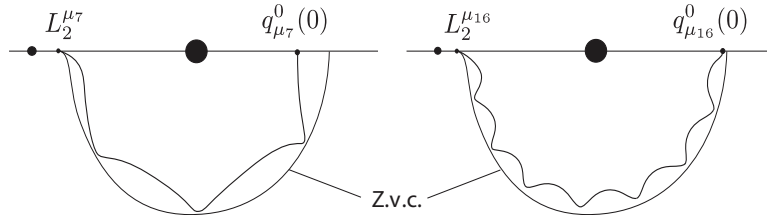


Figure 2. The unstable manifold of L_2^μ in the x, y coordinates.

The PRC3BP admits the following reversing symmetry

$$S(x, y, p_x, p_y) = (x, -y, -p_x, p_y). \quad (7)$$

We will say that an orbit $q(t)$ is S -symmetric when

$$S(q(t)) = q(-t). \quad (8)$$

In PRC3BP the Lyapounov orbits are S -symmetric (we have to choose the initial time so that orbits start from the section $\{y = 0\}$ at time $t = 0$).

We have the following results about the stable and unstable manifolds of L_2^μ and $l_\mu(C)$.

Theorem 3 ([20, Theorem A]) For μ sufficiently small the branch of $W_{L_2^\mu}^u$ contained in the Sun region (see Figures 1 and 2) has a projection on the bounded component of $R(\mu, C)$ given by

$$d(t) = \mu^{1/3} \left(\frac{2}{3}N(\infty) - 3^{1/6} + M(\infty) \cos t + o(1) \right), \quad (9)$$

$$\alpha(t) = -\pi + \mu^{1/3} (N(\infty)t + 2M(\infty) \sin t + o(1)), \quad (10)$$

where d is the distance to the z.v.c., α the angular coordinate, $N(\infty)$ and $M(\infty)$ are constants and the expressions remain true out of a given neighborhood of L_2^μ . The parameter t means the physical time from a suitable origin. The terms $o(1)$ tend to zero when μ does and they are uniform in t for $t = O(\mu^{-1/3})$.

In particular the first intersection with the x axis is orthogonal to that axis, giving a S -symmetric homoclinic orbit for a sequence of values μ which has the following asymptotic expression:

$$\mu_k = \frac{1}{N(\infty)^3 k^3} (1 + o(1)). \quad (11)$$

Let us now introduce a notation for the S -symmetric homoclinic orbit to $L_2^{\mu_k}$ obtained in Theorem 3 for the parameters μ_k given in (11). We will denote such an orbit by $q_{\mu_k}^0(t)$ (see Figure 2). We assume that such an orbit starts at a section $\{y = 0\}$ at time $t = 0$.

Theorem 4 ([20, Theorem B]) For μ and $\Delta C = C - C_2^\mu$ sufficiently small, the branch of $W^u(l_\mu(C))$ contained in the Sun region intersects the plane $y = 0$ for $x > 0$ in a curve diffeomorphic to a circle (see Figure 3) given by

$$\begin{aligned} x &= x_w - \sqrt{\Delta C} (N + 2M \cos \tau)^{-1} (2M + N \cos M_f) (K_1 \cos \tau \cos \sigma - K_2 \sin \tau \sin \sigma) \\ &\quad + \mu^{1/3} M (1 - \cos M_f) \\ &\quad + \mu^{2/3} \left\{ -\frac{2MN}{3} (1 - \cos M_f) + M^2 \sin^2 M_f - \frac{2}{9} N \alpha - \frac{M}{3} \alpha \cos M_f \right\} + O(\mu), \\ \dot{x} &= \sin M_f [\sqrt{\Delta C} N (N + 2M \cos \tau)^{-1} (K_1 \cos \tau \cos \sigma - K_2 \sin \tau \sin \sigma) \\ &\quad + \mu^{1/3} M + \mu^{2/3} \left\{ \frac{MN}{3} + 2M^2 \cos M_f + \frac{M}{3} \alpha \right\}] + O(\mu) \end{aligned}$$

where x_w , M , N , τ , K_1 , K_2 are suitable constants, α measures the distance from μ to some μ_k given by Theorem 3, M_f is obtained implicitly from

$$\begin{aligned} &\left(\frac{1}{3} N \alpha + \mu^{-2/3} \sqrt{\Delta C} 3M (N + 2M \cos \tau)^{-1} (K_1 \cos \tau \cos \sigma - K_2 \sin \tau \sin \sigma) \right) \frac{\pi}{N} \\ &\quad + N M_f + 2M \sin M_f = o(1), \end{aligned} \quad (12)$$

and σ , ranging from 0 to 2π , is the parameter of the curve.

Moreover for points in the (μ, C) plane such that there exists a μ_k of Theorem 3 for which

$$\Delta C > L \mu_k^{4/3} (\mu - \mu_k)^2 \quad (13)$$

holds (where L is a constant), there exist S -symmetric transversal homoclinic orbits. In particular, for $\mu = \mu_k$ there exist symmetrical transversal homoclinic orbits $q_{\mu_k C}^0$ for the periodic orbit $l_{\mu_k}(C)$ for $C > C_2^{\mu_k}$ arbitrarily close to $C_2^{\mu_k}$.

Remark 5 In the original version of Theorems 3 and 4 the C was taken as the Jacobi constant $C = F$ where

$$F(x, y, p_x, p_y) = -2H(\mu, x, y, p_x, p_y) = 2\Omega(x, y) - (\dot{x}^2 + \dot{y}^2). \quad (14)$$

In this paper we have rewritten the Theorems with C as the Hamiltonian of the PRC3BP, which means that we have a change of sign in C compared with the original version.

Remark 6 Using a standard dynamical system theory argument, from the Birkhoff-Smale homoclinic theorem, the transversal homoclinic connections to Lyapounov orbits imply chaotic symbolic dynamics of the system. This is a content of Theorem C in [20]. Since the system is autonomous this dynamics is restricted to the constant energy manifold.

Remark 7 For sufficiently small $\mu = \mu_k$, the curves (obtained in Theorem 4) on $\{y = 0\}$ associated with the stable and unstable manifolds of $l_{\mu_k}(C)$ intersect transversally at an angle $O(\mu_k^{1/3})$. This can be derived based on the parameterization of the curves from Theorem 4 combined with the symmetry property (8) of the PRC3BP. This is done in the Appendix. For more details on the interpretation of the curves from the theorem see also [20].

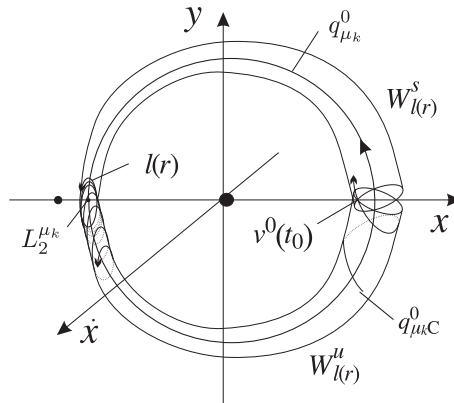


Figure 3. The intersections of the stable and unstable manifolds of $l_{\mu}(C)$ in the PRC3BP.

2.2. The Hill's Problem

Let us consider a change of coordinates which shifts the origin to the smaller body of the mass μ and rescales the coordinates by the factor $\mu^{-1/3}$. We will refer to the following as the Hill's coordinates.

$$\bar{\mathbf{x}} = \mu^{-1/3} (\mathbf{x} - (\mu - 1, 0, 0, \mu - 1)). \quad (15)$$

We will rewrite the Hamiltonian (3) and derive the formula of the Hill's problem and list a number of facts which will be relevant for us in the future.

Let us start with a simple lemma.

Lemma 8 *Consider a Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ and a transformation $\bar{p} = \beta p$, $\bar{q} = \beta q$. Then $(p(t), q(t))$ is a solution of the Hamiltonian system for $H(p, q)$ if and only if $(\bar{p}(t), \bar{q}(t))$ is a solution of the Hamiltonian system for $\bar{H}(\bar{p}, \bar{q}) = \beta^2 H\left(\frac{\bar{p}}{\beta}, \frac{\bar{q}}{\beta}\right)$.*

The shift of the origin present in the transformation (15) is clearly canonical hence we can use Lemma 8 to obtain the Hamiltonian in new variables with $\beta = \mu^{-1/3}$

$$\bar{H}(\mu, \bar{\mathbf{x}}) = \mu^{-2/3} H\left(\mu, \mu^{1/3} \bar{\mathbf{x}} + (\mu - 1, 0, 0, \mu - 1)\right). \quad (16)$$

By expanding Ω in the new coordinates around zero we can rewrite our family of Hamiltonians as

$$\begin{aligned} \bar{H}(\mu, \bar{\mathbf{x}}) &= \bar{H}(\mu, \bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y) = \frac{(\bar{p}_x + \bar{y})^2 + (\bar{p}_y - \bar{x})^2}{2} + \\ &\quad - \frac{1}{\bar{r}} - \frac{3}{2} \bar{x}^2 + O(\mu^{1/3}) + C(\mu), \quad \text{for } \bar{r} < \alpha \mu^{-1/3} \end{aligned} \quad (17)$$

where $C(\mu) = \mu^{-2/3} ((1 - \mu) + (1 - \mu)^2 / 2)$ and $\bar{r} = \sqrt{\bar{x}^2 + \bar{y}^2}$. The term $O(\mu^{1/3})$ depends on (\bar{x}, \bar{y}) and can be written as a function $a(\mu, \bar{x}, \bar{y})$, which for any sufficiently small $\alpha < 1$ and $\mu \in [0, 1]$, $\bar{r} < \alpha \mu^{-1/3}$ satisfies

$$\frac{|a(\mu, \bar{x}, \bar{y})|}{\bar{r}^3} \leq M(\alpha).$$

The reason for introducing α is, that we have to be away from the Sun in order for Taylor series of $\frac{1}{r_1}$ to be convergent. Observe that we can drop the term $C(\mu)$.

It should be stressed that \bar{H} depends analytically on $\bar{\mathbf{x}}$ and $\mu^{1/3}$, hence the derivatives of the $O(\mu^{1/3})$ with respect to $\bar{\mathbf{x}}$ are still $O(\mu^{1/3})$. Therefore for fixed μ

$$\bar{\mathbf{x}}' = J \nabla \bar{H}(\mu, \bar{\mathbf{x}}) = J \nabla \bar{H}(0, \bar{\mathbf{x}}) + O(\mu^{1/3}).$$

The term $O(\mu^{1/3})$ is uniform in $\bar{\mathbf{x}}$ for $|\bar{\mathbf{x}}| \leq \alpha \mu^{-1/3}$ for any fixed sufficiently small $\alpha < 1$. The Hamiltonian $\bar{H}(0, \bar{\mathbf{x}})$ is the Hamiltonian of the Hill's problem

$$H^{\text{Hill}}(\bar{\mathbf{x}}) = \bar{H}(0, \bar{\mathbf{x}}). \quad (18)$$

If we denote by $q^{\text{Hill}}(t)$ the solution of the Hill problem and by $q^\mu(t)$ the solution of the PCR3BP, both expressed in Hill's coordinates (15) and both starting from the same initial condition, then the following holds

$$|q^{\text{Hill}}(t) - q^\mu(t)| \leq e^{l|t|} O(\mu^{1/3}), \quad (19)$$

provided there exist $0 < \alpha < 1$ and a compact convex set $Z \subset B(0, \alpha \mu^{-1/3}) \times \mathbb{R}^2$, such that Z contains both $q^{\text{Hill}}([0, t])$ and $q^\mu([0, t])$ and $(0, 0) \notin \pi_{x,y}(Z)$. The constant l depends on Z .

Let us now list a few properties of the Hill's problem. The problem has two equilibrium points, $L_1^{\text{Hill}} = (-3^{-1/3}, 0, 0, -3^{-1/3})$ and $L_2^{\text{Hill}} = (3^{-1/3}, 0, 0, 3^{-1/3})$. The linearization of $x' = J\nabla H^{\text{Hill}}$ at L_2^{Hill} is given by $x' = Ax$, where

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 8 & 0 & 0 & 1 \\ 0 & -4 & -1 & 0 \end{pmatrix}. \quad (20)$$

The eigenvalues of A are: $\pm\alpha_1, \pm\alpha_2$ with $\alpha_1 = \sqrt{1 + 2\sqrt{7}}$ and $\alpha_2 = \sqrt{1 - 2\sqrt{7}}$. The Hill problem has a reversing symmetry S given by (7).

2.3. The Planar Restricted Elliptic Three Body Problem

The planar restricted elliptic three body problem (PRE3BP) differs from the PRC3BP by the fact that the two larger bodies move on elliptic orbits of eccentricities e instead of circular orbits. The period of these orbits is 2π and the Hamiltonian of the PRE3BP is analogous to (1), with the only difference that in $r_1(t)$ and $r_2(t)$ we take the distance from the elliptic instead of the circular orbits of the two larger masses. The trajectories of these orbits can be written as (see [30]) $((\mu - 1)x(t), (\mu - 1)y(t))$ for the body μ and $(\mu x(t), \mu y(t))$ for $(1 - \mu)$, where

$$\begin{aligned} x(t) &= (1 - e \cos \psi) \cos \psi + O(e^2), \\ y(t) &= (1 - e \cos \psi) \sin \psi + O(e^2), \\ \psi(t) &= t + 2e \sin t + O(e^2). \end{aligned} \quad (21)$$

If one changes into the rotating coordinates (2), which is a canonical transformation (see [21]), then the Hamiltonian (1) becomes (for a detailed derivation see the Appendix)

$$H^e(\mu, \mathbf{x}, t) = H(\mu, \mathbf{x}) + eG(\mu, \mathbf{x}, t) + O(e^2\mu^{-2/3}), \quad (22)$$

where H is the Hamiltonian of the PRC3BP (3), r_2 is given in (4), G is 2π periodic over t and is given by the formula

$$G = \frac{1 - \mu}{(r_1)^3} \bar{g}(\mu, x, y, t) + \frac{\mu}{(r_2)^3} \bar{g}(\mu - 1, x, y, t), \quad (23)$$

$$\bar{g}(\alpha, x, y, t) = \alpha(-2y \sin t + x \cos t) - \alpha^2 \cos t. \quad (24)$$

The term $O(e^2\mu^{-2/3}) = a(x, y, e, \mu, t)$ satisfies $\frac{|a(x, y, e, \mu, t)|}{e^2\mu^{-2/3}} < M(\delta, \kappa, R)$, on the set defined by the following conditions

$$\begin{aligned} \mu &\in [0, 1], \quad t \in \mathbb{R}, \quad e\mu^{-2/3} < \kappa \\ r_1 &> \delta, \quad r_2 \geq \delta\mu^{1/3}, \quad \sqrt{x^2 + y^2} \leq R, \end{aligned} \quad (25)$$

were $\delta > 0$ measures the closest approach to the Sun and to the planet (multiplied by $\mu^{-1/3}$) is a number around $1/2$, R is the radius of the ball containing all orbits of interest in our problem, hence we can take $R = 2$ and κ is sufficiently small number (for more details on κ see the derivation in the Appendix).

We consider points (x, y) inside of the "Sun region" (see Figure 1) which means that $r_2 > \frac{1}{2} \|L_2^\mu - (\mu - 1, 0, 0, \mu - 1)\| > \delta\mu^{1/3}$ for some $\delta > 0$. The Hamiltonian (22) can be rewritten in Hill's coordinates (15) as

$$\bar{H}^e(\mu, \bar{\mathbf{x}}, t) = \bar{H}(\mu, \bar{\mathbf{x}}) + e\bar{G}(\mu, \bar{\mathbf{x}}, t) + O(e^2\bar{r}^2), \quad (26)$$

$$\bar{G}(\mu, \bar{\mathbf{x}}, t) = \mu^{-2/3}G(\mu, \mu^{1/3}\bar{\mathbf{x}} + (\mu - 1, 0, 0, \mu - 1), t), \quad (27)$$

Further, in order to understand the mutual relation between e and μ for which we have interesting dynamical phenomena, it will be important to observe that after rearranging and neglecting terms independent of \bar{x} , (26) can be written as

$$\bar{H}^e(\mu, \bar{\mathbf{x}}, t) = \bar{H}(\mu, \bar{\mathbf{x}}) + e\mu^{-1/3} \frac{2\bar{y} \sin t - \bar{x} \cos t}{\bar{r}^3} + O(e\mu^{2/3}) + O(e^2\mu^{-4/3}), \quad (28)$$

on the following set

$$\begin{aligned} \mu &\in [0, 1], \quad t \in \mathbb{R}, \quad e\mu^{-2/3} < \kappa \\ \bar{r} &> \delta, \quad \bar{r} \leq M_1, \quad M_1\mu^{1/3} < 1. \end{aligned}$$

The Hamiltonian \bar{H}^e generates a differential equation

$$\bar{\mathbf{x}}' = f(\mu, \bar{\mathbf{x}}) + eg(\mu, \bar{\mathbf{x}}, t) + O(e\mu^{1/3}) + O(e^2\mu^{-4/3}) \quad (29)$$

where

$$f(\mu, \mathbf{x}) = J\nabla\bar{H}(\mu, \mathbf{x}) \quad (30)$$

$$g(\mu, \mathbf{x}, t) = J\nabla\bar{G}(\mu, \mathbf{x}). \quad (31)$$

Remark 9 *In our future consideration we shall use equation (28) in a neighborhood of $\bar{L}_2^\mu = \mu^{-1/3}(L_2^\mu - (\mu - 1, 0, 0, \mu - 1))$ of constant radius (later denoted as R_{Hill}), in which Lyapounov orbits reside. In such neighborhood the term $O(e\mu^{2/3})$ of (28) together with higher order terms are uniform. It needs to be emphasized though that in order for (28) to be valid for a given μ , we first need to choose e sufficiently small so that the estimate $e\mu^{-2/3} < \kappa$ in (25) holds true.*

3. From Lyapounov-Moser Theorem to Twist Property at Equilibrium Points

In this section we shall show how one can prove the twist property at an equilibrium point using the Lyapounov-Moser Theorem [23]. First the Theorem will be stated. Next a number of observations on the Theorem in the special case of one real and one pure imaginary eigenvalue with just two degrees of freedom will be made. This will be followed by a brief outline of the construction by which the Theorem was proved [23] from which the twist property will follow.

Theorem 10 (The Lyapounov-Moser Theorem [23]) *Let*

$$\begin{aligned} \dot{q}_\nu &= H_{p_\nu}(q, p) \\ \dot{p}_\nu &= -H_{q_\nu}(q, p) \end{aligned} \quad (32)$$

$\nu = 1, \dots, n$, be an analytic Hamiltonian system with n degrees of freedom and an equilibrium solution $q = p = 0$. Let $\alpha_1, \dots, \alpha_n, -\alpha_1, \dots, -\alpha_n$ be the eigenvalues of the linearization of (32) at the equilibrium point. Assume that the eigenvalues

$$\alpha_1, \dots, \alpha_n, -\alpha_1, \dots, -\alpha_n$$

are $2n$ different complex numbers and that α_1, α_2 are independent over the reals. Let us also assume that for any integer numbers n_1 and n_2

$$\alpha_\nu \neq n_1\alpha_1 + n_2\alpha_2 \quad \text{for } \nu \geq 3.$$

Then there exists a four parameter family of solutions of (32) of the form

$$\begin{aligned} q_\nu &= \phi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) \\ p_\nu &= \psi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) \end{aligned} \quad (33)$$

where

$$\xi_k(t) = \xi_k^0 e^{ta_k(\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0)}, \quad \eta_k(t) = \eta_k^0 e^{-ta_k(\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0)} \quad \text{for } k = 1, 2, \quad (34)$$

and

$$a_1(\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) = \alpha_1 + \dots, \quad a_2(\xi_1^0, \xi_2^0, \eta_1^0, \eta_2^0) = \alpha_2 + \dots \quad (35)$$

are convergent power series. The series ϕ_ν, ψ_ν converge in the neighborhood of the origin and the rank of the matrix

$$\begin{pmatrix} \phi_{\nu\xi_k} & \phi_{\nu\eta_k} \\ \psi_{\nu\xi_k} & \psi_{\nu\eta_k} \end{pmatrix}_{\substack{\nu=1,2,\dots,n \\ k=1,2}}$$

is four. The solutions (33) depend on four small enough complex parameters ξ_k^0, η_k^0 .

If in addition $\alpha_1, \alpha_2, -\alpha_1, -\alpha_2$ contain their complex conjugates, the solution can be chosen to be real, depending on 4 real parameters.

In the case of the PRC3BP, n is simply equal to two and the equations (33), (34) describe all the solutions near the neighborhood of the equilibrium point. We will be interested in the application of the Theorem to the libration point L_2^μ , where α_1 is real and α_2 is pure imaginary. From now on we will restrict our discussion to this particular case. The following remarks and lemmas adapt Theorem 10 to this setting.

Remark 11 *When the system (32) is generated by a real Hamiltonian and if α_1 is real and α_2 is pure imaginary then for the real solutions of (32) of the form (33) the functions $\xi_k(t)$ and $\eta_k(t)$ are invariant under the involution [19, page 102]*

$$J_w(\xi_1, \xi_2, \eta_1, \eta_2) = (\bar{\xi}_1, i\bar{\eta}_2, \bar{\eta}_1, i\bar{\xi}_2). \quad (36)$$

Let us also note that the original version of Theorem 10 in [23] contained an error. There an involution

$$J_w(\xi_1, \xi_2, \eta_1, \eta_2) = (\bar{\xi}_1, \bar{\eta}_2, \bar{\eta}_1, \bar{\xi}_2)$$

was proposed. This stands in conflict with a requirement that the transformation Φ in the proof of Theorem 10 [23] should be canonical (See also equations (51) and (52)).

The reality condition (the fact that a point $(\xi_1, \xi_2, \eta_1, \eta_2)$ given in new coordinates represents a point from \mathbb{R}^4 in original ones) is

$$J_w(\xi_1, \xi_2, \eta_1, \eta_2) = (\xi_1, \xi_2, \eta_1, \eta_2). \quad (37)$$

The subspace of \mathbb{C}^4 of fixed points of J_w , $\text{Fix}(J_w)$ is given by

$$\xi_1 \in \mathbb{R}, \eta_1 \in \mathbb{R}, \xi_2 = r e^{i\varphi}, \eta_2 = i r e^{-i\varphi},$$

where $r, \varphi \in \mathbb{R}$. On $\text{Fix}(J_w)$ we will use the coordinates $(\xi_1, \eta_1, r, \varphi)$.

Remark 12 From the proof of the convergence of the series (33), (35) during the proof of Theorem 10 in [23], it follows that if we consider a family of Hamiltonians

$$H_\lambda : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R},$$

which is an analytic function of all variables including the parameter and possesses for each $\lambda \in I$, where I is a closed interval, a (locally) unique fixed point p_λ of center-saddle type depending analytically on λ , then the radius of convergence of the series (33), (35) around p_λ can be chosen uniformly for close values of λ .

Lemma 13 If α_1 is real and α_2 is pure imaginary then for all real solutions of (32) the series a_1 from the Theorem 10 is real and the series a_2 is pure imaginary. Moreover if we choose a real periodic solution

$$\begin{aligned} q_\nu(t) &= \phi_\nu(0, \xi_2(t), 0, \eta_2(t)) \\ p_\nu(t) &= \psi_\nu(0, \xi_2(t), 0, \eta_2(t)) \end{aligned} \quad \nu = 1, 2 \quad (38)$$

where $\xi_2(t)$ and $\eta_2(t)$ are given by (34), then there exist two real numbers r and φ such that

$$\begin{aligned} \xi_2(t) &= r e^{ta_2(0, ir^2) + i\varphi} \\ \eta_2(t) &= i r e^{-ta_2(0, ir^2) - i\varphi}. \end{aligned}$$

Proof. From Remark 11 we know that the real solutions satisfy the reality condition (37). We therefore have

$$\xi_1^0 e^{ta_1(\xi_1^0 \eta_1^0, \xi_2^0 \eta_2^0)} = \overline{\xi_1^0 e^{ta_1(\xi_1^0 \eta_1^0, \xi_2^0 \eta_2^0)}} \quad (39)$$

$$\xi_2^0 e^{ta_2(\xi_1^0 \eta_1^0, \xi_2^0 \eta_2^0)} = i \left(\overline{\eta_2^0 e^{-ta_2(\xi_1^0 \eta_1^0, \xi_2^0 \eta_2^0)}} \right) \quad (40)$$

$$\eta_1^0 e^{-ta_1(\xi_1^0 \eta_1^0, \xi_2^0 \eta_2^0)} = \overline{\eta_1^0 e^{-ta_1(\xi_1^0 \eta_1^0, \xi_2^0 \eta_2^0)}} \quad (41)$$

$$\eta_2^0 e^{-ta_2(\xi_1^0 \eta_1^0, \xi_2^0 \eta_2^0)} = i \left(\overline{\xi_2^0 e^{ta_2(\xi_1^0 \eta_1^0, \xi_2^0 \eta_2^0)}} \right) \quad (42)$$

if we choose $t = 0$ then from the above we can see that ξ_1^0 and η_1^0 are real and that $\xi_2^0 = i\overline{\eta_2^0}$. Using the fact that $\xi_1^0, \eta_1^0 \in \mathbb{R}$ with (39) or (41) we can see that a_1 must be real. Using (40) or (42) and the fact that $\xi_2^0 = i\overline{\eta_2^0}$ we can see that a_2 is pure imaginary.

All periodic solutions have the initial conditions $\xi_1^0 = \eta_1^0 = 0$. If in addition we choose ξ_2^0 of the form $\xi_2^0 = re^{i\varphi}$ then for the solution to be real, from (40), we must have $\xi_2^0 = i\overline{\eta_2^0}$. In such case equation (34) gives us the periodic solutions as

$$\begin{aligned}\xi_2(t) &= \xi_2^0 e^{ta_2(0, \xi_2^0 \eta_2^0)} = re^{ta_2(0, ir^2) + i\varphi}, \\ \eta_2(t) &= \eta_2^0 e^{-ta_2(0, \xi_2^0 \eta_2^0)} = ire^{-ta_2(0, ir^2) - i\varphi}.\end{aligned}\tag{43}$$

■

Lemma 13 shows that all periodic solutions of (32) which are real and lie close to the equilibrium point, are given by the equation

$$l(r, t) = (0, re^{ta_2(0, ir^2) + i\varphi}, 0, ire^{-ta_2(0, ir^2) - i\varphi}),\tag{44}$$

when seen in the ξ_ν, η_ν coordinates. Let us denote the set which contains these orbits by

$$B_R = \{(0, re^{i\varphi}, 0, ire^{-i\varphi}) \mid \varphi \in [0, 2\pi), 0 \leq r \leq R\},\tag{45}$$

where R is sufficiently small for the series $a_2(0, ir^2)$ to be convergent for $r \leq R$.

Let $P : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be the time 2π shift along the trajectory of (32) i.e.

$$P(q(t)) = q(t + 2\pi),\tag{46}$$

where $q(t)$ is a solution of (32).

Lemma 14 *If in the series a_2 from Theorem 10 i.e.*

$$a_2(\xi_1\eta_1, \xi_2\eta_2) = \alpha_2 + a_{2,1}\xi_1\eta_1 + a_{2,2}\xi_2\eta_2 + \dots\tag{47}$$

for the coefficient $a_{2,2} \in \mathbb{R}$ we have $a_{2,2} \neq 0$, then for a sufficiently small R , the time 2π shift along the trajectory P restricted to the set B_R is an analytic twist map i.e.

$$\begin{aligned}P(r, \varphi) &= (r, \varphi + f(r)) \\ \frac{df}{dr} &\neq 0.\end{aligned}\tag{48}$$

Proof. In the ξ, η coordinates on B_R from (44) we can see that the map P takes form

$$P(0, re^{i\varphi}, 0, ire^{-i\varphi}) = \left(0, re^{2\pi a_2(0, ir^2) + i\varphi}, 0, ire^{-2\pi a_2(0, ir^2) - i\varphi}\right).$$

Keeping in mind that a_2 is pure imaginary we can see that $P(r, \varphi) = (r, \varphi - i2\pi a_2(0, ir^2))$. Since $a_2(0, ir^2) = \alpha_2 + a_{2,2}ir^2 + O(r^4)$ it is evident that if $a_{2,2} \neq 0$, then for sufficiently small r

$$\frac{d}{dr}(a_2(0, ir^2)) \neq 0.\tag{49}$$

■

Remark 15 Note that from Lemma 14 follows the twist property in action-angle coordinates (I, φ) with $I = r^2/2$. This observation will play a role when applying the KAM Theorem 29. Observe also that in action-angle coordinates the map P has the following form $P(I, \varphi) = (I, \varphi + 2\pi \text{im}(\alpha_2) + a_{2,2}I + O(I^2))$, which means that and the twist is more uniform and does not converge to zero as $I \rightarrow 0$, because $\frac{\partial P_\varphi}{\partial I}(I, \varphi) \rightarrow 2\pi a_{2,2}$ for $I \rightarrow 0$.

We will now show how to determine whether for a given problem (32) we have $a_{2,2} \neq 0$. For this we will quickly outline the construction of Moser [23] in order to obtain a formula for $a_{2,2}$. The construction is performed in the following two steps

$$\begin{array}{ccccc} \mathbb{C}^4 & \xrightarrow{\Psi} & \mathbb{C}^4 & \xrightarrow{\Phi} & \mathbb{C}^4, \\ (\xi_1, \xi_2, \eta_1, \eta_2) & \xrightarrow{\Psi} & (x_1, x_2, y_1, y_2) & \xrightarrow{\Phi} & (q_1, q_2, p_1, p_2), \end{array}$$

where the transformation Φ changes the system (32) in the q_1, q_2, p_1, p_2 coordinates into a system with a simplified form

$$\begin{aligned} \dot{x}_\nu &= \alpha_\nu x_\nu + f_\nu(x, y) \\ \dot{y}_\nu &= -\alpha_\nu y_\nu + g_\nu(x, y) \end{aligned} \quad \nu = 1, 2, \quad (50)$$

where α_1 and α_2 are the eigenvalues of the equilibrium point and f and g are power series starting from quadratic terms. From the simplified form (50) the transformation Ψ determines the series from Theorem 10.

The transformation Φ is a linear function which changes the coordinates so that the linear part of the equations (32) in the new coordinates becomes generated by a diagonal matrix. Moreover, the transformation Φ should be canonical i.e.

$$\Phi^T J \Phi = J \quad (51)$$

where

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \quad \text{and} \quad Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Moreover, Φ should satisfy the following reality condition [19, 23], which expresses the fact that J_w is simply the map describing how the complex conjugation works in new coordinates

$$J_z \Phi = \Phi J_w, \quad (52)$$

where

$$\begin{aligned} J_z(q_1, q_2, p_1, p_2) &= (\bar{q}_1, \bar{q}_2, \bar{p}_1, \bar{p}_2) \\ J_w(x_1, x_2, y_1, y_2) &= (\bar{x}_1, i\bar{y}_2, \bar{y}_1, i\bar{x}_2). \end{aligned} \quad (53)$$

The construction of the transformation Ψ is done by comparison of coefficients. We look for

$$\Psi = (\phi_1(\xi, \eta), \phi_2(\xi, \eta), \psi_1(\xi, \eta), \psi_2(\xi, \eta)),$$

with power series $\phi_\nu, \psi_\nu, a_\nu$, $\nu = 1, 2$ of the form

$$\begin{aligned}\phi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) &= \sum_{k=1}^2 \delta_{\nu k} \xi_k + h.o.t. \\ \psi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) &= \sum_{k=1}^2 \delta_{\nu k} \eta_k + h.o.t.\end{aligned}\quad (54)$$

such that

$$\begin{aligned}x_\nu &= \phi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) \\ y_\nu &= \psi_\nu(\xi_1, \xi_2, \eta_1, \eta_2),\end{aligned}\quad (55)$$

satisfy (50) if

$$\begin{aligned}\dot{\xi}_k &= a_k(\xi_1 \eta_1, \xi_2 \eta_2) \xi_k \\ \dot{\eta}_k &= -a_k(\xi_1 \eta_1, \xi_2 \eta_2) \eta_k.\end{aligned}\quad (56)$$

To construct $\phi_\nu, \psi_\nu, a_\nu$ one can rewrite using (55) and (56) the equation (50) as

$$\begin{aligned}\dot{x}_\nu &= \sum_{k=1}^2 \left(\frac{\partial \phi_\nu}{\partial \xi_k} a_k \xi_k - \frac{\partial \phi_\nu}{\partial \eta_k} a_k \eta_k \right) = \alpha_\nu \phi_\nu + f_\nu(\phi, \psi) \\ \dot{y}_\nu &= \sum_{k=1}^2 \left(\frac{\partial \psi_\nu}{\partial \xi_k} a_k \xi_k - \frac{\partial \psi_\nu}{\partial \eta_k} a_k \eta_k \right) = -\alpha_\nu \psi_\nu + g_\nu(\phi, \psi),\end{aligned}\quad \nu = 1, 2. \quad (57)$$

and compare the coefficients in (57). Let us denote by $\phi_{\nu,N}, \psi_{\nu,N}, a_{\nu,N}$ the coefficients in the series $\phi_\nu, \psi_\nu, a_\nu$ which come from the homogenous polynomials of order N . We can rewrite the part of (57) which contains all the terms of order N as

$$\begin{aligned}\sum_{k=1}^2 \alpha_k \left(\xi_k \frac{\partial}{\partial \xi_k} - \eta_k \frac{\partial}{\partial \eta_k} \right) \phi_{\nu,N} + \dots + \delta_{\nu k} \xi_k a_{k,N-1} &= \alpha_\nu \phi_{\nu,N} + \dots \\ \sum_{k=1}^2 \alpha_k \left(\xi_k \frac{\partial}{\partial \xi_k} - \eta_k \frac{\partial}{\partial \eta_k} \right) \psi_{\nu,N} + \dots - \delta_{\nu k} \eta_k a_{k,N-1} &= -\alpha_\nu \psi_{\nu,N} + \dots\end{aligned}\quad (58)$$

where the dots indicate all the terms which can be computed from $\phi_{\nu,l}, \psi_{\nu,l}, a_{\nu,l-1}$ with $l = 1, \dots, N-1$.

The nature of equations (58) suggests that the series can be constructed by induction starting with the lowest terms. It turns out though that not all of the coefficients can be computed from (58). This is because some of the terms in (58) cancel each other out. If we consider a homogenous polynomial $c \xi_1^{n_1} \eta_1^{m_1} \xi_2^{n_2} \eta_2^{m_2}$ of order N from $\phi_{\nu,N}$, such term will cancel out in (58) if

$$\sum_{k=1}^2 \alpha_k \left(\xi_k \frac{\partial}{\partial \xi_k} - \eta_k \frac{\partial}{\partial \eta_k} \right) c \xi_1^{n_1} \eta_1^{m_1} \xi_2^{n_2} \eta_2^{m_2} - \alpha_\nu c \xi_1^{n_1} \eta_1^{m_1} \xi_2^{n_2} \eta_2^{m_2} = 0.$$

This can happen only if we have

$$\sum_{k=1}^2 \alpha_k (n_k - m_k) - \alpha_\nu = 0. \quad (59)$$

By the assumption of the Theorem 10 that for any $t \in \mathbb{R}$ we have $t\alpha_1 + \alpha_2 \neq 0$, we can see that (59) is true only for the terms of the form $c \xi_\nu (\xi_1 \eta_1)^{n_1} (\xi_2 \eta_2)^{n_2}$. The value of the

coefficient c corresponding to such terms is chosen from an appropriate normalization condition [23]. In our case though the choice of the normalization does not play an important role. We are interested in computation of the term $a_{2,2}$ and this can be done by induction starting with $N = 1$ and stopping at $N = 3$. For $N = 1$ all the terms are uniquely determined. For $N = 2$ there are no terms which would cancel out in (58). For $N = 3$ we can use the first equation from (58) to find $a_{2,2}$. There the coefficient $a_{2,2}$ stands together with ξ_2 ($\xi_2\eta_2$) and the term for ξ_2 ($\xi_2\eta_2$) in $\phi_{\nu,3}$ will cancel out from the equation. Therefore $a_{2,2}$ is uniquely determined since it depends only on $\phi_{\nu,1}$, $\psi_{\nu,1}$, $\phi_{\nu,2}$ and $\psi_{\nu,2}$. Using this procedure we can obtain a direct formula for the term $a_{2,2}$.

Lemma 16 *If*

$$\begin{aligned} f_\nu(x_1, x_2, y_1, y_2) &= \sum_{i,j,k,l \geq 1} f_{ijkl}^\nu x_1^i x_2^j y_1^k y_2^l \\ g_\nu(x_1, x_2, y_1, y_2) &= \sum_{i,j,k,l \geq 1} g_{ijkl}^\nu x_1^i x_2^j y_1^k y_2^l \end{aligned} \quad \nu = 1, 2.$$

then

$$\begin{aligned} a_{2,2} &= \frac{1}{\alpha_2} (-f_{1,1,0,0}^2 f_{0,1,0,1}^1 - f_{0,1,1,0}^2 g_{0,1,0,1}^1 + f_{0,0,1,1}^2 g_{0,2,0,0}^1 - f_{0,2,0,0}^2 f_{0,1,0,1}^2 \\ &\quad + 2g_{0,2,0,0}^2 f_{0,0,0,2}^2 + f_{1,0,0,1}^2 f_{0,2,0,0}^1 - g_{0,1,0,1}^2 f_{0,1,0,1}^2) + f_{0,2,0,1}^2 \end{aligned} \quad (60)$$

Proof. The above can be checked from the formula (58) by direct computation. ■

This ends our construction of the coefficient $a_{2,2}$.

Let us now briefly turn to the relation between the energy and the radius r of the periodic orbits (44).

Lemma 17 *For sufficiently small r the energy of the orbit l_r (44) i.e.*

$$h(r) = H(\Phi(\Psi(l_r))), \quad (61)$$

is equal to

$$h(r) = H(0) + \frac{1}{2} D^2 H(0) (\Phi(0, 1, 0, i)) r^2 + o(r^2), \quad (62)$$

where $D^2 H(0) (\Phi(0, 1, 0, i))$ denote the value of the quadratic form $D^2 H(0)$ on the vector $\Phi(0, 1, 0, i)$.

Proof. Since the problem (32) is autonomous the energy is constant along the orbit l_r . Without any loss of generality we can therefore assume that φ in equation (44) for l_r is zero and compute

$$h(r) = H(\Phi(\Psi(l(r, 0)))).$$

Let us first note that the construction of $\Psi = (\phi_1, \phi_2, \psi_1, \psi_2)$ produced power series of the form (54), hence

$$\Psi(l(r, 0)) = (\phi_1, \phi_2, \psi_1, \psi_2)(l(r, 0)) = (0, r, 0, ir) + O(r^2).$$

The transformation Φ is linear and therefore

$$\Phi(\Psi(l(r, 0))) = r\Phi(0, 1, 0, i) + O(r^2). \quad (63)$$

We can compute $h(r)$ as

$$\begin{aligned} h(r) &= H(\Phi(\Psi(l(r, 0)))) \\ &= H(0) + DH(0)(\Phi(\Psi(l(r, 0)))) \\ &\quad + \frac{1}{2}D^2H(0)(\Phi(\Psi(l(r, 0))))^2 + o(|\Phi(\Psi(l(r, 0)))|^2). \end{aligned} \quad (64)$$

Since zero is an equilibrium point we know that $DH(0) = 0$, thus by substituting (63) into (64) we obtain our claim. ■

4. Twist in the PRC3BP at L_2^μ

As mentioned in Section 2, we have a family of periodic Lyapounov orbits $l_\mu(C)$ around L_2^μ for energies larger than and sufficiently close to the energy C_2^μ of the libration point L_2^μ . This family of orbits corresponds to the set B_R (see (45)) of orbits constructed in the previous section. In this section we will show that for sufficiently small μ we have a twist property on this family of periodic orbits. The main idea for the proof is to approximate the PRC3BP (with Hamiltonian (16)) expressed in Hill's coordinates (15) with the Hill's problem (18). First we shall consider the Hill's problem (18) where we have explicit formulas for the libration point L_2^{Hill} , the linearized vector field at L_2^{Hill} its eigenvalues etc., which will allow us to compute the twist coefficient $a_{2,2}^{\text{Hill}}$. We will then show that the coefficients $a_{2,2}^\mu$ computed for the PRC3BP in Hill's coordinates, converges to $a_{2,2}^{\text{Hill}}$ as μ tends to zero.

Lemma 18 *Let P^{Hill} be the time 2π shift along the trajectory Poincaré map of the Hill's problem (18). Then there exists a radius $R_{\text{Hill}} \in \mathbb{R}$, such that the map P^{Hill} expressed in in radius angle coordinates on the set of Lyapounov orbits around L_2^{Hill}*

$$P^{\text{Hill}} : [0, R_{\text{Hill}}] \times S^1 \rightarrow [0, R_{\text{Hill}}] \times S^1,$$

is a twist map.

Proof. We will apply the procedure from the previous section and compute $a_{2,2}^{\text{Hill}}$ for the equilibrium point $L_2^{\text{Hill}} = (3^{-1/3}, 0, 0, 3^{-1/3})$. The linear terms of (18) in L_2^{Hill} are given by (20) with eigenvalues $\pm\alpha_1, \pm\alpha_2$, $\alpha_1 = \sqrt{1 + 2\sqrt{7}}$ and $\alpha_2 = \sqrt{1 - 2\sqrt{7}}$. The first of the two is real and the second is pure imaginary. We will choose the function Φ^{Hill} composed of the eigenvectors of the eigenvalues $\pm\alpha_1$ and $\pm\alpha_2$

$$\Phi^{\text{Hill}} = \begin{pmatrix} \lambda_1 & \beta & \lambda_2 & -i\bar{\beta} \\ -9\lambda_1 \frac{1}{\alpha_1(\sqrt{7}+4)} & -\beta 9 \frac{1}{\alpha_2(\sqrt{7}-4)} & 9\lambda_2 \frac{1}{\alpha_1(\sqrt{7}+4)} & -i\bar{\beta} 9 \frac{1}{\alpha_2(\sqrt{7}-4)} \\ 9\lambda_1 \frac{\sqrt{7}+3}{\alpha_1(\sqrt{7}+4)} & -\beta 9 \frac{\sqrt{7}-3}{\alpha_2(\sqrt{7}-4)} & -9\lambda_2 \frac{\sqrt{7}+3}{\alpha_1(\sqrt{7}+4)} & -i\bar{\beta} 9 \frac{\sqrt{7}-3}{\alpha_2(\sqrt{7}-4)} \\ -\lambda_1 \frac{2}{3+\sqrt{7}} & \beta \frac{2}{\sqrt{7}-3} & -\lambda_2 \frac{2}{3+\sqrt{7}} & -i\bar{\beta} \frac{2}{\sqrt{7}-3} \end{pmatrix}, \quad (65)$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$, $\beta \in \mathbb{C}$. The above transformation Φ^{Hill} satisfies the reality condition (52), and if we choose the coefficients $\lambda_1, \lambda_2, \beta$ as

$$\begin{aligned}\lambda_1 = -\lambda_2 &= \frac{1}{6} \sqrt{\frac{\alpha_1 (\sqrt{7} + 4)}{\sqrt{7}}}, \\ \beta &= \frac{1}{6} \sqrt{\frac{i\alpha_2 (\sqrt{7} - 4)}{\sqrt{7}}},\end{aligned}\tag{66}$$

then Φ^{Hill} is also canonical. Computing the power series f_ν and g_ν from Lemma 16 at L_2^{Hill} and using (60) to compute the term $a_{2,2}^{\text{Hill}}$, by rather laborious computations (performed in Maple) one will obtain

$$a_{2,2}^{\text{Hill}} = \frac{\sqrt[3]{9}}{224} (102\sqrt{7} - 57) \approx 1.9767.$$

Since $a_{2,2}^{\text{Hill}} \neq 0$, by Lemma 14 we have the twist property in the radius angle coordinates for all r such that $0 < r < R_{\text{Hill}}$, where R_{Hill} is sufficiently small. ■

Remark 19 *Let us stress that R_{Hill} is independent of μ . From our attempts of rigorous estimation of R_{Hill} , following the estimates conducted during the proof of Theorem 10 in [23], we obtain that that $R_{\text{Hill}} \approx 10^{-4}$.*

One can also apply the procedure outlined in Section 3 to compute the coefficient $a_{2,2}^\mu$ for the PRC3BP with Hamiltonian (16) expressed in Hill' coordinates (15). To do so one needs to compute the libration point $\bar{L}_2^\mu = \mu^{-1/3} (L_2^\mu - (\mu - 1, 0, 0, \mu - 1))$, compute the expansion of the vector field at $\bar{L}_2^{\mu_k}$ up to the order three, compute the linear change of coordinates $\Phi = \Phi(\mu)$ and compute $a_{2,2}^\mu$ using Lemma 16. Numerical results of such computations for a selection of parameters from the family $\{\mu_k\}$ from Theorem 3 are given in the below table. The values μ_k chosen in the table are the numerical approximations of the series (11) from [20].

k	μ_k	$a_{2,2}^{\mu_k}$
2	0.4253863522E-2	1.967649155
3	0.6752539971E-3	1.968237635
4	0.2192936884E-3	1.970039000
10	0.92907436E-5	1.973971883
11	0.68212830E-5	1.974219993
12	0.51549632E-5	1.974426964
50	0.582146E-7	1.976164111
60	0.336890E-7	1.976252225
70	0.212152E-7	1.976315175
200	0.9096E-9	1.976563023

$a_{2,2}^{Hill} \approx 1.9767$

Table 1. The twist coefficient $a_{2,2}$ for various masses μ_k .

Theorem 20 *Let P^μ denote the the time 2π shift along the trajectory P^μ of the PRC3BP in Hill's coordinates (15). Then for sufficiently small μ the map P^μ expressed in in radius angle coordinates on the set of Lyapounov orbits around $\bar{L}_2^\mu = \mu^{-1/3} (L_2^\mu - (\mu - 1, 0, 0, \mu - 1))$ is an analytic twist map i.e. for $r < R_{Hill}$*

$$P^\mu(r, \varphi) = (r, \varphi + f(r))$$

and

$$\frac{df}{dr} \neq 0 \quad \text{for all } r \in [0, R_{Hill}].$$

Proof. We observe that \bar{L}_2^μ depends analytically on $\mu^{1/3}$. First let us note that since the operator Φ from our construction brings the derivative of the vector field to the Jordan form, the operator $\Phi_{3\text{Body}}^\mu$ for the PRC3BP can be chosen close (depending analytically on $\mu^{1/3}$) to the operator Φ^{Hill} . The same can be said about the coefficients f_ν and g_ν from Lemma 16, since those come from the Taylor expansion up to the third order of the vector field at \bar{L}_2^μ . Moreover, from the proof of the Lyapounov Moser Theorem 10 in [23], we know that the radius of convergence of the series from the Theorem 10 can be chosen to be independent from μ , for μ sufficiently close to zero. This means that the coefficient $a_{2,2}^\mu$ constructed for he PRC3BP will tend to the coefficient $a_{2,2}^{Hill}$ of the Hill's problem

$$\lim_{\mu \rightarrow 0} a_{2,2}^\mu = a_{2,2}^{Hill} \approx 1.9767. \quad (67)$$

■

Remark 21 From Theorem 20 and the results from Table 1, it is reasonable to believe that we will have twist for all μ_k for $k \geq 2$. Let us point out that in the Hill's coordinates the radius of convergence can be chosen to be independent from μ . In the original coordinates of the PRC3BP though, since $r_{\text{Hill}} = \mu^{1/3}r$, this radius will depend and decrease with μ as $R^\mu = \mu^{1/3}R_{\text{Hill}}$. This means that in the original coordinates we will have the twist property only for orbits with a radius smaller than $\mu^{1/3}R_{\text{Hill}}$.

5. Normal hyperbolicity, KAM theorem and the persistence of Lyapounov orbits

In this Section we briefly recall some facts from the normal hyperbolicity theory and a version of the K.A.M. (Kolmogorov, Arnold, Moser) Theorem. We then apply the results to obtain the persistence result of a Cantor family of Lyapounov orbits around L_2^μ for the perturbation from PRC3BP to PRE3BP. Our approach closely follows the method of [8]. We will therefore rewrite the theorems used in [8] and verify that their assumptions are satisfied in our particular setting.

First let us recall the results concerning normal hyperbolicity.

Definition 22 ([8, A1]) Let M be a manifold in \mathbb{R}^n and Φ_t a C^r , $r \geq 1$ flow on it. We say that a (smooth) manifold $\Lambda \subset M$ – possibly with boundary – invariant under Φ_t is α - β normally hyperbolic when there is a bundle decomposition

$$TM = T\Lambda \oplus E^s \oplus E^u,$$

invariant under the flow, and numbers $C > 0$, $0 < \beta < \alpha$, such that for $x \in \Lambda$

$$v \in E_x^s \Leftrightarrow |D\Phi_t(x)v| \leq Ce^{-\alpha t}|v| \quad \forall t > 0, \quad (68)$$

$$v \in E_x^u \Leftrightarrow |D\Phi_t(x)v| \leq Ce^{\alpha t}|v| \quad \forall t < 0, \quad (69)$$

$$v \in T_x\Lambda \Leftrightarrow |D\Phi_t(x)v| \leq Ce^{\beta|t|}|v| \quad \forall t. \quad (70)$$

Theorem 23 ([8, A7]) Let Λ be a compact α - β normally hyperbolic manifold (possibly with a boundary) for the C^r flow Φ_t , satisfying the Definition 22. Then there exists a sufficiently small neighborhood U of Λ and a sufficiently small $\delta > 0$ such that

(i) The manifold Λ is $C^{\min(r, r_1 - \delta)}$, where $r_1 = \alpha/\beta$.

(ii) For any x in Λ , the set

$$\begin{aligned} W_x^s &= \{y \in U : \text{dist}(\Phi_t(y), \Phi_t(x)) \leq Ce^{(-\alpha+\delta)t} \text{ for } t > 0\} \\ &= \{y \in U : \text{dist}(\Phi_t(y), \Phi_t(x)) \leq Ce^{(-\beta-\delta)t} \text{ for } t > 0\} \end{aligned}$$

is a C^r manifold and $T_x W_x^s = E_x^s$.

(iii) The bundles E_x^s are $C^{\min(r, r_0 - \delta)}$ in x , where $r_0 = (\alpha - \beta)/\beta$, and

$$\begin{aligned} W_\Lambda^s &= \{y \in U : \text{dist}(\Phi_t(y), \Lambda) \leq Ce^{(-\alpha+\delta)t} \text{ for } t > 0\} \\ &= \{y \in U : \text{dist}(\Phi_t(y), \Lambda) \leq Ce^{(-\beta-\delta)t} \text{ for } t > 0\} \end{aligned}$$

is a $C^{\min(r, r_0 - \delta)}$ manifold. Moreover $T_x W_\Lambda^s = E_x^s$. Finally

$$W_\Lambda^s = \bigcup_{x \in \Lambda} W_x^s.$$

Moreover, we can find a $\rho > 0$ sufficiently small and a $C^{\min(r, r_0 - \delta)}$ diffeomorphism from the bundle of balls of radius ρ in E_Λ^s to $W_\Lambda^s \cap U$.

Remark 24 An analogous theorem can be stated for W_Λ^u by considering the flow Φ_{-t} .

The following Theorem and two Remarks concern the persistence of the normally hyperbolic manifold and its stable and unstable manifolds.

Theorem 25 ([8, A.14]) *Let $\Lambda \subset M$ (Λ not necessarily compact) be α - β normally hyperbolic for the flow Φ_t generated by the vector field X , which is uniformly C^r in a neighborhood U of Λ such that $\text{dist}(M \setminus U, \Lambda) > 0$. Let Ψ_t be the flow generated by another vector field Y which is C^r and sufficiently C^1 close to X . Then we can find a manifold Γ which is α' - β' normally hyperbolic for Y and $C^{\min(r, r_1 - \delta)}$ close to Λ , where $r_1 = \alpha/\beta$.*

The constants α' , β' are arbitrarily close to α , β if Y is sufficiently C^1 close to X .

The manifold Γ is the only $C^{\min(r, r_1 - \delta)}$ normally hyperbolic manifold C^0 close to Λ and locally invariant under the flow of Y .

The above Theorem is extended to give us a smooth dependence on the parameter by the following two remarks.

Remark 26 (see [8, observation 1. page 390]) *Assume that we have a family of flows $\Phi_{t,e}$, generated by vector fields X_e which are jointly C^r in all its variables (the base point x and the parameter e). Let Λ_e be the normally hyperbolic manifold Γ from Theorem 25 for the flow $\Phi_{t,e}$. Then there exists a $C^{\min(r, r_1 - \delta)}$ mapping $F : \Lambda \times I \rightarrow M$, where $r_1 = \alpha/\beta$ and $I \subset \mathbb{R}$ is an interval containing zero, such that $F(\Lambda, e) = \Lambda_e$ and $F(\cdot, 0)$ is the identity.*

Remark 27 (see [8, observation 2. page 390]) *For a family of flows $\Phi_{t,e}$ with the same assumptions as in Remark 26, there exists a $C^{\min(r, r_1 - \delta)}$ ($r_1 = \alpha/\beta$) mapping $R^s : W_\Lambda^s \times I \rightarrow M$ such that $R^s(W_\Lambda^s, e) = W_{\Lambda_e}^s$, $R^s(\cdot, e)|_\Lambda = F(\cdot, e)$, $R^s(W_x^s, e) = W_{F(x,e)}^s$.*

An analogous mapping R^u also exists for W_Λ^u .

Let us now turn to a quantitative version of the KAM Theorem used in [8]. Let us recall that a real number ω is called a Diophantine number of exponent θ if there exists a constant $C > 0$ such that

$$\left| \omega - \frac{p}{q} \right| \geq \frac{C}{q^{\theta+1}}$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

Definition 28 Let $(M, \omega_M), (N, \omega_N)$ be two symplectic manifolds of same dimension. If $(M, \omega_M), (N, \omega_N)$ are exact symplectic (i.e. there exist one-forms α_M, α_N such that $\omega_M = d\alpha_M, \omega_N = d\alpha_N$) then we say that a diffeomorphism

$$F : M \rightarrow N$$

is exact symplectic when there exists a real valued function G on M such that

$$F^*\alpha_N - \alpha_M = dG.$$

Theorem 29 (KAM Theorem [8, Theorem 4.8]) Let $f : [0, 1] \times \mathbb{T}^1 \rightarrow [0, 1] \times \mathbb{T}^1$ be an exact symplectic C^l map with $l \geq 6$.

Assume that $f = f_0 + ef_1$, where $e \in \mathbb{R}$,

$$f_0(I, \varphi) = (I, \varphi + A(I)), \quad (71)$$

A is C^l , $|\frac{dA}{dI}| \geq M$, and $\|f_1\|_{C^l} \leq 1$.

Then, if $e^{1/2}M^{-1} = \rho$ is sufficiently small, for a set of Diophantine numbers σ of exponent $\theta = 5/4$, we can find invariant tori which are the graph of C^{l-3} functions u_σ , the motion on them is C^{l-3} conjugate to the rotation by σ and the tori cover the whole annulus except a set of measure smaller than $O(M^{-1}e^{1/2})$.

Moreover, we can find expansions

$$u_\sigma = u_\sigma^0 + eu_\sigma^1 + r_\sigma, \quad (72)$$

with $u_\sigma^0 = A^{-1}(\sigma)$, $\|r_\sigma\|_{C^{l-4}} \leq O(e^2)$, and $\|u_\sigma^1\|_{C^{l-4}} \leq O(1)$.

All of the above results have been taken from [8]. Now we will apply them to the setting of the PRE3BP. We will first show that the set of the Lyapounov orbits of the PRC3BP is normally hyperbolic.

Let $\phi_{t,s}^e : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ be given by

$$\phi_{t,s}^e(x) = q(s+t),$$

where $q(\cdot)$ is the solution for the PRE3BP in Hill's coordinates (15) (generated by the Hamiltonian (16)), with an initial condition $q(s) = \bar{x}$. We will define the flow on the extended phase space $\Phi_t^e : \mathbb{R}^4 \times \mathbb{R} \rightarrow \mathbb{R}^4 \times \mathbb{R}$ as

$$\Phi_t^e(\bar{x}, s) = (\phi_{t,s}^e(\bar{x}), s+t). \quad (73)$$

Observe that the flow Φ_t^e is 2π periodic with respect to s variable, hence may be equivalently treated as a flow on $\mathbb{R}^4 \times S^1$. This will later give us uniform C^r estimates. We shall use a notation

$$l(r) = l(r, S^1) \times S^1$$

to denote a torus of all trajectories of Lyapounov orbits $l(r, t)$ (see (44)) of radius r in the extended phase space.

Lemma 30 For a sufficiently small mass μ the set

$$\Lambda = \{l(r) | r \in [0, R_{\text{Hill}}]\}$$

of Lyapounov orbits of the PRC3BP (in the extended phase space) is α - β normally hyperbolic, where $\alpha > 0$ is close to the real eigenvalue α_1 at the Libration point $\bar{L}_2^\mu = \mu^{-1/3} (L_2^\mu - (\mu - 1, 0, 0, \mu - 1))$ and $\beta > 0$ can be chosen arbitrarily close to zero.

Proof. Consider the $\xi_1, \xi_2, \eta_1, \eta_2$ coordinates from the previous section together with a time t coordinate. The ξ_1, η_1 are the coordinates of the hyperbolic expansion and ξ_1, η_1 are the coordinates of the twist rotation around the libration point \bar{L}_2^μ . We have

$$M = \{(\xi_1, \xi_2, \eta_1, \eta_2, t) | \xi_1, \eta_1 \in \mathbb{R}, \xi_2 = re^{i\varphi}, \eta_2 = ire^{-i\varphi}, \\ r \in [0, R_{\text{Hill}}], \varphi \in [0, 2\pi), t \in [0, 2\pi)\}.$$

We can define

$$E^u = \{(\xi_1, 0, 0, 0, 0) | \xi_1 \in \mathbb{R}\}, \\ E^s = \{(0, 0, \eta_1, 0, 0) | \eta_1 \in \mathbb{R}\}, \\ T\Lambda = \{(0, \xi_2, 0, \eta_2, t) | \xi_2 = re^{i\varphi}, \eta_2 = ire^{-i\varphi}, r \in \mathbb{R}, \varphi \in [0, 2\pi), t \in [0, 2\pi)\},$$

then we will have $TM = E^u \oplus E^s \oplus T\Lambda$. The conditions (68) and (69) are satisfied with a coefficient $\alpha > 0$ close to the eigenvalue α_1 at \bar{L}_2^μ because the coordinates ξ_1 and η_1 are the coordinates of hyperbolic expansion and contraction. For sufficiently small μ the eigenvalue α_1 is close to $\alpha_1^{\text{Hill}} = \sqrt{1 + 2\sqrt{7}} \approx 2.5083$ of the Hill's problem.

Let Φ_t be the flow in the extended phase space. From (44) for $x = (0, \xi_2, 0, \eta_2, t) \in \Lambda$ with $\xi_2 = re^{i\varphi}$, $\eta_2 = ire^{-i\varphi}$ we have

$$\|D\Phi_t(x)\| \leq 1 + r \left| \frac{d}{dr} a_2(0, ir^2) \right| t,$$

where a_2 is the function given by (35) in Theorem 10. The growth of derivative of $D\Phi_t(x)$ is at most linear in t . For any $\beta > 0$ there exists a constant $C > 0$, such that for all $v \in T_x\Lambda$ and all t

$$|D\Phi_t(x)v| \leq Ce^{\beta|t|}|v|. \quad (74)$$

■

We will now define the time 2π shift along a trajectory Poincaré map and later apply the KAM Theorem to it. By Lemma 30 for $e = 0$ we have an α - β normally hyperbolic invariant manifold for Φ_t^0 of the form $\Lambda = \{l(r) | r \in [0, R_{\text{Hill}}]\}$. Let U be an open neighborhood of Λ . We will define time 2π shift along a trajectory Poincaré map $P_{t_0}^e : U \cap \{t = t_0\} \rightarrow \mathbb{R}^4$ as

$$P_{t_0}^e(x) = \phi_{2\pi, t_0}^e(x).$$

We are now ready to apply Theorems 25 and 29 to obtain the following persistence result for the family of the Lyapounov orbits.

Theorem 31 *Let $R < R_{\text{Hill}}$ be a fixed number and κ be the parameter from (25). If we choose sufficiently small $\mu^* > 0$ then for all $\mu > 0$ and $e > 0$ for which $e\mu^{-2/3} < \kappa$ the normally hyperbolic manifold (with a boundary; considered in the extended phase space) $\Lambda = \{l(r) | r \in [0, R]\}$ of the PRC3BP persists under the perturbation to PRE3BP with the parameter e , to a normally hyperbolic manifold (with a boundary) Λ_e . Moreover, for any such e there exists a Cantor set $\mathcal{C} \subset [0, R]$ such that for any $r \in \mathcal{C}$ there exists an invariant (two dimensional) torus*

$$l^e(r) = \{l_{t_0}^e(r) | t_0 \in S^1\},$$

where $l_{t_0}^e(r)$ is an one-dimensional torus invariant under the map $P_{t_0}^e$. The family of tori $l^e(r)$ for $r \in \mathcal{C}$, covers Λ_e except a set of a measure smaller than $O((e\mu^{-1/3})^{1/2})$.

Proof. The PRE3BP in Hill's coordinates (15) is generated by the Hamiltonian \bar{H}^e from (28). In the neighborhood U of Λ the \bar{r} from (17) and (28) is bounded and separated from zero, which means that the PRE3BP is a uniform perturbation of the Hill's problem (18). By Lemma 30 we know that Λ is normally hyperbolic for the PRC3BP. Applying Theorem 25 and Remark 26 we obtain a family of normally hyperbolic manifolds Λ_e locally invariant under Φ_t^e , and a function $F : \Lambda \times [0, e_0(\mu)] \rightarrow \mathbb{R}^4 \times S^1$ such that

$$F(\Lambda, e) = \Lambda_e = \{(\Lambda_{t,e}, t) | \Lambda_{t,e} \subset \mathbb{R}^4, t \in S^1\}.$$

From the Implicit Function Theorem we know that the libration point \bar{L}_2^μ continues for small values of e to a 2π periodic orbit $\bar{L}_2^{\mu,e}(t)$. We can modify F so that $F(\bar{L}_2^\mu, t, e) = \bar{L}_2^{\mu,e}(t)$. By Remark 26 the function F is C^{r_1} , where $r_1 = \alpha/\beta$. From the proof of Lemma 30 we know that for sufficiently small μ we have $\alpha \approx \sqrt{1 + 2\sqrt{7}}$ and that $\beta > 0$ can be chosen to be arbitrarily close to zero. This means that for sufficiently small μ , the function F is C^k for any given $k > 0$. Since Λ_e is locally invariant under Φ_t^e and the flow is 2π periodic, for any $t_0 \in S^1$ the manifold $\Lambda_{t_0,e}$ is locally invariant under $P_{t_0}^e$.

Let us fix $t_0 = 0$, fix small $\mu > 0$ and fix a Poincaré map $P^e := P_{t_0=0}^e$ (here we could consider a map $P_{t_0}^e$ for any $t_0 \in S^1$, but we fix $t_0 = 0$ for simplicity) and consider e such that $e\mu^{-2/3} < \kappa$, which ensures that (28) is valid. We shall use a notation $B_R = \pi_{\{t=0\}}\{l(r) | r \leq R\}$ for the set of Lyapounov orbits with radius smaller or equal to R . We will now show that for sufficiently small $e\mu^{-1/3}$ the Poincaré map

$$P^e : F(B_R, 0, e) \rightarrow \Lambda_{0,e} \tag{75}$$

is properly defined and symplectic. By (28), for sufficiently small $e\mu^{-1/3}$ we can see that (75) is properly defined because $\Lambda_{0,e}$ is locally invariant. The map P^e is a restriction to $\Lambda_{0,e}$ of a time 2π shift along a trajectory of a Hamiltonian system. Such shift is symplectic for the standard form $\omega = d\bar{x} \wedge d\bar{p}_x + d\bar{y} \wedge d\bar{p}_y$ (here we use notation $\bar{\mathbf{x}} = (\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y)$ for coordinates since we are working in Hill's coordinates (15)). In order to show that P^e is symplectic it is therefore sufficient to show that ω is non

degenerate on $\Lambda_{0,e}$. For sufficiently small μ and e the manifold $\Lambda_{0,e}$ is arbitrarily close to the manifold of the Lyapounov orbits of the Hill's problem (18), which in turn, for r sufficiently close to zero, is arbitrarily close to the vector space V given by the eigenvectors of the pure complex eigenvalues $\pm\alpha_2^{\text{Hill}} = \pm\sqrt{1-2\sqrt{7}}$. To show that ω is not degenerate on $\Lambda_{0,e}$ it is therefore sufficient to show that ω is not degenerate on V . The eigenvectors corresponding to $\pm\alpha_2^{\text{Hill}}$ are v and $-i\bar{v}$, where v is the second column in Φ^{Hill} (see (65)), which was symplectic, therefore $\omega(v, -i\bar{v}) = 1$. The space V is spanned by x_1 and x_2 , where $v = x_1 + ix_2$. An easy computation shows that $1 = \omega(v, -i\bar{v}) = -2\omega(x_1, x_2)$, which means that ω is not degenerate on V .

Now we will use the KAM Theorem 29 to show that most of the Lyapounov orbits on $\Lambda_{0,e}$ survive under a sufficiently small perturbation. Let us first note that even though the Theorem is stated for a map f on $[0, 1] \times \mathbb{T}^1$, the KAM result is local by its nature and also holds for a map $f : [0, 1] \times \mathbb{T}^1 \rightarrow \mathbb{R} \times \mathbb{T}^1$, as will be the case in our setting. Let ω denote the standard symplectic form in \mathbb{R}^4 i.e. $\omega = d\bar{x} \wedge d\bar{p}_x + d\bar{y} \wedge d\bar{p}_y$. Let ω^e denote the induced form on $\Lambda_{0,e}$. There exist C^{r_1-2} (jointly with the parameter e) close to identity coordinate maps $c_e : \Lambda_{0,e} \rightarrow \Lambda_{0,0}$ which transport the symplectic forms ω^e into the standard one (see [8, page 367]). The map

$$\bar{P}^e := c_e \circ P^e \circ (c_e)^{-1} : B_R \rightarrow B_{R_{\text{Hill}}}$$

is properly defined for sufficiently small e . Clearly for $e = 0$ we have $\bar{P}^0 = P^0$. From the fact that P^e is symplectic and the fact that $P^0 = P_{t_0=0}^{e=0}$ is a twist map follow the same properties for our maps \bar{P}^e and \bar{P}^0 respectively. Now we pass to the action angle coordinates. From Lemma 14 we have that \bar{P}^0 has the form (71). Exact simplecticity of \bar{P}^e is a direct consequence of simplecticity combined with invariance of the origin. To apply the KAM Theorem 29 what is now left is to show that

$$\|\bar{P}^e - \bar{P}^0\|_{C^{r_1-2}} = O(e\mu^{-1/3}). \quad (76)$$

This comes from the fact that in the neighborhood of \bar{L}_2^μ the perturbing term in (28) is uniformly $O(e\mu^{-1/3})$ in the C^l norm. Observe that this estimate holds both in the original coordinates and in the action angle coordinates, because the origin is the fixed point for \bar{P}^e . Therefore the time 2π shift maps $P_{t_0}^e$ and $P_{t_0}^0$ are also $O(e\mu^{-1/3})$ close, from which (76) follows. This gives us a Cantor family of invariant tori $l_0^e(r)$ for $r \in \mathfrak{C}$. Now for $r \in \mathfrak{C}$ we can define

$$\begin{aligned} l^e(r) &:= \{\Phi_t^e(x, 0) \mid x \in l_0^e(r), t \in [0, 2\pi]\} \\ l_{t_0}^e(r) &:= \Phi_{t_0}^e(l_0^e(r), 0). \end{aligned}$$

The fact that the complement of the Cantor set \mathfrak{C} is $O((e\mu^{-1/3})^{1/2})$ follows from the KAM Theorem (see also [24] for more details).

In the above argument we require that $e\mu^{-1/3} < c$ with some sufficiently small c (which is independent of μ) so that both the normally hyperbolic theorem and KAM can be applied. Let finish by observing that for any given c , by choosing sufficiently

small $\mu^* > 0$ and requiring that $\mu < \mu^*$ and $e\mu^{-2/3} < \kappa$ the estimate $e\mu^{-1/3} < c$ follows.

■

Remark 32 *An identical argument to the above proof can be performed to obtain a mirror result to Theorem 31 for any fixed parameter μ_k . To do so though one would have to verify the twist condition. This has been verified numerically for a sequence of parameters in Table 1. It is visible that as the masses μ_k decrease the twist coefficient increases to infinity. It is therefore reasonable to believe that the twist condition holds for all parameters μ_k . Let us emphasize that for sufficiently large k we have rigorously verified the twist condition in Lemma 20, hence the claim of Theorem 31 holds for μ_k sufficiently large k .*

Remark 33 *We can set R from Theorem 31 to be the radius of one of the surviving tori. This will ensure that Λ_e is an invariant manifold with a boundary $l^e(R)$. We should also emphasize that the choice of R is independent of μ .*

If for two surviving tori $l^e(r_1)$ and $l^e(r_2)$ with $r_1 < r_2$ there does not exist an $r \in (r_1, r_2)$ for which the torus $l(r)$ is perturbed to an invariant torus of the PRE3BP, then we say that there exists a gap between the tori $l^e(r_1)$ and $l^e(r_2)$.

Proposition 34 *Let $e\mu^{-1/3}$ be sufficiently small so that the claim of Theorem 31 holds. Then there exists an interval $I \subset [0, R]$ with measure of order $(e\mu^{-1/3})^{1/2}$, for which the set $I \cap \mathfrak{C}$ for which Lyapounov orbits persist under perturbation has gaps smaller than $\zeta e\mu^{-1/3}$, where $\zeta > 0$ is any given constant.*

Proof. The fact that such an interval exists will follow from the fact that the complement of the Cantor set \mathfrak{C} is of the measure $O((e\mu^{-1/3})^{1/2})$. Let us divide the interval $[0, R]$ into n equal parts. If on every interval the set \mathfrak{C} contains gaps larger than $\zeta e\mu^{-1/3}$, then from the fact that the measure of the complement of \mathfrak{C} is $O((e\mu^{-1/3})^{1/2})$ (let us say that this $O((e\mu^{-1/3})^{1/2})$ is equal to $M (e\mu^{-1/3})^{1/2}$ for some $M > 0$) the number of such intervals n must satisfy

$$n\zeta e\mu^{-1/3} \leq M (e\mu^{-1/3})^{1/2},$$

which means that $n \leq \frac{1}{\zeta} M (e\mu^{-1/3})^{-1/2}$. If we divide the interval $[0, R]$ into a slightly larger number \tilde{n} of equal intervals then at least one of them (this will be our interval I) cannot contain a gap larger than $\zeta e\mu^{-1/3}$. The size of such an interval is equal to

$$\frac{R}{\tilde{n}} \approx \frac{1}{M} R \zeta (e\mu^{-1/3})^{1/2}.$$

■

6. Melnikov method

In the previous section we have shown that the normally hyperbolic manifold with a boundary $\Lambda = \{l(r) : r \in [0, R]\}$ (considered in the extended phase space) of the PRC3BP (28) in Hill's coordinates (16) persists under perturbation to Λ_e which is a normally hyperbolic invariant manifold with a boundary of the PRE3BP with eccentricity e . Moreover, we have shown that Λ_e contains a Cantor set of two dimensional invariant KAM tori. In this section we will consider the problem of intersections of the stable and unstable manifolds of such tori.

In this section we shall once again work in the Hill's coordinates (15) and Hamiltonian (26). It will also be convenient for us to parameterize the manifolds Λ and Λ_e using the radius angle coordinates r, φ of the Lyapounov orbits from Section 3 together with time $t \in S^1$. For $e = 0$ we thus use a natural parameterization of the Lyapounov orbits by their Birkhoff normal form coordinates (coordinates obtained from Theorem 10). After the perturbation it will be enough for us to use the fact that the parametrization is smooth (in fact, from the proof of Theorem 31 we know that it will be C^{r_1} with $r_1 = \alpha/\beta$) and that we can parametrize (see proof of Theorem 31) the perturbed libration point $\bar{L}_2^\mu = \mu^{-1/3} (L_2^\mu - (\mu - 1, 0, 0, \mu - 1))$ (which continues to a 2π periodic orbit) by $r = 0$. We can not assume though that the surviving perturbed KAM tori are parameterized by r . Our parametrization simply follows from the Normally Hyperbolic Invariant Manifold Theorem (see Theorem 25 and Remarks 26, 27) without involving the KAM Theorem.

Let us recall that for μ close to μ_k from Theorem 3 prior to the perturbation the fibres W_p^s for $p \in \Lambda$ intersect transversally with the section $\{\bar{y} = 0\}$ (see Section 2, Theorems 3, 4 and also [20]). The same goes for the unstable fibers W_p^u . This means that for sufficiently small $0 < e$ the same will hold for the stable fibres $W_p^{s,e}$ and unstable fibres $W_p^{u,e}$ of points $p \in \Lambda_e$. Each point $p \in \Lambda_e$ can be parameterized by μ, r, φ and t_0 . The fibres of such points $W_p^{u,e}, W_p^{s,e}$ are one dimensional and contained in sections $\Sigma_{t_0} = \{(q, t_0) | q \in \mathbb{R}^4\}$. The intersections of $W_p^{u,e}$ and of $W_p^{s,e}$ with $\{\bar{y} = 0\}$ are functions of $(\mu, r, \varphi, t_0, e)$. For a point $p \in \Lambda_e$ parameterized by μ, r, φ and t_0 we introduce the following notation for the first intersections of $W_p^{s,e}$ and of $W_p^{u,e}$ with $\{\bar{y} = 0\}$

$$\begin{aligned} (p^s(r, \varphi, t_0, e), t_0) &= W_p^{s,e} \cap \{y = 0\}, \\ (p^u(r, \varphi, t_0, e), t_0) &= W_p^{u,e} \cap \{y = 0\}. \end{aligned}$$

Let $q^s(r, \varphi, t_0, e, t)$ and $q^u(r, \varphi, t_0, e, t)$ be the orbits (considered in the standard (not extended) phase space) of the PRE3BP, which start from the points $p^s(r, \varphi, t_0, e)$ and $p^u(r, \varphi, t_0, e)$ respectively at time $t = t_0$ i.e.

$$\begin{aligned} q^s(r, \varphi, t_0, e, t_0) &= p^s(r, \varphi, t_0, e), \\ q^u(r, \varphi, t_0, e, t_0) &= p^u(r, \varphi, t_0, e). \end{aligned}$$

(p^s, p^u, q^s, q^u depend on the choice of μ , but we omit this in our notations for simplicity).

Let us note that for parameters $\mu = \mu_k$ from Theorem 3

$$\begin{aligned} q^s(0, \varphi, t_0, 0, t) &= q^u(0, \varphi, t_0, 0, t) \\ &= \bar{q}_{\mu_k}^0(t - t_0) \\ &= \mu^{-1/3} (q_{\mu_k}^0(t - t_0) - (\mu_k - 1, 0, 0, \mu_k - 1)), \end{aligned} \quad (77)$$

where $q_{\mu_k}^0(t)$ is the homoclinic orbit to $L_2^{\mu_k}$ in the PRC3BP defined just after the statement of Theorem 3. Let us remind the reader that $q_{\mu_k}^0(0) \in \{y = 0\}$.

Lemma 35 For fixed μ and $i \in \{s, u\}$

$$q^i(r, \varphi, t_0, 0, t_0) = q^i(0, \varphi, t_0, 0, t_0) + O(r), \quad (78)$$

$$q^i(r, \varphi, t_0, e, t_0) = q^i(r, \varphi, t_0, 0, t_0) + e \frac{\partial q^i}{\partial e}(r, \varphi, t_0, 0, t_0) + o(e), \quad (79)$$

$$\frac{\partial q^i}{\partial e}(r, \varphi, t_0, 0, t_0) = \frac{\partial q^i}{\partial e}(0, \varphi, t_0, 0, t_0) + O(r), \quad (80)$$

where the bounds $o(e)$ and $O(r)$ are independent from t_0 .

In addition for f, g from (29)

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial q^i}{\partial e}(0, \varphi, t_0, 0, t) \right) &= Df(\mu, q^i(0, \varphi, t_0, 0, t)) \frac{\partial q^i}{\partial e}(0, \varphi, t_0, 0, t) \\ &\quad + g(q^i(0, \varphi, t_0, 0, t), t), \end{aligned} \quad (81)$$

and $\frac{\partial q^i}{\partial e}(0, \varphi, t_0, 0, t)$ is bounded for all $t \in [t_0, +\infty)$ for $i = s$ (or $t \in (-\infty, t_0]$ for $i = u$).

Proof. The normally hyperbolic manifold and the foliation of its stable and unstable manifolds behave smoothly under perturbation and equations (78–80) are a simple consequence of this.

It remains to prove (81). Let $i = s$. We have

$$\begin{aligned} \frac{d}{dt} \left(\frac{\partial q^s}{\partial e}(0, \varphi, t_0, 0, t) \right) &= \frac{\partial}{\partial e} \frac{d}{dt} q^s(0, \varphi, t_0, 0, t) \\ &= \frac{\partial}{\partial e} \left(f(\mu, q^s(0, \varphi, t_0, e, t)) + eg(q^s(0, \varphi, t_0, e, t), t) + O((e\mu^{-1/3})^2) \right) \Big|_{e=0} \\ &= Df(\mu, q^s(0, \varphi, t_0, 0, t)) \frac{\partial q^s}{\partial e}(0, \varphi, t_0, 0, t) + g(q^s(0, \varphi, t_0, 0, t)). \end{aligned}$$

The points $p^s(0, \varphi, t_0, e)$ lie on stable fibres of the periodic orbit perturbed from L_2^μ . The points $p^s(0, \varphi, t_0, e)$ and $p^s(0, \varphi, t_0, 0)$ are therefore $O(e)$ close. Also the periodic orbit perturbed from L_2^μ lies $O(e)$ close to L_2^μ . Since orbits $q^s(0, \varphi, t_0, e, t)$ start from $p^s(0, \varphi, t_0, e)$ this gives us

$$|q^s(0, \varphi, t_0, e, t) - q^s(0, \varphi, t_0, 0, t)| = O(e) \quad (82)$$

for $t \in [t_0, +\infty)$. This means that

$$\left| \frac{\partial q^s}{\partial e}(0, \varphi, t_0, 0, t) \right| = \left| \lim_{e \rightarrow 0} \frac{q^s(0, \varphi, t_0, e, t) - q^s(0, \varphi, t_0, 0, t)}{e} \right|$$

is bounded.

For $i = u$ the argument is analogous. ■

Remark 36 *Let us note that the bound $O(r)$ in (78) is independent of μ .*

Proof. This follows from formulas for the parameterization of the intersection of the manifolds with $\{y = 0\}$ from Theorem 4. Each curve from Theorem 4 gives an intersection of an unstable manifold of a Lyapounov orbit. A point $q^i(r, \varphi, t_0, 0, t_0)$ is a point of intersection of the unstable manifold of a Lyapounov orbit $l(r)$ with $\{y = 0\}$, and in x, \dot{x} coordinates is represented by a point $(x(\sigma, \sqrt{\Delta C}), \dot{x}(\sigma, \sqrt{\Delta C}))$ on a curve from Theorem 4, for some $\sigma \in S^1$ and $\Delta C > 0$. The point $q^i(0, \varphi, t_0, 0, t_0)$ is the point of intersection of the homoclinic orbit to $L_2^{\mu_k}$ with $\{y = 0\}$ and in x, \dot{x} coordinates is given by $(x(\sigma, 0), \dot{x}(\sigma, 0))$ (where the choice of σ plays no role since for $\Delta C = 0$ equations from Theorem 4 give a single point).

From Theorem 4

$$x(\sigma, \sqrt{\Delta C}) - x(\sigma, 0) = O(\sqrt{\Delta C}) \quad \text{and} \quad \dot{x}(\sigma, \sqrt{\Delta C}) - \dot{x}(\sigma, 0) = O(\sqrt{\Delta C}). \quad (83)$$

By (16) and Lemma 17

$$\begin{aligned} \sqrt{\Delta C} &= \sqrt{H(\mu, \mu^{1/3}l(r) + (\mu - 1, 0, 0, \mu - 1)) - H(\mu, L_2^\mu)} \\ &= \sqrt{\mu^{2/3}\bar{H}(\mu, l(r)) - \mu^{2/3}\bar{H}(\mu, l(0))} \\ &= \mu^{1/3}O(r). \end{aligned} \quad (84)$$

The points $x(\sigma, \sqrt{\Delta C}), x(\sigma, 0), \dot{x}(\sigma, \sqrt{\Delta C}), \dot{x}(\sigma, 0)$ are considered in original coordinates of the system on the section $\{y = 0\}$. For a point $(x, \dot{x}) = (x(\sigma, \sqrt{\Delta C}), \dot{x}(\sigma, \sqrt{\Delta C}))$ on the section $\{y = 0\}$ by (3) we have $p_x = \dot{x} - y = \dot{x}$ and

$$p_y = \sqrt{2(H(L_2^{\mu_k}) + \Delta C + \Omega(x, 0)) - p_x^2} = \sqrt{2H(L_2^{\mu_k})} + \mu^{1/3}O(r).$$

Recall that the points $q^i(r, \varphi, t_0, 0, t_0), q^i(0, \varphi, t_0, 0, t_0)$ are given in Hill's coordinates (15). By (15), (83), (84)

$$\begin{aligned} &|q^i(r, \varphi, t_0, 0, t_0) - q^i(0, \varphi, t_0, 0, t_0)| \\ &= \mu^{-1/3} \left| (x, 0, p_x, p_y)(\sigma, \sqrt{\Delta C}) - (x, 0, p_x, p_y)(\sigma, 0) \right| = O(r). \end{aligned}$$

■

Remark 37 We believe that with techniques similar to the ones used for the proof of Theorem 4 in [20], but involving additionally terms coming from the perturbation from the PRC3BP to the PRE3BP, it should be possible to prove that the terms $o(e)$ and $O(r)$ from (79), (80) can be chosen independently of μ_k . Such statement requires a detailed proof which is rather technical. We skip this intentionally since in later parts of our argument it will turn out that even if this had been done by us, we still cannot obtain uniform bounds for the radius of set on which we have structural stability for the PRE3BP. This is due to the fact that we have not obtained uniform bounds for the Melnikov integral (85) (see Remark 40). Such bounds seem even harder to obtain than proving that the terms $o(e)$ and $O(r)$ from (79), (80) are independent of μ_k .

Remark 36 will allow us though to extract some information as to the radius for which we shall have structural stability for the PRE3BP. It will turn out that this radius needs to converge to zero with μ_k going to zero at least fast enough so that $\mu_k^{-1/3}R(\mu_k)$ is bounded. From our proof though it is not transparent how small exactly it shall need to be chosen.

We now have the following lemma regarding the energy of the points $p^s(r, \varphi, t_0, e)$ and $p^u(r, \varphi, t_0, e)$.

Lemma 38 Assume that μ is one of the parameters μ_k for which a homoclinic orbit $\bar{q}_{\mu_k}^0(t) = \mu^{-1/3}(q_{\mu_k}^0(t) - (\mu_k - 1, 0, 0, \mu_k - 1))$ to $\bar{L}_2^\mu = \mu^{-1/3}(L_2^\mu - (\mu_k - 1, 0, 0, \mu_k - 1))$ exists (See Theorem 3). For any two points $p_1, p_2 \in \Lambda_e$ with coordinates (r_1, φ_1, t_0) and (r_2, φ_2, t_0) respectively, we have

$$\begin{aligned} \bar{H}(\mu_k, p^s(r_1, \varphi_1, t_0, e)) - \bar{H}(\mu_k, p^u(r_2, \varphi_2, t_0, e)) \\ = \bar{H}(\mu_k, l(r_1)) - \bar{H}(\mu_k, l(r_2)) + eM_{\mu_k}(t_0) \\ + O(e \max\{|r_1|, |r_2|\}) + o(e), \end{aligned}$$

where

$$M_{\mu_k}(t_0) = \int_{-\infty}^{+\infty} \{\bar{H}, \bar{G}\}(\mu_k, \bar{q}_{\mu_k}^0(t - t_0), t) dt. \quad (85)$$

Proof. Let \cdot denote the scalar product and let Δ_s and Δ_u denote the following functions

$$\begin{aligned} \Delta_s(t, t_0) &:= \nabla \bar{H}(\mu_k, \bar{q}_{\mu_k}^0(t - t_0)) \cdot \frac{\partial q^s}{\partial e}(0, \varphi_1, t_0, 0, t) \\ \Delta_u(t, t_0) &:= \nabla \bar{H}(\mu_k, \bar{q}_{\mu_k}^0(t - t_0)) \cdot \frac{\partial q^u}{\partial e}(0, \varphi_2, t_0, 0, t). \end{aligned}$$

Using the facts that $\bar{H} = \bar{H}_r = \bar{H}(\mu_k, l(r))$ is constant along the solutions $q^s(r, \varphi, t_0, 0, t)$

of the PRC3BP, from (79) and (77) we can compute

$$\begin{aligned}
& \bar{H}(\mu_k, q^s(r_1, \varphi_1, t_0, e, t_0)) \\
&= \bar{H}(\mu_k, q^s(r_1, \varphi_1, t_0, 0, t_0)) \\
&\quad + e \nabla \bar{H}(\mu_k, q^s(r_1, \varphi_1, t_0, 0, t_0)) \cdot \frac{\partial q^s}{\partial e}(r_1, \varphi_1, t_0, 0, t_0) + o(e) \\
&= \bar{H}_{r_1} + e \nabla \bar{H}(\mu_k, \bar{q}_{\mu_k}^0(0) + O(r_1)) \cdot \left(\frac{\partial q^s}{\partial e}(0, \varphi_1, t_0, 0, t_0) + O(r_1) \right) + o(e) \\
&= \bar{H}_{r_1} + e \nabla \bar{H}(\mu_k, \bar{q}_{\mu_k}^0(0)) \cdot \frac{\partial q^s}{\partial e}(0, \varphi_1, t_0, 0, t_0) + O(er_1) + o(e) \\
&= \bar{H}_{r_1} + e \Delta_s(t_0, t_0) + O(er_1) + o(e), \tag{86}
\end{aligned}$$

and similarly one can show that

$$\bar{H}(\mu_k, q^u(r_2, \varphi_2, t_0, e, t_0)) = \bar{H}_{r_2} + e \Delta_u(t_0, t_0) + O(er_2) + o(e). \tag{87}$$

Let us stress that, by Lemma 35 we know that $O(er_1)$, $O(er_2)$ and $o(e)$ are uniform with respect to t_0 .

Let us investigate the evolution of $\Delta_s(t, t_0)$ and $\Delta_u(t, t_0)$ in time. Let us concentrate on the term $\Delta_s(t, t_0)$. Using (81), (77) and $\nabla \bar{H} = -Jf$ (see (30)) we can compute

$$\begin{aligned}
-\frac{d}{dt}(\Delta_s(t, t_0)) &= \left(JDf(\mu_k, \bar{q}_{\mu_k}^0(t-t_0)) \frac{d}{dt} \bar{q}_{\mu_k}^0(t-t_0) \right) \cdot \frac{\partial q^s}{\partial e}(0, \varphi_1, t_0, 0, t) \\
&\quad + (Jf(\mu_k, \bar{q}_{\mu_k}^0(t-t_0))) \cdot \frac{d}{dt} \frac{\partial q^s}{\partial e}(0, \varphi_1, t_0, 0, t) \\
&= (JDf(\mu_k, \bar{q}_{\mu_k}^0(t-t_0)) f(\mu_k, \bar{q}_{\mu_k}^0(t-t_0))) \cdot \frac{\partial q^s}{\partial e}(0, \varphi_1, t_0, 0, t) \\
&\quad + (Jf(\mu_k, \bar{q}_{\mu_k}^0(t-t_0))) \cdot \left(Df(\mu_k, \bar{q}_{\mu_k}^0(t-t_0)) \frac{\partial q^s}{\partial e}(0, \varphi_1, t_0, 0, t) \right) \\
&\quad + (Jf(\mu_k, \bar{q}_{\mu_k}^0(t-t_0))) \cdot g(\mu_k, \bar{q}_{\mu_k}^0(t-t_0), t) \\
&= (Jf(\mu_k, \bar{q}_{\mu_k}^0(t-t_0))) \cdot g(\mu_k, \bar{q}_{\mu_k}^0(t-t_0), t) \\
&= -\{\bar{H}, \bar{G}\}(\mu_k, \bar{q}_{\mu_k}^0(t-t_0), t),
\end{aligned} \tag{88}$$

where the third equality comes from the fact that for any $p, q \in \mathbb{R}^4$

$$(JDf(\mu_k, \bar{q}_{\mu_k}^0(t-t_0))p) \cdot q + (Jp) \cdot (Df(\mu_k, \bar{q}_{\mu_k}^0(t-t_0))q) = 0, \tag{89}$$

with $p = f(\mu_k, \bar{q}_{\mu_k}^0(t-t_0))$ and $q = \frac{\partial q^s}{\partial e}(0, \varphi_1, t_0, 0, t)$. Equation (89) follows from the fact that $\omega(p, q) = Jp \cdot q$ is the standard symplectic form which is invariant under the flow $\phi(t, x)$ of (16) i.e.

$$\omega \left(\frac{\partial}{\partial x} \phi(t, x) p, \frac{\partial}{\partial x} \phi(t, x) q \right) = \omega(p, q), \tag{90}$$

hence by differentiating (90) with respect to t and setting $t = 0$ we obtain (89).

We can now compute $\Delta_s(t_0, t_0)$ using (88)

$$\Delta_s(+\infty, t_0) - \Delta_s(t_0, t_0) = \int_{t_0}^{+\infty} \{\bar{H}, \bar{G}\}(\mu_k, \bar{q}_{\mu_k}^0(t-t_0), t) dt.$$

Since $\lim_{t \rightarrow +\infty} \bar{q}_{\mu_k}^0(t-t_0) = \bar{L}_2^{\mu_k}$ at geometric rate and $f(\mu_k, \bar{L}_2^{\mu_k}) = 0$, from the fact that $\frac{\partial q^s}{\partial e}(0, \varphi_1, t_0, 0, t)$ is bounded on $[t_0, +\infty)$ we have

$$\Delta_s(+\infty, t_0) = \lim_{t \rightarrow +\infty} Jf(\mu_k, \bar{q}_{\mu_k}^0(t-t_0)) \cdot \frac{\partial q^s}{\partial e}(0, \varphi_1, t_0, 0, t) = 0$$

and therefore

$$-\Delta_s(t_0, t_0) = \int_{t_0}^{+\infty} \{\bar{H}, \bar{G}\}(\mu_k, \bar{q}_{\mu_k}^0(t-t_0), t) dt, \quad (91)$$

and Δ is uniformly with respect to t_0 absolutely convergent.

Analogous computations give

$$\Delta_u(t_0, t_0) = \int_{-\infty}^{t_0} \{\bar{H}, \bar{G}\}(\bar{q}_{\mu_k}^0(t-t_0), t) dt. \quad (92)$$

From (86), (87), (91) and (92) we obtain our claim. ■

Theorem 39 *Consider the PRE3BP with a sufficiently small parameter $\mu = \mu_k$ for which a homoclinic orbit to $L_2^{\mu_k}$ exists (See Theorem 3). Assume that*

$$M_{\mu_k}(t_0) = \int_{-\infty}^{+\infty} \{\bar{H}, \bar{G}\}(\mu_k, \bar{q}_{\mu_k}^0(t-t_0), t) dt \quad (93)$$

has simple zeros. Let $R(\mu_k) \in \mathbb{R}$, be such that $0 < R(\mu_k)$ and $\mu_k^{-1/3}R(\mu_k)$ is sufficiently close to zero. Then for any $R \in (0, R(\mu_k))$ there exists an $e_0(R)$ such that for all $e\mu_k^{-1/3} \in [0, \min(e_0(R), \kappa\mu_k^{1/3})]$ (where κ is the constant from (25)) and all $r \in \mathfrak{C} \cap [R, R(\mu_k)]$ for which $l(r)$ is perturbed to an invariant torus $l^e(t)$

$$l^e(r) = \{(l_{t_0}^e(r), t_0) | t_0 \in S^1\}$$

the manifolds $W_{l^e(r)}^s$ and $W_{l^e(r)}^u$ (considered in the extended phase space) intersect transversally.

Proof. We consider the PRE3BP (29) in Hill's coordinates (15). In [20] it has been shown (see also Theorem 4 in Section 2.1) that for the unperturbed PRC3BP $W_{l(r)}^s$ intersects transversally with $W_{l(r)}^u$ at $\{\bar{y} = 0\}$. Let $\Sigma_{t_0} = \mathbb{R}^4 \times \{t_0\}$ be the time t_0 section in the extended phase space. Let $v^0(t_0)$ denote some point for which

$$v^0(t_0) \in W_{l(r)}^s \cap W_{l(r)}^u \cap \{\bar{y} = 0\} \cap \Sigma_{t_0}. \quad (94)$$

There can be more than just one such point (see Figures 3, 4), namely, if we consider $\pi_{x, \bar{p}_x}(W_{l(r)}^u \cap \Sigma_{t_0} \cap \{\bar{y} = 0\})$ and $\pi_{\bar{x}, \bar{p}_x}(W_{l(r)}^s \cap \Sigma_{t_0} \cap \{y = 0\})$ then the two sets are homeomorphic to two circles, which intersect transversally at least one point

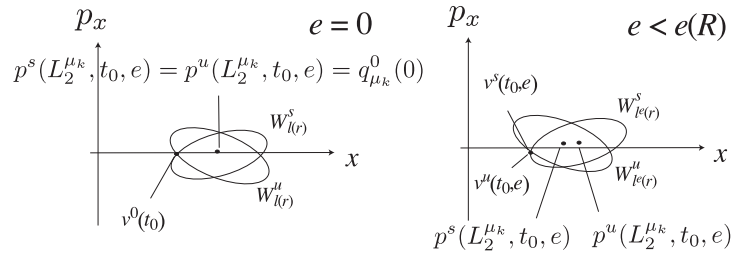


Figure 4. The manifolds $W_{l(r)}^s$, $W_{l(r)}^u$, $W_{l^e(r)}^s$ and $W_{l^e(r)}^u$ for $r > R$, intersected with Σ_{t_0} and $\{y = 0\}$, and projected onto the x, p_x coordinates.

($\bar{x} = \bar{x}_0, \bar{p}_x = 0$) (see Theorem 4). Fixing any one of such points of intersection will be sufficient for our proof. From the construction we have $\pi_{x,y,p_x,t}(v^0(t_0)) = (\bar{x}_0, 0, 0, t_0)$. The extended phase space of the PRC3BP is five dimensional. We can choose the energy \bar{H} to be the remaining fifth coordinate in the neighborhood of $v^0(t_0)$. In these coordinates

$$v^0(t_0) = (\bar{x} = \bar{x}_0, \bar{y} = 0, \bar{p}_x = 0, \bar{H} = \bar{H}(\mu_k, l(r)), t = t_0).$$

Now we perturb from the PRC3BP to the PRE3BP. Consider sufficiently small μ_k and perturbation e satisfying $e\mu_k^{-2/3} < \kappa$ so that by Theorem 31 we have a Cantor set \mathfrak{C} of KAM tori $l^e(r)$. Let us consider some small $R(\mu_k) \in \mathbb{R}$ (the size of $R(\mu_k)$ needs to be chosen small compared with $M_{\mu_k}(t)$. This is discussed later on in our argument). Let $0 < R < R(\mu_k)$ and $r \in \mathfrak{C} \cap [R, R(\mu_k)]$. Consider now the following sets

$$\pi_{\bar{x}, \bar{p}_x} (W_{l^e(r)}^u \cap \Sigma_{t_0} \cap \{\bar{y} = 0\}) \quad \text{and} \quad \pi_{\bar{x}, \bar{p}_x} (W_{l^e(r)}^s \cap \Sigma_{t_0} \cap \{\bar{y} = 0\}).$$

For sufficiently small e these sets remain homeomorphic to circles. For $e = 0$ the curves intersect transversally at an angle $O(\mu_k^{1/3})$ (see Remark 7). This means that by choosing $e\mu_k^{-1/3} > 0$ sufficiently small the curves shall intersect at some point ($\bar{x} = \bar{x}_0(t_0, e), \bar{p}_x = \bar{p}_x^0(t_0, e)$) which is close to $(x_0, 0)$. The choice of $e\mu_k^{-1/3}$ also depends on R , since in order to ensure that the curves intersect we assume that their radius is greater than R . Hence we have $e_0(R)$ in the formulation of our theorem. For the PRE3BP the energy \bar{H} is no longer preserved. This means that the intersection of the circles on the \bar{x}, \bar{p}_x plane does not imply an intersection in the extended phase space. Namely we have two points $v^s(t_0, e) \in W_{l^e(r)}^s \cap \Sigma_{t_0} \cap \{\bar{y} = 0\}$ and $v^u(t_0, e) \in W_{l^e(r)}^u \cap \Sigma_{t_0} \cap \{\bar{y} = 0\}$ which may differ on the energy coordinate (see Figure 5)

$$\begin{aligned} v^s(t_0, e) &= (\bar{x}_0(t_0, e), 0, \bar{p}_x^0(t_0, e), h^s(t_0, e), t_0) \\ v^u(t_0, e) &= (\bar{x}_0(t_0, e), 0, \bar{p}_x^0(t_0, e), h^u(t_0, e), t_0). \end{aligned} \tag{95}$$

We will show that assumptions of our theorem imply that for some t_0 the points $v^s(t_0, e)$ and $v^u(t_0, e)$ coincide. This will imply intersection between $W_{l^e(r)}^s$ and $W_{l^e(r)}^u$. Later we will also show that such intersection is transversal.

The points $v^s(t_0, e)$ and $v^u(t_0, e)$ are both contained in $\{\bar{y} = 0\}$. Moreover $v^s(t_0, e) \in W_{l^e(r)}^s \cap \Sigma_{t_0} = W_{l_0^e(r)}^s$ and $v^u(t_0, e) \in W_{l^e(r)}^u \cap \Sigma_{t_0} = W_{l_0^e(r)}^u$. This means that there exist r^s, φ^s and r^u, φ^u (these depend on t_0 and e but we omit this in our notations for simplicity) such that

$$\begin{aligned} v^s(t_0, e) &= (p^s(r^s, \varphi^s, t_0, e), t_0) \\ v^u(t_0, e) &= (p^u(r^u, \varphi^u, t_0, e), t_0). \end{aligned} \quad (96)$$

In the proof of Theorem 31 $l_0^e(r)$ is constructed from continuation along trajectories of a KAM torus $l_0^e(r)$. Therefore from (72) in KAM Theorem 29 we have that

$$\begin{aligned} r^s &= r + O(e\mu^{-1/3}), \\ r^u &= r + O(e\mu^{-1/3}), \end{aligned} \quad (97)$$

and the bound $O(e\mu^{-1/3})$ is uniform for all $r \in \mathfrak{C}$. Applying Lemma 38 we have that

$$\begin{aligned} h^u(t_0, e) - h^s(t_0, e) &= \bar{H}(\mu_k, p^s(r^s, \varphi^s, t_0, e)) - \bar{H}(\mu_k, p^u(r^u, \varphi^u, t_0, e)) \\ &= \bar{H}(\mu_k, l(r^s)) - \bar{H}(\mu_k, l(r^u)) + eM_{\mu_k}(t_0) + O(eR(\mu_k)) + o(e). \end{aligned} \quad (98)$$

The Hamiltonian in coordinates $(q, p) = (q_1, q_2, p_1, p_2) = (\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y) - \bar{L}_2^{\mu_k}$ centered in $\bar{L}_2^{\mu_k}$ is

$$H(p, q) := \bar{H}(\mu_k, (q, p) + \bar{L}_2^{\mu_k}).$$

By Lemma 17 we hence know that

$$\bar{H}(\mu_k, l(r)) = \bar{H}(\mu_k, \bar{L}_2^{\mu_k}) + \frac{1}{2}D^2H(0)(\Phi(0, 1, 0, i))r^2 + o(r^2)$$

hence from (97) and (98) we have

$$\begin{aligned} h^u(t_0, e) - h^s(t_0, e) &= O(re\mu^{-1/3}) + eM_{\mu_k}(t_0) + O(eR(\mu_k)) + o(e) \\ &= eM_{\mu_k}(t_0) + O(e\mu^{-1/3}R(\mu_k)) + o(e). \end{aligned} \quad (99)$$

Setting first $R(\mu_k)$ sufficiently small (so that $\mu_k^{-1/3}R(\mu_k)$ is small in comparison to $M_{\mu_k}(t)$) and then reducing e sufficiently close to zero implies that since $M_{\mu_k}(t_0)$ has simple zeros, for some parameters t_0 (close to these zeros) we will have $h^u(t_0, e) - h^s(t_0, e) = 0$, which implies that $v^s(t_0, e) = v^u(t_0, e)$ and in turn ensures that

$$W_{l^e(r)}^s \cap W_{l^e(r)}^u \neq \emptyset.$$

Now we will show that this intersection is transversal. First note that from the analyticity of the functions $v^s(t_0, e)$ and $v^u(t_0, e)$ using the same argument as in the proof of Lemma 38 we also have that

$$\frac{\partial}{\partial t}(h^u(t_0, e) - h^s(t_0, e)) = e\frac{\partial}{\partial t}M_{\mu_k}(t) + O(e\mu_k^{-1/3}R(\mu_k)) + o(e). \quad (100)$$

We know that prior to our perturbation $W_{l(r)}^s$ and $W_{l(r)}^u$ intersect transversally at $v^0(t_0)$ (94). This intersection is not transversal in the full extended phase space, it is only transversal in the constant energy manifold

$$M = \{(\bar{x}, \bar{y}, \bar{p}_x, \bar{H}, t) | \bar{H} = \bar{H}(\mu_k, l(r))\} \subset \mathbb{R}^3 \times \{\bar{H}(\mu_k, l(r))\} \times S^1.$$

To be more precise, we know that $v^0(t_0) \in W_{l_0^s(r)}^s \cap W_{l_0^u(r)}^u$, that $W_{l_0^s(r)}^s, W_{l_0^u(r)}^u \subset \Sigma_{t_0}$ and that [20]

$$T_{v^0(t_0)}(W_{l_0^s(r)}^s) + T_{v^0(t_0)}(W_{l_0^u(r)}^u) = \mathbb{R}^3 \times \{0\} \times \{0\}.$$

These properties are preserved under small perturbation $e > 0$, hence for $v^s(t_0, e) = v^u(t_0, e) =: v(t_0, e)$ we have

$$T_{v(t_0, e)}(W_{l_0^s(r)}^s) + T_{v(t_0, e)}(W_{l_0^u(r)}^u) = \mathbb{R}^3 \times \{0\} \times \{0\}.$$

We need to show that we also have transversality on the t_0 and energy coordinate. For a fixed e the curves $v^s(t, e)$ and $v^u(t, e)$ belong to $W_{l^e(r)}^s$ and $W_{l^e(r)}^u$ respectively. At the time $t = t_0$ for which $v^s(t_0, e) = v^u(t_0, e) = v(t_0, e)$ we have $M'_{\mu_k}(t_0) \neq 0$. This means that using (100), for sufficiently small e ,

$$\begin{aligned} \frac{\partial}{\partial t} (\pi_H(v^s(t, e)) - \pi_H(v^u(t, e))) |_{t=t_0} &= \frac{\partial}{\partial t} (h^u(t, e) - h^s(t, e)) |_{t=t_0} \\ &= eM'_{\mu_k}(t_0) + O(e\mu_k^{-1/3}R(\mu_k)) + o(e) \\ &\neq 0. \end{aligned}$$

We also have for $i \in \{u, s\}$, $\frac{\partial}{\partial t} (\pi_{t_0} v^i(t, e)) = \frac{\partial}{\partial t} t = 1$. This since $\frac{d}{dt} v^s(t, e) |_{t=t_0} \in T_{v(t_0, e)}(W_{l^e(r)}^s)$ and $\frac{d}{dt} v^u(t, e) |_{t=t_0} \in T_{v(t_0, e)}(W_{l^e(r)}^u)$ implies transversality, which finishes our proof.

The order of choice of parameters in the above argument is important, so let us quickly run through how it should be conducted. We first choose sufficiently small μ_k so that we can apply Theorem 31. Then by choosing small $R(\mu_k)$ we ensure that $\mu_k^{-1/3}R(\mu_k)$ is sufficiently small compared with M_{μ_k} and M'_{μ_k} . We then choose e so that $e\mu_k^{-1/3}$ is sufficiently small so that we have transversal intersections of $\pi_{x, p_x}(W_{l^e(r)}^u \cap \Sigma_{t_0} \cap \{\bar{y} = 0\})$ and $\pi_{x, p_x}(W_{l^e(r)}^s \cap \Sigma_{t_0} \cap \{\bar{y} = 0\})$. The parameter e needs also to be sufficiently small so that $M_{\mu_k}(t_0)$ and $M'_{\mu_k}(t_0)$ dominate in (99) and (100) respectively. ■

Remark 40 *In the proof of Theorem 39 we see that we need to choose the radius $R(\mu_k)$ to be sufficiently small so that $\mu_k^{-1/3}R(\mu_k)$ is small in comparison to the Melnikov integral M_{μ_k} and its derivative. Since we do not have uniform bounds on the size of the Melnikov integral with respect to μ_k , from our argument we cannot say how small $R(\mu_k)$ needs to be. From our numerical investigation which will follow in Table 2 we can see that for t_0 for which we have a simple zero of the Melnikov integral, the bound on the derivative is independent of μ_k . This means that we need to choose the radius $R(\mu_k)\mu_k^{1/3}$ to be sufficiently small (hence $R(\mu_k)$ converges to zero with μ_k going to zero).*

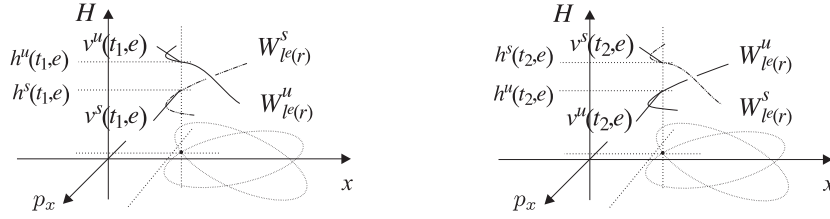


Figure 5. The manifolds $W_{l^e(r)}^s$ and $W_{l^e(r)}^u$ intersected with $\Sigma_{t_1}, \Sigma_{t_2}$ and $\{y = 0\}$, in the x, p_x, H coordinates for $t_1 < t_0 < t_2$.

Corollary 41 *If μ_k is sufficiently small and the Melnikov integral has a simple zero then for sufficiently small $\mu_k^{-1/3} R(\mu_k)$ there exists a $\zeta > 0$, such that for any two radii r_1 and r_2 from $\mathfrak{C} \cap [R, R(\mu_k)]$*

$$|r_1 - r_2| < \zeta e \mu_k^{-1/3},$$

the manifolds $W_{l^e(r_i)}^s$ and $W_{l^e(r_j)}^u$ intersect transversally for $i, j \in \{1, 2\}$.

Proof. The proof of this fact is a mirror argument to the proof of Theorem 39. Below we restrict our attention to pointing out the difference we have connected with the derivation of (99) in the setting where we have two radii.

Let $h_1^u(t_0, e)$ and $h_2^s(t_0, e)$ stand for energies of points of potential intersection (constructed analogously to $h^u(t_0, e)$ and $h^s(t_0, e)$ in (95)). Let r_1^u and r_2^s stand for radii constructed analogously to r^u and r^s (see (96)), but coming from the unstable and stable manifold of $l^e(r_1)$ and $l^e(r_2)$ respectively. Using Lemma 17 and the fact that $r_1^u - r_1 = O(e \mu_k^{-1/3})$ and $r_2^s - r_2 = O(e \mu_k^{-1/3})$ we have

$$\begin{aligned} |\bar{H}(\mu_k, l(r_1^u)) - \bar{H}(\mu_k, l(r_2^s))| &\leq O(|(r_1^u)^2 - (r_2^s)^2|) \\ &\leq O(R(\mu_k) |r_1^u - r_2^s|) \\ &\leq O(R(\mu_k) |r_1 - r_2|) + O(e \mu_k^{-1/3} R(\mu_k)). \end{aligned}$$

Using an identical argument to the derivation of (99) this gives us

$$h_1^u(t_0, e) - h_2^s(t_0, e) = e M_{\mu_k}(t_0) + O(R(\mu_k) |r_1 - r_2|) + O(e \mu_k^{-1/3} R(\mu_k)) + o(e).$$

Using this estimate and following the proof of Theorem 39 we obtain our claim. ■

Remark 42 *Mirror arguments to the proof of Theorem 39 and Corollary 41 give transversal intersections of invariant manifolds for invariant tori of the PRE3BP with $\mu = \mu_k$ for $k = 2, 3, \dots$. Here we state this as a separate remark since Theorem 39 and Corollary 41 are fully rigorous and do not rely on any numerical computations. For the argument with an arbitrary μ_k we would need to use the fact that in the PRC3BP we have a twist property on the family of Lyapounov orbits and apply Remark 32. The twist for arbitrary μ_k has only been demonstrated numerically (see Table 1).*

7. Computation of the Melnikov integral.

In this section we will demonstrate that for $t_0 = 0$ and for all parameters μ_k from Theorem 3 the Melnikov integral $M_{\mu_k}(t_0)$ (93) is zero and also that $\frac{dM_{\mu_k}}{dt_0}(0) \neq 0$. The fact that the Melnikov integral is zero will follow directly from the S -symmetry (8) of the homoclinic orbit $q_{\mu_k}^0(t)$. The fact that $\frac{dM_{\mu_k}}{dt_0}(0) \neq 0$ will be demonstrated numerically. We will compute the integral for the first few parameters μ_k and then demonstrate that for sufficiently small parameters μ_k the integral converges to an integral along an unstable manifold of the Hill's problem.

7.1. The Melnikov integral and its derivative at $t_0 = 0$

We start with a lemma which ensures the convergence of the Melnikov integral (93).

Lemma 43 *The Melnikov integral (93) and its derivative is absolutely convergent uniformly with respect to t_0 . The Melnikov function can be expressed as*

$$M_{\mu_k}(t_0) = \int_{-\infty}^{+\infty} \left[\frac{\partial \bar{G}}{\partial t}(\mu_k, \bar{q}_{\mu_k}^0(t), t + t_0) - \frac{\partial \bar{G}}{\partial t}(\mu_k, \bar{L}_2^{\mu_k}, t + t_0) \right] dt, \quad (101)$$

and also

$$\frac{dM_{\mu_k}}{dt}(t_0) = \int_{-\infty}^{+\infty} \left[\frac{\partial^2 \bar{G}}{\partial t^2}(\mu_k, \bar{q}_{\mu_k}^0(t), t + t_0) - \frac{\partial^2 \bar{G}}{\partial t^2}(\mu_k, \bar{L}_2^{\mu_k}, t + t_0) \right] dt. \quad (102)$$

Proof. The orbit $\bar{q}_{\mu_k}^0(t)$ is the homoclinic orbit to the Libration point $\bar{L}_2^{\mu_k}$. Let us note that the velocity \bar{x}' and \bar{y}' of $\bar{q}_{\mu_k}^0(t)$ exponentially tends to zero as t tends to plus infinity and minus infinity. Moreover the partial derivatives of \bar{G} on $\bar{q}_{\mu_k}^0(t)$ are uniformly bounded. This means that the integral over

$$\int_{-\infty}^{+\infty} |\{\bar{H}, \bar{G}\}|(\bar{q}_{\mu_k}^0(t), t + t_0) dt = \int_{-\infty}^{+\infty} |\bar{x}' \frac{\partial \bar{G}}{\partial \bar{x}} + \bar{y}' \frac{\partial \bar{G}}{\partial \bar{y}}|(\mu_k, \bar{q}_{\mu_k}^0(t), t + t_0) dt,$$

is convergent uniformly with respect to t_0 .

The orbit $\bar{q}_{\mu_k}^0(t)$ is the solution of the PRC3BP, hence differentiating gives

$$\frac{d\bar{G}}{dt}(\mu_k, \bar{q}_{\mu_k}^0(t), t + t_0) = \frac{\partial \bar{G}}{\partial t}(\mu_k, \bar{q}_{\mu_k}^0(t), t + t_0) + \{\bar{G}, \bar{H}\}(\mu_k, \bar{q}_{\mu_k}^0(t), t + t_0). \quad (103)$$

From (103) we have

$$\begin{aligned} M_{\mu_k}(t_0) &= \int_{-\infty}^{+\infty} \{\bar{H}, \bar{G}\}(\mu_k, \bar{q}_{\mu_k}^0(t), t + t_0) dt \\ &= \lim_{T \rightarrow \infty} \int_{-T}^T \left(\frac{\partial \bar{G}}{\partial t}(\mu_k, \bar{q}_{\mu_k}^0(t), t + t_0) - \frac{d\bar{G}}{dt}(\mu_k, \bar{q}_{\mu_k}^0(t), t + t_0) \right) dt \\ &= \lim_{T \rightarrow \infty} \left[\bar{G}(\mu_k, \bar{q}_{\mu_k}^0(-T), -T + t_0) - \bar{G}(\mu_k, \bar{q}_{\mu_k}^0(T), T + t_0) \right. \\ &\quad \left. + \int_{-T}^T \frac{\partial \bar{G}}{\partial t}(\mu_k, \bar{q}_{\mu_k}^0(t), t + t_0) dt \right]. \end{aligned} \quad (104)$$

To complete the proof of (101) observe that that from $\lim_{T \rightarrow \pm\infty} \bar{q}_{\mu_k}^0(T) = \bar{L}_2^{\mu_k}$ it follows that

$$\begin{aligned} \lim_{T \rightarrow \infty} \left(\int_{-T}^T \frac{\partial \bar{G}}{\partial t} (\mu_k, \bar{L}_2^{\mu_k}, t + t_0) dt - \bar{G} (\mu_k, \bar{q}_{\mu_k}^0(T), T + t_0) \right. \\ \left. + \bar{G} (\mu_k, \bar{q}_{\mu_k}^0(-T), -T + t_0) \right) = 0 \end{aligned} \quad (105)$$

uniformly with respect to t_0 .

From (104) and (105) we obtain (101).

To prove (102) it is enough to observe that the formal integration of (101) is correct, because the integral on the right hand side of (102) is uniformly convergent with respect to t_0 for the same reasons as the integral in formula (101). ■

It turns out that the computation of the Melnikov integral is not the major obstacle. The fact that we have a zero for $t_0 = 0$ follows directly from the S -symmetry (8) of the homoclinic orbit $q_{\mu_k}^0(t)$ and G . This fact is shown in the below lemma. Later though we will need to show that this zero is nontrivial by computing $\frac{dM_{\mu_k}}{dt}(0)$, which turns out to be a much harder task. The lemma also provides the formula for the needed integral.

Lemma 44 *The Melnikov integral (93) at $t_0 = 0$ is equal to zero and*

$$\frac{dM_{\mu_k}}{dt}(0) = -2\mu_k^{-2/3} \int_{-\infty}^0 (G(\mu_k, q_{\mu_k}^0(t), t) - G(\mu_k, L_2^{\mu_k}, t)) dt. \quad (106)$$

Proof. First let us observe that from (27)

$$\bar{G}(\mu_k, \bar{q}_{\mu_k}^0(t), t) = \mu_k^{-2/3} G(\mu_k, q_{\mu_k}^0(t), t).$$

The orbit $q_{\mu_k}^0(t)$ and fixed point $L_2^{\mu_k}$ are S -symmetric with respect to the symmetry (8). From (7), (23), by direct computation one can check that $\frac{\partial G}{\partial t}(\mu, S(\cdot), -t) = -\frac{\partial G}{\partial t}(\mu, \cdot, t)$ and that $\frac{\partial^2 G}{\partial t^2}(\mu, S(\cdot), -t) = \frac{\partial^2 G}{\partial t^2}(\mu, \cdot, t)$ hence we have

$$\begin{aligned} \int_{-\infty}^0 \left(\frac{\partial \bar{G}}{\partial t}(\mu_k, \bar{q}_{\mu_k}^0(t), t) - \frac{\partial \bar{G}}{\partial t}(\mu_k, \bar{L}_2^{\mu_k}, t) \right) dt = \\ = \mu_k^{-2/3} \int_0^{+\infty} \left(\frac{\partial G}{\partial t}(\mu_k, q_{\mu_k}^0(-t), -t) - \frac{\partial G}{\partial t}(\mu_k, L_2^{\mu_k}, -t) \right) dt \\ = \mu_k^{-2/3} \int_0^{+\infty} \left(\frac{\partial G}{\partial t}(\mu_k, S(q_{\mu_k}^0(t)), -t) - \frac{\partial G}{\partial t}(\mu_k, S(L_2^{\mu_k}), -t) \right) dt \\ = -\mu_k^{-2/3} \int_0^{+\infty} \left(\frac{\partial G}{\partial t}(\mu_k, q_{\mu_k}^0(t), t) - \frac{\partial G}{\partial t}(\mu_k, L_2^{\mu_k}, t) \right) dt, \end{aligned}$$

which gives $M_{\mu_k}(0) = 0$. From an analogous computation using $\frac{\partial^2 G}{\partial t^2}(S(\cdot), -t) = \frac{\partial^2 G}{\partial t^2}(\cdot, t)$ follows

$$\frac{dM_{\mu_k}}{dt}(0) = 2\mu_k^{-2/3} \int_{-\infty}^0 \left(\frac{\partial^2 G}{\partial t^2}(\mu_k, q_{\mu_k}^0(t), t) - \frac{\partial^2 G}{\partial t^2}(\mu_k, L_2^{\mu_k}, t) \right) dt.$$

Form (23) we have $\frac{\partial^2 G}{\partial t^2} = -G$, which gives (106). ■

Remark 45 *The verification of the fact that $\frac{dM_{\mu_k}}{dt}(0)$ is nonzero is not straightforward. In this paper we will restrict ourselves to numerical verification of this fact. We would like to highlight that for a given parameter μ_k it is possible to obtain a rigorous-computer-assisted estimate on $\frac{dM_{\mu_k}}{dt}(0)$. To do so one first needs to obtain a rigorous bound $[\underline{\mu}_k, \bar{\mu}_k]$ which contains the parameter μ_k for which we have a homoclinic orbit. Then one needs to obtain rigorous enclosures on the trajectories $q_\mu^0(t)$ for all $\mu \in [\underline{\mu}_k, \bar{\mu}_k]$. Using these, a bound on $\frac{dM_{\mu_k}}{dt}(0)$ can be computed from (106).*

We have successfully conducted such computations for the parameter μ_2 . We have used the fact that if one extends the system by including μ as an additional variable, then for any interval I the set $\{L_2^\mu\}_{\mu \in I}$ is a normally hyperbolic invariant manifold. We have applied a topological method, given in [6], [7], for detection of normally hyperbolic manifolds, combined with a parameterization method [5]. Based on these we proved the following. We have shown that the parameter μ_2 for which we have the homoclinic orbit is contained in $[0.0042538631, 0.0042538639]$. We then proved that $\frac{dM_{\mu_2}}{dt}(0) \in [1.301020122, 1.865308899]$. A detailed proof of this fact, along with results for other parameters, will be the subject of a forthcoming publication.

In Table 2 we enclose the (nonrigorous) numerical results for the computation of (106) obtained for k up to 13.

k	$\frac{dM_{\mu_k}}{dt}(0)$
2	1.57396
3	-0.396727
4	0.395119
5	-0.396931
6	0.395784
7	-0.395511
8	0.393389
9	-0.393253
10	0.390924
11	-0.391194
12	0.388961
13	-0.389459

Table 2. Numerical results for the derivation of $\frac{dM_{\mu_k}}{dt}(0)$ for various mass parameters.

To obtain the numerical results from Table 2 we have used a parameterization method [5] to obtain an expansion of the manifold around $L_2^{\mu_k}$ as a polynomial of degree 20. Then we integrated the system numerically using a Taylor method of order 20. Numerical evidence points to M_{μ_k} having a nontrivial zero for $t = 0$.

8. Transition Chains and Main Result

In this section we will show that there exists a sequence of Lyapounov orbits $l(r_i)$ for $i = 1, \dots, N$ which survive for a sufficiently small perturbation e , and such that their stable and unstable manifolds intersect transversally

$$W_{l^e(r_i)}^s \pitchfork W_{l^e(r_{i+1})}^u \quad \text{and} \quad W_{l^e(r_i)}^u \pitchfork W_{l^e(r_{i+1})}^s. \quad (107)$$

Such sequences are referred to as transition chains. Existence of such chains ensures that we have an orbit which shadows the homo- or heteroclinic connections between the surviving tori generating rich symbolic dynamics (see [10],[11],[12]). Apart from this, from our argument it will also follow that for any surviving orbit $l(r)$ we also have transversal intersection of its stable and unstable manifold

$$W_{l^e(r)}^u \pitchfork W_{l^e(r)}^s. \quad (108)$$

This ensures that the chaotic dynamics of the PRC3BP which is implied by the existence of a transversal homoclinic orbit to $l(r)$ (see Theorem 4 and Remark 6) survives.

We are now ready to rigorously reformulate our main theorem (Theorem 2).

Theorem 46 *For any μ from the sequence of masses $\{\mu_k\}_{k=2}^\infty$ from Theorem 3, if the twist property is satisfied, then there exists a radius R_{Hill} , which is independent of μ , such that for eccentricities e of the elliptic problem, with $e\mu^{-2/3} < \kappa$ (see 25 for interpretation of κ) and sufficiently small $e\mu^{-1/3}$, there exists a Cantor set $\mathfrak{C} \subset [0, R_{\text{Hill}}]$ such that for all $r \in \mathfrak{C}$ the Lyapounov orbits $l(r)$ are perturbed to invariant tori $l^e(r)$ in the extended phase space. Moreover, if the derivative of the Melnikov integral (106) is nonzero, then there exists a radius $R(\mu)$ of order at most $O(\mu^{1/3})$ and a transition chain $l^e(r_i)$ for a sequence of radii $0 < r_1 < r_2 < \dots < r_N < R(\mu)$ such that the difference $r_N - r_1$ is of order $(e\mu^{-1/3})^{1/2}$.*

The transversal intersections of the transition chain (107), (108) lead to a homo- and heteroclinic tangle of the stable/unstable manifolds of $l^e(r_i)$, which in turn leads to symbolic dynamics involving diffusion in energy.

Remark 47 *Let us note now that we can verify assumptions of the above theorem based on some numerical results. The twist property for the family of Lyapounov orbits has been rigorously proved for sufficiently small μ_k in Theorem 31, but the fact that we have twist for μ_k with $k = 1, 2, \dots$ has only been demonstrated numerically (see Section 4, Table 1 and Remark 32). Secondly, assumptions of Theorem 46 require that the derivative of the Melnikov function (106) at zero is nonzero. This has only been demonstrated numerically in Section 7.*

We believe that the above can be verified using rigorous-computer-assisted methods. This is currently a subject of ongoing work (see Remark 45).

Remark 48 *All radiuses considered in Theorem 46 are given in the Hill's coordinates (15). This means that in the original coordinates of our system (given by the Hamiltonian (22)) the radiuses are reduced by a factor of $\mu^{1/3}$.*

Proof of Theorem 46. Let us fix a $\mu = \mu_k$. For sufficiently small μ by Theorem 31, and if the twist condition holds for all μ_k by Remark 32, we have the radius R_{Hill} and a Cantor $\mathfrak{C} \subset [0, R_{\text{Hill}}]$ of radii for which the Lyapounov orbits survive the perturbation. From Theorem 39, Remark 42 (see also Corollary 41) it follows that by choosing $R(\mu) < R_{\text{Hill}}$, for which $R(\mu)\mu^{-1/3}$ is sufficiently small, we know that there exists a $\zeta > 0$ such that for radii $r_1, r_2 \in \mathfrak{C}$ such that $|r_1 - r_2| < \zeta e\mu^{-1/3}$, for e with sufficiently small $e\mu^{-1/3}$, we have

$$W_{I^e(r_i)}^u \pitchfork W_{I^e(r_i)}^s \quad \text{for } i \in \{1, 2\}.$$

We now need to show that we can find a sequence $r_1 < r_2 < \dots < r_N$ such that $r_i \in \mathfrak{C}$ and the difference $r_N - r_1$ is of order $(e\mu^{-1/3})^{1/2}$, for which the gaps between r_i and r_{i+1} are smaller than $\zeta e\mu^{-1/3}$. The existence of such a sequence follows from Proposition 34.

The last claim of Theorem 46 follows from [10], [11]. ■

9. Concluding remarks, future work

In this paper we have shown that the chaotic dynamics observed for the planar restricted circular three body problem survives the perturbation into the planar restricted elliptic three body problem, when its eccentricity is sufficiently small. We have also shown that this dynamics is extended to include diffusion in energy. The diffusion proved in this paper covers a small range of energies. This is due to the fact that in our argument we use a Melnikov type method which does not allow us to jump between the "large gaps" between the KAM tori. An interesting problem which could be addressed is whether these large gaps can be overcome (this potentially could be done using techniques similar to [9] or [12]).

Our result holds only for a specific family $\{\mu_k\}$ of masses of the primaries. The choice of these masses is such that they ensure the existence of the homoclinic orbit to $L_2^{\mu_k}$, which is then used for the Melnikov argument. An interesting question is whether one can observe similar dynamics in real life setting, say in the Jupiter-Sun system. In such a case we will no longer have a homoclinic connection for the point L_2^μ . For the (circular) Jupiter-Sun system though we know that we have a transversal homoclinic connection for Lyapounov orbits (see [17] and [28]). Such orbits could be used for a similar construction. Our argument also required that we have sufficiently small eccentricities. It would be interesting to find out if the dynamics persists for the actual eccentricity of the Jupiter-Sun system. For this problem it is quite likely that applying the mechanism discussed in this paper would be very hard. Our argument relies on the use of the KAM theorem, which works for sufficiently small perturbations. To apply it for an explicit eccentricity seems a difficult task. Other methods could be exploited though. Instead of proving the persistence of the tori and trying to detect intersections of their invariant manifolds, one could focus on detection of symbolic dynamics for the diffusing orbits in the spirit of [28]. This seems a far more realistic target for the near future and is being currently considered as an extension of this work.

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11. Appendix

11.1. Splitting of manifolds associated to Lyapounov orbits of the PRC3BP

Here we investigate the dependence on parameter μ of the splitting of the intersection of curves from Theorem 4. Let $\mu = \mu_k$, which means that in (12) $\alpha = 0$. The transversal intersection of the curves is on $\{\dot{x} = 0\}$. The curve intersects $\{\dot{x} = 0\}$ for $\sigma = \sigma_0$ for which

$$(K_1 \cos \tau \cos \sigma_0 - K_2 \sin \tau \sin \sigma_0) = O(\mu^{4/3}). \quad (109)$$

This is because from (12) only then can we have $M_f(\sigma_0)$ sufficiently close to zero so that $\dot{x} = 0$. In particular, (12) and (109) gives

$$M_f(\sigma_0) = O(\mu^{2/3}). \quad (110)$$

By implicit function theorem we know that

$$\begin{aligned} \gamma(\mu, \sqrt{\Delta C}) &:= \frac{\partial M}{\partial \sigma}(\sigma_0) \\ &= -\frac{\mu^{-2/3} \sqrt{\Delta C} 3M (N + 2M \cos \tau)^{-1} (-K_1 \cos \tau \sin \sigma_0 - K_2 \sin \tau \cos \sigma_0)}{N + 2M \cos(M_f(\sigma_0))} \\ &\in \mu^{-2/3} \sqrt{\Delta C} [a_1, a_2], \end{aligned} \quad (111)$$

where a_1, a_2 are constants, $0 \notin [a_1, a_2]$ (for sufficiently small μ and ΔC these constants are arbitrarily close to one another. For their precise values one would need to substitute the constants M, τ, K_1, K_2, N from [20] into (111)). By (109), (110), (111)

$$\begin{aligned} &\frac{\partial x}{\partial \sigma}(\sigma_0) \\ &= \sqrt{\Delta C} (N + 2M \cos \tau)^{-1} N \left(-\sin(M_f(\sigma_0)) \gamma(\mu, \sqrt{\Delta C}) \right) \\ &\cdot (K_1 \cos \tau \cos \sigma_0 - K_2 \sin \tau \sin \sigma_0) \\ &+ \sqrt{\Delta C} (N + 2M \cos \tau)^{-1} (2M + N \cos M_f(\sigma_0)) (-K_1 \cos \tau \sin \sigma_0 - K_2 \sin \tau \cos \sigma_0) \\ &+ \mu^{1/3} M \sin(M_f(\sigma_0)) \gamma(\mu, \sqrt{\Delta C}) \\ &+ \mu^{2/3} \left\{ -\frac{2MN}{3} \sin(M_f(\sigma_0)) \gamma(\mu, \sqrt{\Delta C}) + M^2 2 \sin(M_f(\sigma_0)) \cos(M_f(\sigma_0)) \gamma(\mu, \sqrt{\Delta C}) \right\} \\ &\in \sqrt{\Delta C} [b_1, b_2], \end{aligned}$$

where b_1, b_2 are constants, $0 \notin [b_1, b_2]$ (the second term in the above equation plays a dominant role). Also

$$\begin{aligned} \frac{\partial \dot{x}}{\partial \sigma}(\sigma_0) &= \cos(M_f(\sigma_0)) \gamma(\mu, \sqrt{\Delta C}) \left[\frac{\sqrt{\Delta C} N (K_1 \cos \tau \cos \sigma_0 - K_2 \sin \tau \sin \sigma_0)}{N + 2M \cos \tau} \right. \\ &\quad \left. + \mu^{1/3} M + \mu^{2/3} \left\{ \frac{MN}{3} + 2M^2 \cos M_f + \frac{M}{3} \alpha \right\} \right] \\ &\quad + \sin(M_f(\sigma_0)) [\sqrt{\Delta C} N (N + 2M \cos \tau)^{-1} (-K_1 \cos \tau \sin \sigma_0 - K_2 \sin \tau \cos \sigma_0) \\ &\quad - 2M^2 \sin(M_f(\sigma_0)) \gamma(\mu, \sqrt{\Delta C})] \\ &\in \mu^{-1/3} \sqrt{\Delta C} [c_1, c_2], \end{aligned}$$

where c_1, c_2 are constants, $0 \notin [c_1, c_2]$. This means that

$$\frac{\partial x}{\partial \sigma}(\sigma_0) / \frac{\partial \dot{x}}{\partial \sigma}(\sigma_0) \in \mu^{1/3} [d_1, d_2], \quad (112)$$

for constant d_1, d_2 , $0 \notin [d_1, d_2]$. Stable/unstable manifolds are S symmetric (see (8) and Theorem 4), hence for a curve coming from the intersection of the stable manifold with $\{y = 0\}$ we shall have same estimates for $\frac{\partial x}{\partial \sigma}$, and estimates with reversed sign for $\frac{\partial \dot{x}}{\partial \sigma}$. This by (112) gives a splitting of the two manifolds with angle of order $\mu^{1/3}$.

11.2. Derivation of equations for the PRE3BP in rotating coordinates

Here we derive the Hamiltonian (22) of the PRE3BP in rotating coordinate system. Let $R(\varphi)$ be the rotation by the angle φ

$$R(\varphi) = \begin{bmatrix} \cos \varphi & -\sin \varphi \\ \sin \varphi & \cos \varphi \end{bmatrix}.$$

The ellipse which is obtained when solving the 2-body problem is given by (see (21))

$$\begin{aligned} r(t) &= \frac{1 - e^2}{1 + e \cos \psi(t)} = 1 - e \cos \psi(t) + O(e^2), \\ z(t) &= r(t) R(\psi(t)) \cdot [1, 0]^T, \\ \psi(t) &= t + 2e \sin t + O(e^2). \end{aligned} \quad (113)$$

The primary with the mass μ (the planet) has the following location $z_p(t) = (\mu - 1)z(t)$, while the primary mass $1 - \mu$ (the Sun) is located at the point $z_s(t) = \mu z(t)$.

We shall now compute the distances between the comet and the primaries in rotating coordinates. Let $r_1(t)$ be the square of the distance between the comet and the Sun, and $r_2(t)$ between the comet and the planet. Let $(x(t), y(t))$ denote the 'rotating' coordinates of the comet and $q(t)$ be the position of the comet in the 'static' coordinate frame $(x(t), y(t)) = R(-t)q(t)$. Using the fact the the length of the vector is not changed

by the rotation and (113) we have

$$\begin{aligned}
 r_1(t)^2 &= \left\| R(t) \cdot (x, y)^T - \mu r(t) R(\psi(t)) \cdot [1, 0]^T \right\|^2 \\
 &= \left\| (x, y)^T - \mu r(t) R(\psi(t) - t) \cdot [1, 0]^T \right\|^2 \\
 &= x^2 + y^2 - 2\mu r(t) (x \cos(\psi(t) - t) + y \sin(\psi(t) - t)) + \mu^2 r(t)^2 \\
 &= x^2 + y^2 - 2\mu (x \cos(\psi(t) - t) + y \sin(\psi(t) - t)) + \mu^2 + \\
 &\quad 2\mu e \cos \psi(t) (x \cos(\psi(t) - t) + y \sin \psi(t) - t) - 2\mu^2 e \cos \psi(t) + O(\mu e^2).
 \end{aligned}$$

Using

$$\begin{aligned}
 \psi(t) &= t + 2e \sin t + O(e^2), \\
 \cos(\psi(t)) &= \cos t - 2e \sin^2 t + O(e^2), \\
 \cos(\psi(t) - t) &= 1 + O(e^2), \\
 \sin(\psi(t) - t) &= 2e \sin t + O(e^2),
 \end{aligned}$$

$$\begin{aligned}
 x \cos(\psi(t) - t) + y \sin(\psi(t) - t) &= x + 2ey \sin t + O(e^2), \\
 \cos(\psi(t)) (x \cos(\psi(t) - t) + y \sin(\psi(t) - t)) &= x \cos t + O(e),
 \end{aligned}$$

in the expression for $r_1(t)$ we obtain

$$r_1(t)^2 = (x - \mu)^2 + y^2 + 2e\bar{g}(\mu, x, y, t) + O(\mu e^2), \quad (114)$$

where \bar{g} is given in (24). Expression for $r_2(t)$ is obtained from (114) with the substitution $\mu \mapsto (\mu - 1)$. Observe that

$$\frac{1}{\sqrt{r^2 + c}} = \frac{1}{r\sqrt{1 + \frac{c}{r^2}}} = \frac{1}{r} \left(1 - \frac{c}{2r^2} + O\left(\left(\frac{c}{r^2}\right)^2\right) \right). \quad (115)$$

We shall use notation r_1, r_2 from (4), with $r_1 > \delta$ and $r_2 > \mu^{1/3}\delta$. Note that to apply (115) for $r = r_2$ we need to have $e\bar{g}(\mu - 1, x, y, t) + O(\mu e^2) < r_2^2$, which means that we need to take e sufficiently small so that $e\mu^{-2/3} < \kappa$ for sufficiently small κ . Equations (114), (115) give

$$\frac{1}{r_1(t)} = \frac{1}{r_1} - \frac{e\bar{g}(\mu, x, y, t)}{r_1^3} + O(\mu^2 e^2), \quad (116)$$

$$\frac{1}{r_2(t)} = \frac{1}{r_2} - \frac{e\bar{g}(\mu - 1, x, y, t)}{r_2^3} + O(e^2 \mu^{-5/3}). \quad (117)$$

Substituting (116), (117) into (1) gives (22).

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