

Existence of a Center Manifold in a Practical Domain around L_1 in the Restricted Three-Body Problem*

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Abstract. We present a method of proving existence of center manifolds within specified domains. The method is based on a combination of topological tools, normal forms, and rigorous computer-assisted computations. We apply our method to obtain a proof of a center manifold in an explicit region around the equilibrium point L_1 in the Earth–Sun planar restricted circular three-body problem.

Key words. center manifolds, normal forms, celestial mechanics, restricted three-body problem, covering relations, cone conditions

AMS subject classifications. 37D10, 37G05, 37N05, 34C20, 34C45, 70F07, 70F15, 70K45

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1. Introduction. In this paper we give a method for proving existence of center manifolds for systems with an integral of motion. The aim of the paper is not to give yet another proof of the center manifold theorem, but to provide a practical tool which can be applied to nontrivial systems. There are a number of advantages to the method. First, the method is not perturbative. We thus do not need to start with an invariant manifold and then perturb it. All that is required is a good numerical approximation of the position of a center manifold. The conditions required in order to ensure existence of the manifold in the vicinity of the numerical approximation are such that it is possible to verify them using (rigorous, interval-based) computer-assisted computations. This is another advantage, since it allows for application to problems which cannot be treated analytically. The method gives explicit bounds on the position and on the size of the manifold. Moreover, under appropriate assumptions we can also prove that the manifold is unique.

Our proof of existence of the center manifold is performed using purely topological arguments. This means that it can be applied to treat nonanalytic invariant manifolds. The main disadvantage of using topological tools, though, is that the proof ensures only Lipschitz continuity of the manifold even for manifolds with higher order regularity.

To apply the method one needs to have a good numerical approximation of the position

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of the center manifold. In our application we obtain such a numerical approximation using a normal form around a fixed point. For us the normal form plays a role of a suitable change of coordinates, which “straightens out” the center manifold. In our approach we are not concerned with convergence of the normal form. We are also not concerned with estimates of the high order terms in our normal form expansion. As long as a truncated change of coordinates to normal form gives an accurate enough approximation of the position of the manifold, it can be combined with our topological method to give a proof of existence of the manifold.

To demonstrate that the proposed method is indeed applicable, we use it to prove the existence of a center manifold around a fixed point L_1 in the Earth–Sun planar restricted three-body problem (RTBP). The system is two degrees of freedom Hamiltonian, and L_1 is a saddle \times center fixed point. Existence of the center manifold around L_1 is well known and extensively investigated numerically. Sufficiently close to L_1 it is a straightforward consequence of the Lyapunov–Moser theorem. It is also well known that the center manifold around L_1 is foliated by periodic orbits. In our proof, though, we neither make use of nor prove this fact. Our objective is to apply our general method, for which such a particular structure is not required. The aim is to prove existence of the manifold over an explicitly given domain. The size of the manifold that we prove can be considered as “large” (see Figures 5 and 6). It is well known from numerical evidence that the manifold is by far larger than this. Nevertheless, to the best of our knowledge, this is the first rigorous proof of its existence over a given range of nonnegligible size.

We emphasize that the method is much more general than the considered example. It is not restricted to two degrees of freedom systems and works in arbitrary dimension. Even though we apply the method to a well-known and relatively simple system, it can be applied to situations for which there is no previous proof. All that is required for our method is an integral of motion and a fixed point. Around the fixed point one can perform a change of coordinates to a normal form and use it to validate the assumptions of our topological theorem.

The paper is organized as follows. In section 2 we give the setup of the problem and state our main theorem (Theorem 2.4). Assumptions of the theorem are based on estimates on the derivatives of the vector field within the investigated region. Based on these the existence of an invariant manifold is established. In section 3 we give a topological proof of the existence of an invariant manifold for maps with saddle-center-type properties. In section 4 we use the result obtained for maps to prove Theorem 2.4. In section 5 we apply our Theorem 2.4 to prove the existence of a center manifold around an equilibrium point L_1 in the RTBP. To do so we first introduce the problem and present a procedure of transforming the system into a normal form. We then discuss how normal forms provide very accurate approximations of center manifolds. Finally, we combine Theorem 2.4 and normal forms with rigorous interval-arithmetic-based computer-assisted computations to prove the existence of the manifold. Section 6 contains concluding remarks and an outline of future work.

2. Setup. We will consider the following problem. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and

$$(2.1) \quad \mathbf{x}' = F(\mathbf{x})$$

be an ODE (we impose the usual assumptions implying existence and uniqueness of solutions) with a fixed point \mathbf{x}_0 and an integral of motion $H : \mathbb{R}^n \rightarrow \mathbb{R}$. By this we mean that for any solution $q(t)$ of (2.1) we have

$$(2.2) \quad H(q(t)) = c,$$

where c is some constant dependent on the initial condition $q(0)$. Since in our applications we shall deal with the RTBP, which is a Hamiltonian system where H is the Hamiltonian, we shall refer to H as the energy from now on. We shall use the notation $\Phi(t, \mathbf{x})$ for the flow induced by (2.1).

2.1. Well-aligned coordinates. We will investigate the dynamics of (2.1) in some compact set D , contained in an open subset U of \mathbb{R}^n , such that the fixed point $\mathbf{x}_0 \in D$, and whose image by a diffeomorphism

$$(2.3) \quad \phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$$

is

$$(2.4) \quad \phi(D) = D_\phi = \bar{B}_c^R \times \bar{B}_u^r \times \bar{B}_s^r,$$

where \bar{B}_i^r (for $i \in \{c, u, s\}$) stand for i -dimensional closed balls around zero of radius r . We assume that $n = c + u + s$. We will refer to $p = \phi(\mathbf{x})$ as the *aligned coordinates*. In these coordinates we will use the notation $p = (\theta, x, y)$ with $\theta \in \bar{B}_c^R$, $x \in \bar{B}_u^r$, and $y \in \bar{B}_s^r$. We will refer to θ as the central coordinate, to x as the unstable coordinate, and to y as the stable coordinate (the subscripts c, u, s standing for central, unstable, and stable, respectively).

The motivation behind the above setup is the following. We will search for a center manifold of (2.1) homeomorphic to a c -dimensional disc inside the set D . Such manifolds have associated stable and unstable vector bundles, which in the coordinate system ϕ are given approximately by the coordinates of the balls \bar{B}_s^r and \bar{B}_u^r , respectively. We do not assume though that the coordinates x and y align exactly with directions of hyperbolic expansion and contraction. It will turn out that it is enough that they point roughly in these directions. The remaining coordinates θ are the central coordinates of our system. We need to have a good approximation of where the center manifold is. This approximation is given by $\phi^{-1}(\bar{B}_c^R \times \{0\}) \subset \mathbb{R}^n$. The change of coordinates ϕ can be obtained from some nonrigorous numerical computation (in our application for the RTBP, normal forms). It is important to emphasize that we will not assume that $\phi^{-1}(\bar{B}_c^R \times \{0\})$ is invariant under the flow (2.1). Allowing for errors, we expect the true manifold to lie in $\phi^{-1}(\bar{B}_c^R \times \bar{B}_u^r \times \bar{B}_s^r)$. This means that we take an enclosure of radius r of our initial guess and look for the invariant manifold in this enclosure.

We will search for the part of the center manifold with energy $H \leq h$ for some $h \in \mathbb{R}$. We assume that the center coordinate is well aligned with the energy H in the sense that we have

$$(2.5) \quad H(\phi^{-1}(\bar{B}_c^{R-v} \times \bar{B}_u^r \times \bar{B}_s^r)) < h < H(\phi^{-1}(\partial\bar{B}_c^R \times \bar{B}_u^r \times \bar{B}_s^r))$$

for some $v > 0$ (here we use the notation ∂A to denote the boundary of a set A).

Our detection of the center manifold in the RTBP is going to be carried out in two stages. First we obtain ϕ as a change of coordinates into a normal form, after which we shall employ our topological theorem (Theorem 2.4) to prove the existence of the manifold.

2.2. Local bounds on the vector field and the statement of the main result. We are now ready to state the main assumptions needed for our method. These will be expressed in terms of local bounds on the derivative of the vector field (2.1). First, let us introduce a notation F^ϕ for the vector field in the aligned coordinates, i.e.,

$$(2.6) \quad F^\phi(p) = D\phi(\phi^{-1}(p))F(\phi^{-1}(p)),$$

and a notation $[dF^\phi(N)]$ for an interval enclosure of the derivative on a set $N \subset D_\phi$,

$$[dF^\phi(N)] = \left\{ A \in \mathbb{R}^{n \times n} \mid A_{ij} \in \left[\inf_{p \in N} \frac{dF_i^\phi}{dp_j}, \sup_{p \in N} \frac{dF_i^\phi}{dp_j} \right] \text{ for all } i, j = 1, \dots, n \right\}.$$

For any point $p = (\theta, 0, 0)$ from $\bar{B}_c^R \times \{0\} \times \{0\}$ we define a set

$$(2.7) \quad N_p := \bar{B}_c(\theta, \rho) \times \bar{B}_u^r \times \bar{B}_s^r \cap D_\phi,$$

where $\bar{B}_c(\theta, \rho)$ is a c -dimensional ball of radius $\rho > 0$ centered at θ . We introduce the following notation for the bound on the derivatives of F^ϕ on the sets N_p :

$$(2.8) \quad [dF^\phi(N_p)] \subset \begin{pmatrix} \mathbf{C} & \varepsilon_c & \varepsilon_c \\ \varepsilon_m & \mathbf{A} & \varepsilon_u \\ \varepsilon_m & \varepsilon_s & \mathbf{B} \end{pmatrix}.$$

Here \mathbf{A} , \mathbf{B} , \mathbf{C} , ε_c , ε_m , ε_s , and ε_u are interval matrices, that is, matrices with interval coefficients. Here we slightly abuse notation since the pairs of matrices ε_c and ε_m need not be equal; they even have different dimension when $u \neq s$. We use the same notation since later we shall assume uniform bounds for both of matrices ε_c and both of ε_m . Let us also note that the bounds \mathbf{A} , \mathbf{B} , \mathbf{C} , ε_c , ε_m , ε_s , and ε_u may be different for different p . We do not indicate this in our notation to keep it relatively simple.

Remark 2.1. *If the system possesses a center manifold and the adjusted coordinates are well aligned in the sense of section 2.1, then the interval matrices ε_i in (2.8), with $i \in \{c, m, s, u\}$, should turn out to be small. The matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are the bounds on derivatives of the vector field in the unstable, stable, and central directions, respectively. If the alignment of our coordinates is correct, then we expect the contraction/expansion rates associated with \mathbf{C} to be weaker than for \mathbf{A} and \mathbf{B} .*

We will use the following notation to express our assumptions about $[dF^\phi(N_p)]$. Let $\delta^u, \delta^s, c^u, c^s, \varepsilon_i > 0$ denote contraction/expansion rates, such that for any matrix $A \in \mathbf{A}$, $B \in \mathbf{B}$, $e_i \in \varepsilon_i$ for $i \in \{m, c, u, s\}$, we have

$$(2.9) \quad \inf\{x^T Ax : \|x\| = 1\} > \delta^u,$$

$$(2.10) \quad \sup\{y^T By : \|y\| = 1\} < -\delta^s,$$

$$(2.11) \quad c^s < \inf\{\theta^T C\theta : \|\theta\| = 1\} \leq \sup\{\theta^T C\theta : \|\theta\| = 1\} < c^u,$$

$$(2.12) \quad \|e_i\| < \varepsilon_i \quad \text{for } i \in \{m, c, u, s\},$$

where $\|\cdot\|$ is the standard Euclidean norm (throughout the paper we shall use no other norms). Once again, $\varepsilon_i, c^s, c^u, \mu, \delta^u$, and δ^s can depend on p .

Let $\gamma, \alpha_h, \alpha_v, \beta_h, \beta_v > 0$ be constants such that

$$(2.13) \quad \alpha_h > \alpha_v \quad \text{and} \quad \beta_v > \beta_h$$

and such that the radius ρ considered for the central part of the sets N_p satisfies

$$(2.14) \quad \rho > r \sqrt{\frac{\alpha_h}{\gamma}}, \quad \rho > r \sqrt{\frac{\beta_v}{\gamma}},$$

where r is the radius of the balls \bar{B}_u^r and \bar{B}_s^r in (2.4). Let us define the following constants:

$$(2.15) \quad \begin{aligned} \kappa_c^{\text{forw}} &:= c^u + \frac{1}{2} \left(\frac{\alpha_h}{\gamma} \varepsilon_m + \frac{\beta_h}{\gamma} \varepsilon_m + 2\varepsilon_c \right), \\ \kappa_u^{\text{forw}} &:= \delta^u - \frac{1}{2} \left(\varepsilon_m + \varepsilon_u + \frac{\gamma}{\alpha_h} \varepsilon_c + \frac{\beta_h}{\alpha_h} \varepsilon_s \right), \\ \kappa_s^{\text{forw}} &:= -\delta^s + \frac{1}{2} \left(\varepsilon_m + \frac{\alpha_h}{\beta_h} \varepsilon_u + \frac{\gamma}{\beta_h} \varepsilon_c + \varepsilon_s \right), \end{aligned}$$

$$(2.16) \quad \begin{aligned} \kappa_c^{\text{back}} &:= c^s - \frac{1}{2} \left(\varepsilon_m \frac{\alpha_v}{\gamma} + \varepsilon_m \frac{\beta_v}{\gamma} + 2\varepsilon_c \right), \\ \kappa_u^{\text{back}} &:= \delta^u - \frac{1}{2} \left(\varepsilon_m + \varepsilon_u + \frac{\gamma}{\alpha_v} \varepsilon_c + \frac{\beta_v}{\alpha_v} \varepsilon_s \right), \\ \kappa_s^{\text{back}} &:= -\delta^s + \frac{1}{2} \left(\varepsilon_m + \frac{\alpha_v}{\beta_v} \varepsilon_u + \frac{\gamma}{\beta_v} \varepsilon_c + \varepsilon_s \right). \end{aligned}$$

The superscripts “forw” and “back” in the above constants come from the fact that they shall be associated with estimates on the dynamics induced by the vector field (2.6) for forward and backward evolution in time, respectively. At this stage the subscripts v and h in constants α and β do not have an intuitive meaning. During the course of the proof they shall be associated with horizontal and vertical slopes of constructed invariant manifolds (hence h for “horizontal” and v for “vertical”), and then their meaning will become more natural.

Remark 2.2. *Even though coefficients (2.15), (2.16) are technical in nature, they have a quite natural interpretation in terms of the dynamics of the system. The estimates $\kappa_c^i, \kappa_u^i, \kappa_s^i$ for $i \in \{\text{forw}, \text{back}\}$ are essentially estimates of the contraction/expansion rates associated with the center, unstable, and stable coordinates, respectively. These estimates take into account errors ε_i for $i \in \{s, u, c, m\}$ in the setup of coordinates. Note that when our coordinates are perfectly aligned with the dynamics, then $\varepsilon_i = 0$ for $i \in \{s, u, c, m\}$, and in turn*

$$\kappa_s^{\text{forw}} = \kappa_s^{\text{back}} = -\delta^s, \quad \kappa_c^{\text{back}} = c^s, \quad \kappa_c^{\text{forw}} = c^u, \quad \kappa_u^{\text{forw}} = \kappa_u^{\text{back}} = \delta^u,$$

which are the bounds on the derivative of the vector field in the unstable, stable, and center directions given in (2.9), (2.10), (2.11). The key assumptions of Theorem 2.4 are (2.17) and (2.18). In particular, these assumptions imply

$$\kappa_s^{\text{back}} < \kappa_c^{\text{back}}, \quad \kappa_c^{\text{forw}} < \kappa_u^{\text{forw}},$$

which is equivalent to assuming that the dynamics in the center coordinate is weaker than the dynamics in the stable and unstable directions. These are classical assumptions for center manifold theorems (see [12], for instance).

Remark 2.3. We have a certain freedom in our choice of the constants γ , α_h , α_v , β_h , β_v . They offer flexibility when verifying assumptions of Theorem 2.4. During the course of the proof of Theorem 2.4 it will turn out that they also give Lipschitz-type bounds $L_c = \sqrt{\frac{2\gamma}{\min(\alpha_h - \alpha_v, \beta_v - \beta_h)}}$, $L_s = \sqrt{\frac{1}{\alpha_h} \max(\gamma, \beta_h)}$, $L_u = \sqrt{\frac{1}{\beta_v} \max(\gamma, \alpha_v)}$ for our center, stable, and unstable manifolds, respectively (for more details see Corollary 4.4).

We are now ready to state our main tool for detection of center manifolds.

Theorem 2.4 (main theorem). Let $h \in \mathbb{R}$. Assume that (2.5) holds for some $v > 0$. Assume also that for any $p \in \bar{B}_c^R \times \{0\} \times \{0\}$, for the constants κ_c^{forw} , κ_u^{forw} , κ_s^{forw} , κ_c^{back} , κ_u^{back} , κ_s^{back} , ε_u , ε_s , δ^u , δ^s computed on a set N_p (defined by (2.7)) the inequalities

$$(2.17) \quad \kappa_c^{forw}, \kappa_s^{forw} < \kappa_u^{forw}, \quad 0 < \kappa_u^{forw},$$

$$(2.18) \quad \kappa_s^{back} < 0, \quad \kappa_s^{back} < \kappa_c^{back}, \kappa_u^{back}$$

hold and also that there exist $E_u, E_s > 0$ such that for any $q \in N_p \cap (\bar{B}_c^R \times \{0\} \times \{0\})$

$$(2.19) \quad \|\pi_x F^\phi(q)\| < rE_u, \quad \|\pi_y F^\phi(q)\| < rE_s,$$

and

$$(2.20) \quad E_u + \varepsilon_u < \delta^u,$$

$$(2.21) \quad E_s + \varepsilon_s < \delta^s.$$

If the above assumptions hold, then there exists a C^0 function

$$\chi : \bar{B}_c^{R-v} \rightarrow D_\phi$$

such that the following hold:

1. For any $\theta \in \bar{B}_c^{R-v}$ we have $\pi_\theta \chi(\theta) = \theta$ and

$$\Phi(t, \phi^{-1}(\chi(\theta))) \in D \quad \text{for all } t \in \mathbb{R}.$$

2. If for some $\mathbf{x} \in \phi^{-1}(\bar{B}_c^{R-v} \times \bar{B}_u^r \times \bar{B}_s^r)$ we have

$$\Phi(t, \mathbf{x}) \in D \quad \text{for all } t \in \mathbb{R},$$

then there exists a $\theta \in \bar{B}_c^{R-v}$ such that $\mathbf{x} = \phi^{-1}(\chi(\theta))$.

In subsequent sections we shall present a proof of this theorem, building up auxiliary results along the way. Before we move on to these results let us make a couple of comments on the result.

Remark 2.5. Theorem 2.4 establishes uniqueness of the invariant manifold. This is not a typical scenario in the case of center manifolds, which are usually not unique. Uniqueness in our case follows from condition (2.5), which by our construction will ensure that for any point from our center manifold a trajectory starting from it cannot leave the set D . This means

that dynamics on the center manifold with $H \leq h$ is contained in a compact set. This is the underlying reason that allows us to obtain uniqueness.

In the case of the planar RTBP the uniqueness of the manifold could also be shown by proving that the manifold is foliated by periodic orbits. Our method is more general and can also similarly be applied to more complicated systems (say, to the spatial RTBP where the fixed point is a saddle \times center \times center) where we do not have such foliation.

Remark 2.6. The main strength of our result lies in the fact that it allows us to easily obtain explicit bounds for the position and size of the manifold. The center manifold is contained in $D = \phi^{-1}(\bar{B}_c^R \times \bar{B}_u^r \times \bar{B}_s^r)$. Since the manifold is a graph of χ , from point 1 of Theorem 2.4 we know that it is of the form $\phi^{-1}\{(\theta, \pi_{x,y}\chi(\theta)) \mid \theta \in \bar{B}_c^{R-v}\}$, which ensures that it “fills in” the set D nontrivially. In contrast, the classical center manifold theorem does not provide such explicit bounds.

Remark 2.7. Since the method uses topological arguments only, it can be applied to manifolds with limited differentiability. This makes it more general compared to standard methods such as the “method of majorants” [18], where analyticity is required.

Remark 2.8. It is important to remark that our result only establishes continuity (together with Lipschitz type conditions) of the center manifold. The center manifold theorem clearly indicates that in a sufficiently small neighborhood of a saddle-center fixed point we should have higher order smoothness. We believe, though, that assumptions similar in spirit to those of Theorem 2.4 will imply higher order smoothness. This will be the subject of forthcoming work. The result obtained so far should be regarded as a first step toward this end.

In our application for the RTBP, in a neighborhood sufficiently close to the equilibrium point, our manifold shall inherit all regularity, which follows from the center manifold theorem (see Remark 5.1).

Let us finish the section with a final comment. In order to verify the assumptions of Theorem 2.4 it is sufficient to consider some finite covering $\{\bigcup_{i \in I} U_i\}$ of the set D_ϕ and to verify bounds on local derivatives on sets U_i . It is not necessary to consider an infinite number of points p and their associated sets N_p , as long as for any $p \in \bar{B}_c^R \times \{0\} \times \{0\}$ we have $N_p \subset U_i$ for some $i \in I$. This makes the assumptions of Theorem 2.4 verifiable in practice using rigorous computer-assisted tools.

3. Topological approach to center manifolds for maps. In this section we will state some preliminary results, which will next be used for the proof of Theorem 2.4 in section 4. The results will be stated for maps instead of flows. In section 4 we will take a time shift along a trajectory map for the flow generated by (2.1) and apply the results to it. The main result of this section is Theorem 3.7. The result is in the spirit of versions of normally hyperbolic invariant manifold theorems obtained in [5], [8], and [7]. The main difference is that we do not deal with a normally hyperbolic manifold without boundary, but with a selected part of a center manifold (homeomorphic to a disc) with a boundary. In this section the fact that the dynamics does not diffuse through the boundary along the center coordinate is imposed by assumption. This assumption will later follow from assuming that (2.2), (2.5) hold for (2.1).

We now give the setup for maps. Let $D \subset U \subset \mathbb{R}^n$, the change of coordinates $\phi : U \rightarrow \phi(U)$, and $D_\phi = \phi(D)$ be as in section 2.1. We consider a dynamical system given by a smooth invertible map $f : U \rightarrow U$. In adjusted coordinates we denote the map as $f_\phi := \phi \circ f \circ \phi^{-1}$,

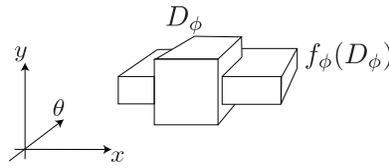


Figure 1. A map f which satisfies covering conditions. The set D_ϕ is contracted in coordinate y and expanded in coordinate x . Note that in the θ coordinate the set may be simply shifted, expanded, or contracted, just as long as conditions (3.3)–(3.7) are satisfied.

$f_\phi : \phi(U) \rightarrow \mathbb{R}^n$. We assume that

$$(3.1) \quad H(p) = H(f(p))$$

for all $p \in D_\phi$ and also that for some $v > 0$ condition (2.5) holds.

We introduce the following sets:

$$(3.2) \quad \begin{aligned} D_\phi^- &= \bar{B}_c^R \times \partial \bar{B}_u^r \times \bar{B}_s^r, \\ D_\phi^+ &= \bar{B}_c^R \times \bar{B}_u^r \times \partial \bar{B}_s^r. \end{aligned}$$

We now introduce a number of definitions. The first is a definition of a covering relation.

Definition 3.1. We say that a map $f : U \rightarrow U$ satisfies covering conditions in D if

$$(3.3) \quad \pi_x(f_\phi(D_\phi^-)) \cap \bar{B}_u^r = \emptyset,$$

$$(3.4) \quad \pi_y(f_\phi^{-1}(D_\phi^+)) \cap \bar{B}_s^r = \emptyset,$$

$$(3.5) \quad \pi_y(f_\phi(D_\phi)) \cap (\mathbb{R}^s \setminus \bar{B}_s^r) = \emptyset,$$

$$(3.6) \quad \pi_x(f_\phi^{-1}(D_\phi)) \cap (\mathbb{R}^u \setminus \bar{B}_u^r) = \emptyset,$$

and for any point $p \in \bar{B}_c^R \times \{0\}$,

$$(3.7) \quad \pi_{(x,y)} f_\phi(p), \pi_{(x,y)} f_\phi^{-1}(p) \in \text{int}(\bar{B}_u^r \times \bar{B}_s^r).$$

Conditions (3.5) and (3.4) mean that, in the y (stable) projection, f_ϕ contracts the set D_ϕ strictly inside \bar{B}_s^r (see Figure 1). Conditions (3.6) and (3.3) mean that, in the x (unstable) projection, f_ϕ expands the set D_ϕ strictly outside \bar{B}_u^r . The final assumption (3.7) is needed to ensure that the image of D_ϕ by f_ϕ intersects D_ϕ . Without assumption (3.7), all other assumptions (3.3)–(3.6) could easily follow from having the image of D disjoint with D .

Covering relations are tools which can be used to ensure existence of an invariant set in D . To prove that this set is a manifold we shall need additional assumptions. These shall be expressed by “cone conditions.” To introduce these conditions, first we need some notation.

Let $Q_h, Q_v : \mathbb{R}^c \times \mathbb{R}^s \times \mathbb{R}^u \rightarrow \mathbb{R}$ be functions defined by

$$(3.8) \quad \begin{aligned} Q_h(\theta, x, y) &= -\gamma \|\theta\|^2 + \alpha_h \|x\|^2 - \beta_h \|y\|^2, \\ Q_v(\theta, x, y) &= -\gamma \|\theta\|^2 - \alpha_v \|x\|^2 + \beta_v \|y\|^2, \end{aligned}$$

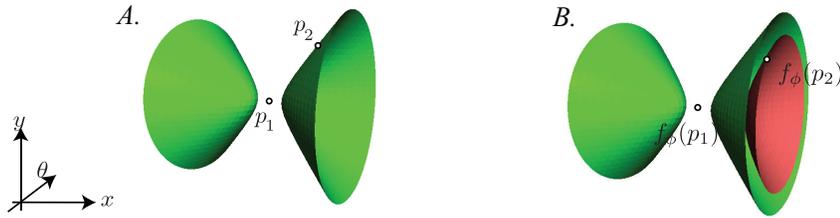


Figure 2. An example of a function f , which satisfies cone conditions: A. Two points p_1, p_2 for which $Q_h(p_1 - p_2) = c > 0$. B. Difference of the images of the points lies on a cone $Q_h(f_\phi(p_1) - f_\phi(p_2)) > mc$. A similar condition (but with reversed roles of the x and y coordinates) needs to hold for the inverse map.

with $\gamma, \alpha_h, \alpha_v, \beta_h, \beta_v > 0$, and

$$(3.9) \quad \alpha_h > \alpha_v \quad \text{and} \quad \beta_v > \beta_h.$$

Definition 3.2. We say that a map $f : U \rightarrow U$ satisfies cone conditions in D if there exists an $m > 1$ such that the following hold:

1. For any two points $p_1, p_2 \in D_\phi$ satisfying $p_1 \neq p_2$ and $Q_h(p_1 - p_2) \geq 0$ we have

$$(3.10) \quad Q_h(f_\phi(p_1) - f_\phi(p_2)) > mQ_h(p_1 - p_2).$$

2. For any two points $p_1, p_2 \in D_\phi$ satisfying $p_1 \neq p_2$ and $Q_v(p_1 - p_2) \geq 0$ we have

$$(3.11) \quad Q_v(f_\phi^{-1}(p_1) - f_\phi^{-1}(p_2)) > mQ_v(p_1 - p_2).$$

Definition 3.2 intuitively states that if we have two points that lie horizontally with respect to each other, then their images are going to be pulled apart in the horizontal, x coordinate (see Figure 2). If, on the other hand, we have two points that lie vertically with respect to each other, then their preimages are going to be pulled apart in the vertical, y coordinate.

We now give definitions of horizontal discs and vertical discs. These will be building blocks in our construction of invariant manifolds.

Definition 3.3. We say that a continuous monomorphism $\mathbf{h} : \bar{B}_u^r \rightarrow D_\phi$ is a horizontal disc if $\pi_x \mathbf{h}(x) = x$ and for any $x_1, x_2 \in \bar{B}_u^r$

$$(3.12) \quad Q_h(\mathbf{h}(x_1) - \mathbf{h}(x_2)) \geq 0.$$

Thus, to any point x in the graph $\mathbf{h}(x)$ we can attach a horizontal cone, so that the graph always remains entirely inside the cone (see Figure 3).

Definition 3.4. We say that a continuous monomorphism $\mathbf{v} : \bar{B}_s^r \rightarrow D_\phi$ is a vertical disc if $\pi_y \mathbf{v}(y) = y$ and for any $y_1, y_2 \in \bar{B}_s^r$

$$Q_v(\mathbf{v}(y_1) - \mathbf{v}(y_2)) \geq 0.$$

Thus, to any point y in the graph $\mathbf{v}(y)$ we can attach a vertical cone, so that the graph always remains entirely inside the cone.

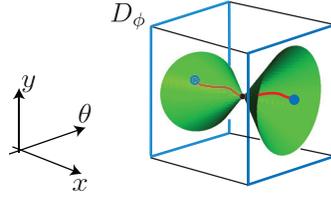


Figure 3. A horizontal disc in D_ϕ .

The following lemma is a key auxiliary result for the proof of Theorem 3.7, which is the main result of this section. Roughly speaking, it states that under appropriate conditions, an image of a horizontal disc is a horizontal disc.

Lemma 3.5. *Let \mathbf{h}_1 be a horizontal disc. If f satisfies covering and cone conditions in D , then there exists a horizontal disc \mathbf{h}_2 such that*

$$\{p : \pi_{x,y}p \in \bar{B}_u^r \times \bar{B}_s^r\} \cap f_\phi(\mathbf{h}_1(\bar{B}_u^r)) = \mathbf{h}_2(\bar{B}_u^r).$$

Moreover, if $H(\phi^{-1}(\mathbf{h}_1(\bar{B}_u^r))) < h$, and for any $p \in D_\phi$ such that $H(\phi^{-1}(p)) < h$ we have

$$(3.13) \quad \pi_\theta(f_\phi(p)) \in \bar{B}_c^R,$$

then

$$\mathbf{h}_2(\bar{B}_u^r) \subset D_\phi \quad \text{and} \quad H(\phi^{-1}(\mathbf{h}_2(\bar{B}_u^r))) < h.$$

Proof. Without loss of generality we can assume that ϕ is equal to identity. Thus we can set $D_\phi = D$ and $f_\phi = f$.

Let \mathbf{h} be any horizontal disc; then by (3.8), (3.12), and (3.10) for $x_1 \neq x_2$

$$(3.14) \quad \begin{aligned} \alpha_h \|\pi_x f(\mathbf{h}(x_1)) - \pi_x f(\mathbf{h}(x_2))\|^2 &\geq Q_h(f(\mathbf{h}(x_1)) - f(\mathbf{h}(x_2))) \\ &> mQ_h(\mathbf{h}(x_1) - \mathbf{h}(x_2)) \\ &\geq 0, \end{aligned}$$

which means that $\pi_x \circ f \circ \mathbf{h}$ is injective.

Let us define a function $F : \bar{B}_u^r \rightarrow \mathbb{R}^u$ as

$$F(x) := \pi_x(f(\mathbf{h}_1(x))).$$

We shall first show that there exists an $x_0 \in \bar{B}_u^r$ such that $F(x_0) \in \bar{B}_u^r$. Using the notation $\mathbf{h}_1(x) = (h_\theta(x), x, h_y(x))$ we can define a family of horizontal discs $\mathbf{h}_\alpha(x) = (\alpha h_\theta(x), x, \alpha h_y(x))$. We define a function $l : [0, 1] \times \bar{B}_u^r \rightarrow \mathbb{R}^u$ as

$$l(\alpha, x) := \pi_x \circ f \circ \mathbf{h}_\alpha(x).$$

By (3.2) and (3.3), since $\mathbf{h}_\alpha(\partial \bar{B}_u^r) \subset D^-$ for any $\alpha \in [0, 1]$, we have $l(\alpha, \partial \bar{B}_u^r) \cap \bar{B}_u^r = \emptyset$. Since, as shown at the beginning of the proof,

$$l(\alpha, \cdot) := \pi_x \circ f \circ \mathbf{h}_\alpha : \bar{B}_u^r \rightarrow \mathbb{R}^u$$

is a continuous monomorphism, we have either $l(\alpha, \bar{B}_u^r) \cap \bar{B}_u^r = \emptyset$ or $\bar{B}_u^r \subset \text{int}(l(\alpha, \bar{B}_u^r))$. This also means that

$$\inf\{\|l(\alpha, 0) - x\| : x \in \partial\bar{B}_u^r\} > 0,$$

and thus the function $\delta : [0, 1] \rightarrow \mathbb{R}$ defined as

$$\delta(\alpha) := \begin{cases} 0, & l(\alpha, 0) \in \bar{B}_u^r, \\ 1, & l(\alpha, 0) \notin \bar{B}_u^r, \end{cases}$$

is continuous. We have

$$\begin{aligned} \pi_x \mathbf{h}_{\alpha=0}(0) &= 0, \\ \pi_y \mathbf{h}_{\alpha=0}(0) &= 0, \end{aligned}$$

so condition (3.7) implies $l(0, 0) = \pi_x \circ f(\mathbf{h}_{\alpha=0}(0)) \in B_u^r$; hence $\delta(0) = 0$. Suppose, to obtain a contradiction, that $F(x) \notin \bar{B}_u^r$ for all $x \in \bar{B}_u^r$. This would mean that, in particular, $F(0) = l(1, 0) \notin \bar{B}_u^r$, and hence $\delta(1) = 1$. This contradicts the fact that $\delta(0) = 0$ and δ is continuous.

We have shown that there exists an $x_0 \in \bar{B}_u^r$ such that $F(x_0) \in \bar{B}_u^r$. From (3.3) it follows that $F(\partial\bar{B}_u^r) \cap \bar{B}_u^r = \emptyset$. We also know that $F = \pi_x \circ f \circ \mathbf{h}_1$ is continuous and injective. Putting these facts together gives $\bar{B}_u^r \subset F(\bar{B}_u^r)$. This means that for any $v \in \bar{B}_u^r$ there exists a unique $x = x(v) \in \bar{B}_u^r$ such that $F(x) = v$. We define

$$\mathbf{h}_2(v) = (h_{2,\theta}(v), v, h_{2,y}(v)) := (\pi_\theta \circ f \circ \mathbf{h}_1(x(v)), v, \pi_y \circ f \circ \mathbf{h}_1(x(v))).$$

For any $v_1 \neq v_2$, $v_1, v_2 \in \bar{B}_u^r$, by (3.12) and (3.10) we have

$$\begin{aligned} (3.15) \quad Q_h(\mathbf{h}_2(v_1) - \mathbf{h}_2(v_2)) &= Q_h(f \circ \mathbf{h}_1(x(v_1)) - f \circ \mathbf{h}_1(x(v_2))) \\ &> mQ_h(\mathbf{h}_1(x(v_1)) - \mathbf{h}_1(x(v_2))) \\ &\geq 0. \end{aligned}$$

Since $Q_h(\mathbf{h}_2(v_1) - \mathbf{h}_2(v_2)) > 0$,

$$\begin{aligned} \alpha_h \|v_1 - v_2\| &> \beta_h \|h_{2,y}(v_1) - h_{2,y}(v_2)\|^2 + \gamma \|h_{2,\theta}(v_1) - h_{2,\theta}(v_2)\|^2 \\ &\geq \min(\beta_h, \gamma) \|(h_{2,\theta}, h_{2,y})(v_1) - (h_{2,\theta}, h_{2,y})(v_2)\|^2, \end{aligned}$$

and therefore \mathbf{h}_2 is continuous.

Finally, let us note that (3.1) and $\mathbf{h}_2(v) = f \circ \mathbf{h}_1(x(v))$ imply $H(\mathbf{h}_2(\bar{B}_u^r)) = H(\mathbf{h}_1(\bar{B}_u^r)) < h$. This by (3.13) implies that $\mathbf{h}_2(\bar{B}_u^r) \subset D$. ■

The next lemma follows from mirror arguments.

Lemma 3.6. *Let \mathbf{v}_1 be a vertical disc. If f satisfies covering and cone conditions in D , then there exists a vertical disc \mathbf{v}_2 such that*

$$\{p : \pi_{x,y} p \in \bar{B}_u^r \times \bar{B}_s^r\} \cap f_\phi(\mathbf{v}_1(\bar{B}_s^r)) = \mathbf{v}_2(\bar{B}_s^r).$$

Moreover, if $H(\phi^{-1}(\mathbf{v}_1(\bar{B}_s^r))) < h$, and for any $p \in D_\phi$ such that $H(\phi^{-1}(p)) < h$ we have

$$(3.16) \quad \pi_\theta(f_\phi^{-1}(p)) \in \bar{B}_c^R,$$

then

$$\mathbf{v}_2(\bar{B}_s^r) \subset D_\phi \quad \text{and} \quad H(\phi^{-1}(\mathbf{v}_2(\bar{B}_s^r))) < h.$$

We are now ready to state our main result for maps, which will be the main tool for the proof of Theorem 2.4.

Theorem 3.7. *If f satisfies covering and cone conditions in D , and in addition for any $p \in D_\phi$ with $H(\phi^{-1}(p)) < h$ we have*

$$(3.17) \quad \pi_\theta f_\phi(p) \in \bar{B}_c^R \quad \text{and} \quad \pi_\theta f_\phi^{-1}(p) \in \bar{B}_c^R,$$

then there exists a C^0 function $\chi : \bar{B}_c^{R-v} \rightarrow D_\phi$ such that the following hold:

1. For any $\theta \in \bar{B}_c^{R-v}$ we have $\pi_\theta \chi(\theta) = \theta$ and

$$f^n(\phi^{-1}(\chi(\theta))) \in D \quad \text{for all } n \in \mathbb{Z}.$$

2. If for some $p \in \phi^{-1}(\bar{B}_c^{R-v} \times \bar{B}_u^r \times \bar{B}_s^r)$ we have

$$f^n(p) \in D \quad \text{for all } n \in \mathbb{Z},$$

then there exists a $\theta \in \bar{B}_c^{R-v}$ such that $p = \phi^{-1}(\chi(\theta))$.

Proof. Without loss of generality we assume that ϕ is equal to identity, which means that $D_\phi = D$ and $f_\phi = f$.

Let $\theta_0 \in \bar{B}_c^{R-v}$ and $y_0 \in \bar{B}_s^r$. Let $\mathbf{h}_1 : \bar{B}_u^r \rightarrow D$ be a horizontal disc defined by

$$\mathbf{h}_1(x) = (\theta_0, x, y_0).$$

Clearly \mathbf{h}_1 satisfies cone conditions and also, by (2.5), $H(\phi^{-1}(\mathbf{h}_1(\bar{B}_u^r))) < h$. Applying inductively Lemma 3.5, we obtain a sequence of horizontal discs $\mathbf{h}_1, \mathbf{h}_2, \dots$ such that

$$f(\mathbf{h}_{i-1}(\bar{B}_u^r)) \cap D = \mathbf{h}_i(\bar{B}_u^r) \quad \text{and} \quad H(\mathbf{h}_i(\bar{B}_u^r)) < h.$$

This by compactness of \bar{B}_u^r ensures existence of a point $x_0^* \in \bar{B}_u^r$ such that for all $n \in \mathbb{N}$

$$(3.18) \quad f^n(\mathbf{h}_1(x_0^*)) \in D.$$

Suppose that we have two points x_0^1 and x_0^2 which satisfy (3.18). Then by (3.15) we have

$$(3.19) \quad \begin{aligned} \alpha_h r^2 &\geq \alpha_h \|\pi_x(f^n(\mathbf{h}_1(x_0^1)) - f^n(\mathbf{h}_1(x_0^2)))\|^2 \\ &> Q_h(f^n(\mathbf{h}_1(x_0^1)) - f^n(\mathbf{h}_1(x_0^2))) \\ &> mQ_h(f^{n-1}(\mathbf{h}_1(x_0^1)) - f^{n-1}(\mathbf{h}_1(x_0^2))) \\ &\dots \\ &> m^n Q_h(\mathbf{h}_1(x_0^1) - \mathbf{h}_1(x_0^2)), \end{aligned}$$

which since $m > 1$ cannot hold for all n . This means that functions $W^{cs} : \bar{B}_c^{R-v} \times \bar{B}_s^r \rightarrow D$, $w^{cs} : \bar{B}_c^{R-v} \times \bar{B}_s^r \rightarrow \bar{B}_u^r$, given as

$$W^{cs}(\theta_0, y_0) = (\theta_0, w^{cs}(\theta_0, y_0), y_0) := (\theta_0, x_0^*, y_0),$$

are properly defined. Note that by an argument similar to (3.19), for any $(\theta_1, y_1) \neq (\theta_2, y_2)$ we must have

$$(3.20) \quad 0 > Q_h(W^{cs}(\theta_2, y_2) - W^{cs}(\theta_1, y_1)).$$

This gives

$$\begin{aligned} \max(\gamma, \beta_h) \|(\theta_2, y_2) - (\theta_1, y_1)\|^2 &\geq \gamma \|\theta_2 - \theta_1\|^2 + \beta_h \|y_2 - y_1\|^2 \\ &> \alpha_h \|w^{cs}(\theta_2, y_2) - w^{cs}(\theta_1, y_1)\|^2, \end{aligned}$$

which means that w^{cs} is Lipschitz with a constant

$$(3.21) \quad L_s = \sqrt{\frac{\max(\gamma, \beta_h)}{\alpha_h}}.$$

Mirror arguments, involving Lemma 3.6, give existence of functions $W^{cu} : \bar{B}_c^{R-v} \times \bar{B}_u^r \rightarrow D$, $w^{cu} : \bar{B}_c^{R-v} \times \bar{B}_u^r \rightarrow \bar{B}_s^r$,

$$W^{cu}(\theta, x) = (\theta, x, w^{cu}(\theta, x)),$$

such that for any point $(\theta, x) \in \bar{B}_c^{R-v} \times \bar{B}_u^r$ and all $n \in \mathbb{N}$

$$f^{-n}(W^{cu}(\theta, x)) \in D.$$

Also, w^{cu} is Lipschitz with a constant

$$(3.22) \quad L_u = \sqrt{\frac{\max(\gamma, \alpha_v)}{\beta_v}}.$$

We shall show that for any $\theta \in \bar{B}_c^{R-v}$ the sets $W^{cs}(\theta, \bar{B}_s^r)$ and $W^{cu}(\theta, \bar{B}_u^r)$ intersect. Let us define $P_\theta : \bar{B}_u^r \times \bar{B}_s^r \rightarrow \bar{B}_u^r \times \bar{B}_s^r$ as

$$P_\theta(x, y) := (\pi_x W^{cs}(\theta, y), \pi_y W^{cu}(\theta, x)).$$

Since P_θ is continuous, by the Brouwer fixed point theorem there exists an (x_0, y_0) such that $P_\theta(x_0, y_0) = (x_0, y_0)$. This means that

$$W^{cs}(\theta, y_0) = (\theta, w^{cs}(\theta, y_0), y_0) = (\theta, x_0, w^{cu}(\theta, y_0)) = W^{cu}(\theta, x_0).$$

Now we shall show that for any given $\theta \in \bar{B}_c^{R-v}$ there exists only a single point of such intersection. Suppose that for some $\theta \in \bar{B}_c^{R-v}$ there exist $(x_1, y_1), (x_2, y_2) \in \bar{B}_u^r \times \bar{B}_s^r$, $(x_1, y_1) \neq (x_2, y_2)$ such that

$$W^{cs}(\theta, y_1) = W^{cu}(\theta, x_1) \quad \text{and} \quad W^{cs}(\theta, y_2) = W^{cu}(\theta, x_2).$$

We would then have $W^{cs}(\theta, y_m) = W^{cu}(\theta, x_m) = (\theta, x_m, y_m)$ for $m = 1, 2$.

From (3.20) it follows that

$$0 > Q_h(W^{cs}(\theta, y_1) - W^{cs}(\theta, y_2)) = Q_h((\theta, x_1, y_1) - (\theta, x_2, y_2)),$$

and by mirror argument

$$0 > Q_v(W^{cu}(x_1, \lambda) - W^{cu}(x_2, \lambda)) = Q_v((\theta, x_1, y_1) - (\theta, x_2, y_2)),$$

which implies that

$$0 > (\alpha_h - \alpha_v) \|x_1 - x_2\|^2 + (\beta_v - \beta_h) \|y_1 - y_2\|^2,$$

which contradicts (3.9).

We now define $\chi(\theta) = (\theta, \chi_{x,y}(\theta)) := (\theta, x_0, y_0)$ for $x_0 = x_0(\theta)$, $y_0 = y_0(\theta)$ such that $W^{cs}(\theta, y_0) = W^{cu}(\theta, x_0)$. By previous arguments we know that χ is properly defined. We need to show continuity. Let us take any $\theta_1, \theta_2 \in \bar{B}_c^{R-v}$. From (3.20) it follows that

$$(3.23) \quad Q_h(\chi(\theta_1) - \chi(\theta_2)) = Q_h(W^{cs}(\theta_1, y_0(\theta_1)) - W^{cs}(\theta_2, y_0(\theta_2))) < 0,$$

and by mirror argument

$$(3.24) \quad Q_v(\chi(\theta_1) - \chi(\theta_2)) = Q_v(W^{cu}(\theta_1, x_0(\theta_1)) - W^{cu}(\theta_2, x_0(\theta_2))) < 0.$$

From (3.23), (3.24) it follows that

$$\begin{aligned} \alpha_h \|x_0(\theta_1) - x_0(\theta_2)\|^2 - \beta_h \|y_0(\theta_1) - y_0(\theta_2)\|^2 &< \gamma \|\theta_1 - \theta_2\|^2, \\ -\alpha_v \|x_0(\theta_1) - x_0(\theta_2)\|^2 + \beta_v \|y_0(\theta_1) - y_0(\theta_2)\|^2 &< \gamma \|\theta_1 - \theta_2\|^2, \\ (\alpha_h - \alpha_v) \|x_0(\theta_1) - x_0(\theta_2)\|^2 + (\beta_v - \beta_h) \|y_0(\theta_1) - y_0(\theta_2)\|^2 &< 2\gamma \|\theta_1 - \theta_2\|^2, \end{aligned}$$

which gives

$$(3.25) \quad \|\chi_{x,y}(\theta_1) - \chi_{x,y}(\theta_2)\| < \sqrt{\frac{2\gamma}{\min(\alpha_h - \alpha_v, \beta_v - \beta_h)}} \|\theta_1 - \theta_2\|$$

and by (3.9) implies Lipschitz bounds for $\chi_{x,y}$ and continuity of χ . ■

4. Proof of the main theorem. In this section we shall show that assumptions of Theorem 2.4 imply that a map induced as a shift along a trajectory of the flow of (2.1) for sufficiently small time satisfies covering and cone conditions. This will allow us to apply Theorem 3.7 to prove Theorem 2.4.

We start with a lemma which shows that assumptions of Theorem 2.4 imply covering conditions for a shift along the trajectory of (2.1).

Lemma 4.1. *Assume that for any $p \in \bar{B}_c^{R-v} \times \{0\} \times \{0\}$ assumptions (2.20), (2.21), (2.19) of Theorem 2.4 hold; then for sufficiently small $\tau > 0$ and all $t \in (0, \tau]$ a function*

$$f(\mathbf{x}) := \Phi(t, \mathbf{x})$$

satisfies covering conditions.

Proof. Without loss of generality we assume that $\phi = id$. Let $q \in B_c^R \times \{0\} \times \{0\} \cap N_p$. By (2.19), for sufficiently small t

$$\begin{aligned} \|\pi_x \Phi(t, q)\| &= \left\| \pi_x \left(\Phi(0, q) + \frac{d}{dt} \Phi(0, q)t + o(t) \right) \right\| \\ &= \|0 + t\pi_x F(q) + o(t)\| \\ (4.1) \qquad \qquad &< |t| r E_u. \end{aligned}$$

Analogous computation yields

$$(4.2) \qquad \qquad \qquad \|\pi_y \Phi(t, q)\| < |t| r E_s.$$

In later parts of the proof we shall use the fact that for any $q_1, q_2 \in N_p$

$$(4.3) \qquad \qquad F(q_1) - F(q_2) = \int_0^1 dF(q_2 + s(q_1 - q_2)) ds (q_1 - q_2).$$

Now we shall prove (3.3). Let $q = (\theta, x, y) \in D_\phi^- \cap N_p$, which means that $\|x\| = r$. Using $\frac{d}{dt} \Phi(t, q)|_{t=0} = F(q)$, $\Phi(0, q) = q$, and (4.3), we have

$$\begin{aligned} &\frac{d}{dt} \|\pi_x (\Phi(t, q) - \Phi(t, (\theta, 0, 0)))\|^2 |_{t=0} \\ &= \frac{d}{dt} \left(\pi_x (\Phi(t, q) - \Phi(t, (\theta, 0, 0)))^T \pi_x (\Phi(t, q) - \Phi(t, (\theta, 0, 0))) \right) \Big|_{t=0} \\ &= 2\pi_x (q - (\theta, 0, 0))^T \pi_x (F(q) - F(\theta, 0, 0)) \\ &= 2x^T \pi_x \left(\int_0^1 dF(\theta, sx, sy) ds (0, x, y) \right) \\ &= 2x^T (Ax + e_u y), \end{aligned}$$

where

$$A = \int_0^1 \frac{\partial(\pi_x F)}{\partial x}(\theta, sx, sy) ds, \qquad e_u = \int_0^1 \frac{\partial(\pi_x F)}{\partial y}(\theta, sx, sy) ds.$$

From bounds (2.9) and (2.12) we thus obtain

$$(4.4) \qquad \frac{d}{dt} \|\pi_x (\Phi(t, q) - \Phi(t, (\theta, 0, 0)))\|^2 |_{t=0} > 2(r^2 \delta^u - \|x\| \|e_u\| \|y\|) > 2r^2 (\delta^u - \varepsilon_u).$$

Using the same arguments, we can also show that for any $q = (\theta, x, y) \in N_p$

$$(4.5) \qquad \frac{d}{dt} \|\pi_y (\Phi(t, q) - \Phi(t, (\theta, 0, 0)))\|^2 |_{t=0} < 2\|y\| (\varepsilon_s r - \|y\| \delta^s).$$

Combining (4.1), (4.4), and (2.20) for sufficiently small $t > 0$ gives

$$\begin{aligned}
(4.6) \quad \|\pi_x f(q)\| &= \|\pi_x \Phi(t, q)\| \\
&\geq \|\pi_x (\Phi(t, q) - \Phi(t, (\theta, 0, 0)))\| - \|\pi_x \Phi(t, (\theta, 0, 0))\| \\
&> \sqrt{\|\pi_x (\Phi(t, q) - \Phi(t, (\theta, 0, 0)))\|^2 - \text{tr} E_u} \\
&> \sqrt{\|\pi_x (\Phi(0, q) - \Phi(0, (\theta, 0, 0)))\|^2 + t2(\delta^u - \varepsilon_u r^2) - \text{tr} E_u} \\
&= \sqrt{r^2 + t2r^2(\delta^u - \varepsilon_u)} - \text{tr} E_u \\
&> r.
\end{aligned}$$

This establishes (3.3). Now we shall show (3.5). For any $q = (\theta, x, y) \in D_\phi$ and sufficiently small $t > 0$, derivation analogous to (4.6) (for these computations we use estimates (4.2), (4.5)) gives

$$(4.7) \quad \|\pi_y f(q)\| < \sqrt{\|y\|^2 + t2\|y\|(\varepsilon_s r - \|y\|\delta^s)} + \text{tr} E_s.$$

Since $\|y\| \leq r$ by (2.21), for sufficiently small $t > 0$, inequality (4.7) implies that $\|\pi_y f(q)\| < r$ and hence establishes (3.5).

Proof of (3.4) and (3.6) follows from arguments analogous to $t < 0$.

Conditions (3.7) hold for sufficiently small t . This follows from continuity of $\Phi(p, t)$ with respect to t since

$$f(p) = \Phi(t, p), \quad f^{-1}(p) = \Phi(-t, p),$$

and for $p \in \bar{B}_c^R \times \{0\} \times \{0\}$

$$\pi_{(x,y)} \Phi(0, p) = \pi_{(x,y)} p = (0, 0) \in \text{int}(\bar{B}_u^r \times \bar{B}_s^r). \quad \blacksquare$$

Now we shall show that assumptions of Theorem 2.4 imply cone conditions for a shift along trajectory of (2.1). Let us start with a simple technical lemma.

Lemma 4.2. *Let $C = (C_{ij})_{i,j=1,2,3}$ be a $(c+u+s) \times (c+u+s)$ matrix. Assume that for $a_i, b_i \in \mathbb{R}$, $i = 1, 2, 3$, we have*

$$(4.8) \quad \inf\{x_i^T C_{ii} x_i : \|x_i\| = 1\} \geq a_i \quad \text{for } i = 1, 2, 3,$$

$$(4.9) \quad \sup\{x_i^T C_{ii} x_i : \|x_i\| = 1\} \leq b_i \quad \text{for } i = 1, 2, 3;$$

then for any $x = (x_1, x_2, x_3) \in \mathbb{R}^{c+u+s}$

$$(4.10) \quad x^T C x \geq (a_1 - c_1) \|x_1\|^2 + (a_2 - c_2) \|x_2\|^2 + (a_3 - c_3) \|x_3\|^2,$$

$$(4.11) \quad x^T C x \leq (b_1 + c_1) \|x_1\|^2 + (b_2 + c_2) \|x_2\|^2 + (b_3 + c_3) \|x_3\|^2,$$

where

$$\begin{aligned}
c_1 &= \frac{1}{2} (\|C_{21}\| + \|C_{31}\| + \|C_{12}\| + \|C_{13}\|), \\
c_2 &= \frac{1}{2} (\|C_{21}\| + \|C_{23}\| + \|C_{12}\| + \|C_{32}\|), \\
c_3 &= \frac{1}{2} (\|C_{31}\| + \|C_{23}\| + \|C_{13}\| + \|C_{32}\|).
\end{aligned}$$

Proof. The estimate (4.10) follows by direct computation from (4.8) and the fact that for any i, j

$$\pm 2q_j^T C_{ji} q_i \geq -\|C_{ji}\| (\|q_j\|^2 + \|q_i\|^2).$$

Similarly (4.11) follows from (4.9) and

$$\pm 2q_j^T C_{ji} q_i \leq \|C_{ji}\| (\|q_j\|^2 + \|q_i\|^2). \quad \blacksquare$$

Let I_k denote a $k \times k$ identity matrix. Let

$$\begin{aligned} Q_1 &= \text{diag}(-\gamma I_c, \alpha_h I_u, -\beta_h I_s), \\ Q_2 &= \text{diag}(-\gamma I_c, -\alpha_v I_u, \beta_v I_s) \end{aligned}$$

be matrices associated with quadratic forms Q_h and Q_v , respectively. Now we are ready to prove that assumptions of Theorem 2.4 imply cone conditions for a time shift along a trajectory map.

Lemma 4.3. *Assume that for any $p \in \bar{B}_c^{R-v} \times \{0\} \times \{0\}$ assumption (2.17) of Theorem 2.4 holds; then for sufficiently small $\tau > 0$ and all $t \in (0, \tau]$ a function*

$$f(\mathbf{x}) := \Phi(t, \mathbf{x})$$

satisfies cone conditions with a coefficient $m = 1 + th$, with some constant $h > 0$.

Proof. Let $p_1, p_2 \in D_\phi$ be such that $p_i = (\theta_i, x_i, y_i)$ for $i = 1, 2$, $p_1 \neq p_2$, and $Q_h(p_1 - p_2) \geq 0$. Let $p = (\theta_1, 0, 0) \in \bar{B}_c^R \times \{0\} \times \{0\}$. Condition (2.14) implies that $p_1, p_2 \in N_p$. We compute

$$\begin{aligned} (4.12) \quad & \frac{d}{dt} ((\Phi(t, p_1) - \Phi(t, p_2))^T Q_1 (\Phi(t, p_1) - \Phi(t, p_2)))|_{t=0} \\ &= 2(p_1 - p_2)^T Q_1 (F(p_1) - F(p_2)) \\ &= 2(p_1 - p_2)^T Q_1 B (p_1 - p_2), \end{aligned}$$

where

$$B = \int_0^1 dF(p_2 + t(p_1 - p_2)) dt \in [dF(N_p)].$$

For $C = Q_1 B$, from (2.11), (2.9), (2.10) we have

$$\begin{aligned} (4.13) \quad & \inf\{x_1^T C_{11} x_1 : \|x_1\| = 1\} \geq -\gamma c^u, \\ & \inf\{x_2^T C_{22} x_2 : \|x_2\| = 1\} \geq \alpha_h \delta^u, \\ & \inf\{x_3^T C_{33} x_3 : \|x_3\| = 1\} \geq \beta_h \delta^s. \end{aligned}$$

Using (4.10) from Lemma 4.2 with (4.13) and (2.12), for $\kappa_c^{\text{forw}}, \kappa_u^{\text{forw}}, \kappa_s^{\text{forw}}$ given by (2.15) and $\mu_1 \in (\max(\kappa_c^{\text{forw}}, \kappa_s^{\text{forw}}), \kappa_u^{\text{forw}})$ we have

$$\begin{aligned} (4.14) \quad & x^T C x \geq -\kappa_c^{\text{forw}} \gamma \|x_1\|^2 + \kappa_u^{\text{forw}} \alpha_h \|x_2\|^2 - \kappa_s^{\text{forw}} \beta_h \|x_3\|^2 \\ & > \mu_1 \left(-\gamma \|x_1\|^2 + \alpha_h \|x_2\|^2 - \beta_h \|x_3\|^2 \right) \\ & = \mu_1 x^T Q_1 x. \end{aligned}$$

The constant $\mu_1 \in (\max(\kappa_c^{\text{forw}}, \kappa_s^{\text{forw}}), \kappa_u^{\text{forw}})$ can be chosen to be greater than zero thanks to assumption (2.17). This means that by (4.12) and (4.14)

$$\frac{d}{dt} ((\Phi(t, p_1) - \Phi(t, p_2))^T Q_1 (\Phi(t, p_1) - \Phi(t, p_2)))|_{t=0} > 2\mu_2 Q_h(p_1 - p_2).$$

For sufficiently small $\tau > 0$ and $t \in (0, \tau)$ we therefore have

$$\begin{aligned} Q_h(f(p_1) - f(p_2)) &= Q_h(\Phi(t, p_1) - \Phi(t, p_2)) \\ &= Q_h(p_1 - p_2) + t \frac{d}{dt} Q_h(\Phi(t, p_1) - \Phi(t, p_2))|_{t=0} + o(t) \\ &> (1 + t2\mu_1) Q_h(p_1 - p_2), \end{aligned}$$

which establishes (3.10) with $m = 1 + t2\mu_1 > 1$.

The proof of (3.11) is obtained analogously with $m = 1 + t2\mu_2 > 1$ for some $\mu_2 < 0$, $\mu_2 \in (\kappa_s^{\text{back}}, \min(\kappa_c^{\text{back}}, \kappa_u^{\text{back}}))$, with negative time $t < 0$.

So far the entire argument has been done for points in N_p . We can choose $h_p = \min\{2|\mu_1|, 2|\mu_2|\}$ so that (3.10) and (3.11) hold for any $p_1, p_2 \in N_p$ with a constant $m = 1 + |t|h_p$. By compactness of D_ϕ we can now choose an $h > 0$ such that (3.10) and (3.11) hold with a constant $m = 1 + |t|h$ for all $p_1, p_2 \in D_\phi$. ■

We are now ready for the proof of our main result.

Proof of Theorem 2.4. By Lemmas 4.1 and 4.3 we know that assumptions of Theorem 2.4 imply cone and covering conditions for a map induced by the flow by a small time shift. Now we just need to show that for a map

$$f(\mathbf{x}) := \Phi(t, \mathbf{x}),$$

with sufficiently small $t > 0$, for any $p \in D_\phi$ with $H(\phi^{-1}(p)) < h$ we have (3.17). This follows from (2.5) and continuity of $\Phi(t, \mathbf{x})$ with respect to t . The claim now follows from Theorem 3.7. ■

By applying Theorem 3.7 in our proof of Theorem 2.4 we have established more than just continuity of our center manifold. We have also obtained existence of its stable and unstable manifolds, together with explicit Lipschitz-type bounds on their slopes. This is summarized in the following corollary.

Corollary 4.4. *During the course of the proof of Theorem 3.7 we have shown that in local coordinates given by ϕ the stable, unstable, and center manifolds obtained by our argument are given in terms of functions*

$$\begin{aligned} W^{cs} &: \bar{B}_c^{R-v} \times \bar{B}_s^r \rightarrow D_\phi, \\ W^{cu} &: \bar{B}_c^{R-v} \times \bar{B}_u^r \rightarrow D_\phi, \\ \chi &: \bar{B}_c^{R-v} \rightarrow D_\phi, \end{aligned}$$

respectively. We have also shown that these functions are of the form

$$\begin{aligned} W^{cs}(\theta, y) &= (\theta, w^{cs}(\theta, y), y), \\ W^{cu}(\theta, x) &= (\theta, x, w^{cu}(\theta, y)), \\ \chi(\theta) &= (\theta, \chi_{x,y}(\theta)), \end{aligned}$$

with functions $w^{cs} : \bar{B}_c^{R-v} \times \bar{B}_s^r \rightarrow \bar{B}_u^r$, $w^{cu} : \bar{B}_c^{R-v} \times \bar{B}_u^r \rightarrow \bar{B}_s^r$, and $\chi_{x,y} : \bar{B}_c^{R-v} \rightarrow \bar{B}_u^r \times \bar{B}_s^r$ by (3.21), (3.22), and (3.25) satisfying Lipschitz conditions with constants

$$L_s = \sqrt{\frac{\max(\gamma, \beta_h)}{\alpha_h}},$$

$$L_u = \sqrt{\frac{\max(\gamma, \alpha_v)}{\beta_v}},$$

$$L_c = \sqrt{\frac{2\gamma}{\min(\alpha_h - \alpha_v, \beta_v - \beta_h)}}.$$

Thus our method gives explicit Lipschitz-type bounds for our invariant manifolds of (2.1).

5. Center manifold around L_1 in the restricted three-body problem. In the following we specialize our study to the center manifold of the equilibrium point L_1 in the RTBP. In our approach we use normal forms to align coordinates. This is a standard approach used to approximate the center manifold around the equilibrium points of the RTBP (see, for example, [4, 3, 14, 10]). Our implementation of normal forms is based on [13].

We shall provide a rigorous estimate on the size of the manifold. It is important to emphasize that the manifold extends much further beyond our bound, as has been numerically investigated in [2].

Section 5.1 describes the RTBP, presents its equations of motion, and specifies the equilibrium point L_1 around which we shall later prove existence of the center manifold. A general reference for this section is Szebehely’s book [19]. Section 5.2 constructs “aligned coordinates” (described in section 2.1) around L_1 in the RTBP using a suitable normal form procedure. A general reference for this section is the paper by Jorba [13] on computation of normal forms with application to the RTBP. In section 5.3 we show how normal forms can be used to obtain a very accurate numerical estimate on where the center manifold is positioned. In section 5.4 we apply Theorem 2.4 to obtain a rigorous enclosure of the center manifold.

5.1. Restricted three-body problem. The problem is defined as follows: two main bodies rotate in the plane about their common center of mass on circular orbits under their mutual gravitational influence. A third body moves in the same plane of motion as the two main bodies, attracted by the gravitation of the previous two but not influencing their motion. The problem is to describe the motion of the third body.

Usually, the two rotating bodies are called the *primaries*. We will consider as primaries the Sun and the Earth. The third body can be regarded as a satellite or a spaceship of negligible mass.

We use a rotating system of coordinates centered at the center of mass. The plane X, Y rotates with the primaries. The primaries are on the X axis; the Y axis is perpendicular to the X axis and contained in the plane of rotation.

We rescale the masses μ_1 and μ_2 of the primaries so that they satisfy the relation $\mu_1 + \mu_2 = 1$. After such rescaling the distance between the primaries is 1. (See Szebehely [19, section 1.5].)

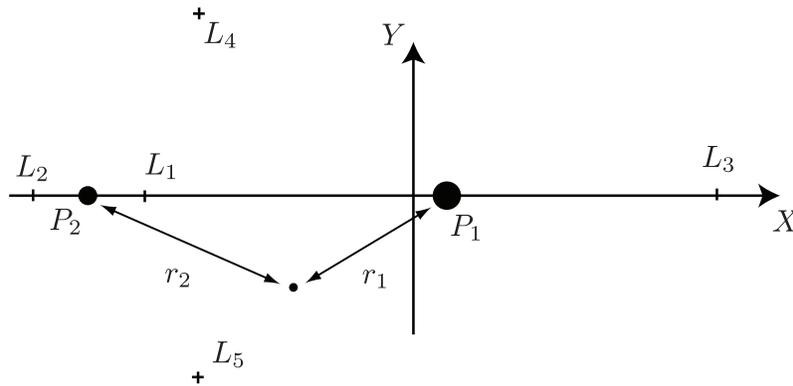


Figure 4. Notation for the rotating system of coordinates with origin at the center of mass. The Sun has the mass $1 - \mu$ and is fixed at $P_1 = (\mu, 0)$. The Earth has the mass μ and is fixed at $P_2 = (\mu - 1, 0)$. The third massless particle moves in the XY plane.

Let the smaller mass be $\mu_2 = \mu \in [3.04041, 3.04043] \times 10^{-6}$ and the larger one be $\mu_1 = 1 - \mu$, corresponding to the values of the Earth and the Sun, respectively. We use a convention in which in the rotating coordinates the Sun is located to the right of the origin at $P_1 = (\mu, 0)$, and the Earth is located to the left at $P_2 = (\mu - 1, 0)$.

The equations of motion of the third body are

$$(5.1a) \quad \ddot{X} - 2\dot{Y} = \Omega_X,$$

$$(5.1b) \quad \ddot{Y} + 2\dot{X} = \Omega_Y,$$

where

$$\Omega = \frac{1}{2}(X^2 + Y^2) + \frac{1 - \mu}{r_1} + \frac{\mu}{r_2}$$

and r_1, r_2 denote the distances from the third body to the larger and the smaller primary, respectively (see Figure 4),

$$\begin{aligned} r_1^2 &= (X - \mu)^2 + Y^2, \\ r_2^2 &= (X - \mu + 1)^2 + Y^2. \end{aligned}$$

These equations have an integral of motion [19] called the Jacobi integral:

$$C = 2\Omega - (\dot{X}^2 + \dot{Y}^2).$$

The equations of motion take Hamiltonian form if we consider positions X, Y and momenta $P_X = \dot{X} - Y, P_Y = \dot{Y} + X$. The Hamiltonian is

$$(5.2) \quad H = \frac{1}{2}(P_X^2 + P_Y^2) + YP_X - XP_Y - \frac{1 - \mu}{r_1} - \frac{\mu}{r_2},$$

with the vector field given by

$$F = J\nabla H,$$

$$J = \begin{pmatrix} 0 & \text{id} \\ -\text{id} & 0 \end{pmatrix}, \quad \text{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Hamiltonian and the Jacobi integral are simply related by $H = -\frac{C}{2}$.

Due to the Hamiltonian integral, the dimensionality of the space can be reduced by one. Trajectories of (5.1) stay on the *energy surface* M given by $H(X, Y, P_X, P_Y) = h = \text{constant}$, a three-dimensional submanifold of \mathbb{R}^4 . Equivalently, M is the level surface

$$(5.3) \quad M \equiv \{C(X, Y, \dot{X}, \dot{Y}) = c = -2h\}$$

of the Jacobi integral.

The RTBP in a rotating frame, described by (5.1), has five equilibrium points (see [19]). Three of them, denoted $L_1, L_2,$ and $L_3,$ lie on the X axis and are usually called the “collinear” equilibrium points (see Figure 4). Notice that we denote L_1 the *interior* collinear point, located between the primaries.

At this point we would like to make it clear that in this paper we focus only on the equilibrium point $L_1,$ though other collinear equilibria points could be investigated in the same manner.

The Jacobian of the vector field at L_1 has two real and two purely imaginary eigenvalues. Since the three-body problem is Hamiltonian it can be shown by the Lyapunov–Moser theorem [15] that in a sufficiently small neighborhood of L_1 there exists a family of periodic orbits which is parameterized by energy. This family of orbits forms a center manifold. Our aim shall be to prove the existence of this manifold in a neighborhood which is far from $L_1.$ As mentioned before, close to L_1 the existence of this manifold follows from the center manifold theorem (or in this case also from the Lyapunov–Moser theorem). The hard task is to prove its existence far from the equilibrium point.

Remark 5.1. *Since the center manifold around L_1 is foliated by periodic orbits, it has to be identical to the invariant manifold obtained through Theorem 2.4 due to point 2 of the theorem. The Lyapunov–Moser theorem ensures the existence of periodic orbits locally. In such a local domain we are guaranteed that the manifold χ from Theorem 2.4 is analytic. Outside of this domain Theorem 2.4 establishes only Lipschitz continuity of $\chi.$*

By showing that the entire manifold is foliated by periodic orbits we would prove that it is analytic. Our method, though, does not provide such result. In a forthcoming publication [6] we shall present an alternative method, based on continuation techniques, that will prove the foliation.

5.2. Normal form. The linearized dynamics around the equilibrium point L_1 is of type saddle \times center for all values of $\mu.$ In this section we use a normal form procedure to approximate the nonlinear dynamics locally around $L_1.$

For the purpose of this paper, the normal form coordinates will be used precisely as the well-aligned coordinates described in section 2.1.

The goal of the normal form procedure is to simplify the Taylor expansion of the Hamiltonian around the equilibrium point using canonical, near-identity changes of variables. This procedure is carried up to a given (finite) degree in the expansion. The resulting Hamiltonian is then truncated to (finite) degree. Such a Hamiltonian is said to be in *normal form.*

We compute a normal form expansion that is as simple as possible, i.e., one that has the minimum number of monomials. This is sometimes called a *full,* or *complete,* normal form. The equations of motion corresponding to the truncated normal form can be integrated

exactly. As a result, *locally* the normal form gives a very accurate approximation of the dynamics.

In particular, here we use the normal form to approximate the local center manifold by a one-parameter family of periodic orbits with increasing energy.

The normal form construction proceeds in three steps. First we perform some convenient translation and scaling of coordinates and expand the Hamiltonian around L_1 as a power series. Then we make a linear change of coordinates to put the quadratic part of the Hamiltonian in a simple form, which diagonalizes the linear part of equations of motion. Finally we use the so-called Lie series method to perform a sequence of canonical, near-identity transformations that simplify nonlinear terms in the Hamiltonian of successively higher degree.

The transformation to well-aligned coordinates $\phi : U \rightarrow \phi(U) \subset \mathbb{R}^n$ is the composition of all the transformations performed during these three steps.

A similar full normal form expansion has been used for the *spatial* RTBP in a previous paper [10]. We refer the reader to the previous paper for the fine details of the normal form construction, which will be left out of the current paper.

5.2.1. Hamiltonian expansion. We start by writing the Hamiltonian (5.2) as a power series expansion around the equilibrium point L_1 . First we translate the origin of coordinates to the equilibrium point. In order to have good numerical properties for the Taylor coefficients, it is also convenient to scale coordinates [17]. The translation and scaling are given by

$$(5.4) \quad X = -\gamma x + \mu - 1 + \gamma, \quad Y = -\gamma y,$$

where γ is the distance from L_1 to its closest primary (the Earth).

Since scalings are not canonical transformations, we apply this change of coordinates to the equations of motion to obtain

$$(5.5a) \quad \ddot{x} - 2\dot{y} = \Omega_x,$$

$$(5.5b) \quad \ddot{y} + 2\dot{x} = \Omega_y,$$

where

$$\Omega = \frac{1}{2}(x^2 + y^2) - \frac{\mu - 1 + \gamma}{\gamma}x + \frac{1}{\gamma^3} \left(\frac{1 - \mu}{r_1} + \frac{\mu}{r_2} \right)$$

and r_1, r_2 denote the (scaled) distances from the third body to the larger and the smaller primaries, respectively.

Defining $p_x = \dot{x} - y$, $p_y = \dot{y} + x$, the libration-point centered equations of motion (5.5) are Hamiltonian, with Hamiltonian function

$$(5.6) \quad H = \frac{1}{2}(p_x^2 + p_y^2) + yp_x - xp_y + \frac{\mu - 1 + \gamma}{\gamma}x - \frac{1}{\gamma^3} \left(\frac{1 - \mu}{r_1} + \frac{\mu}{r_2} \right).$$

Our first change of coordinates can therefore be summarized as $R : \mathbb{R}^4 \rightarrow \mathbb{R}^4$,

$$(5.7) \quad \begin{aligned} (X, Y, P_X, P_Y) &= R(x, y, p_x, p_y) \\ &= (-\gamma x + \mu - 1 + \gamma, -\gamma y, -\gamma p_x, -\gamma p_y + \mu - 1 + \gamma). \end{aligned}$$

The Hamiltonian is then rewritten in the form [13, 14]

$$(5.8) \quad H = \frac{1}{2}(p_x^2 + p_y^2) + yp_x - xp_y - \sum_{n \geq 2} c_n(\mu) \rho^n P_n \left(\frac{x}{\rho} \right),$$

where P_n is the n th Legendre polynomial, and the coefficients $c_n(\mu)$ are given by

$$c_n(\mu) = \frac{1}{\gamma^3} \left(\mu + (-1)^n \frac{(1 - \mu)\gamma^{n+1}}{(1 - \gamma)^{n+1}} \right).$$

This expansion holds when $\rho < \min(|P_1|, |P_2|) = |P_2| = 1$; i.e., it is valid in a ball centered at L_1 that extends up to the Earth.

5.2.2. Linear changes of coordinates. Now we transform the *linear* part of the system into Jordan form, which is convenient for the normal form procedure. This particular transformation is derived in [13, 14], for instance.

Consider the quadratic part H_2 of the Hamiltonian (5.8),

$$(5.9) \quad H_2 = \frac{1}{2}(p_x^2 + p_y^2) + yp_x - xp_y - c_2x^2 + \frac{c_2}{2}y^2,$$

which corresponds to the linearization of the equations of motion. It is well known [14] that the linearized system has eigenvalues of the form $\pm\lambda, \pm i\nu$, where λ, ν are real and positive.

One can find [14, section 2.1] a symplectic linear change of variables

$$C = \begin{pmatrix} \frac{2\lambda}{s_1} & \frac{-2\lambda}{s_1} & 0 & \frac{2\nu}{s_2} \\ \frac{\lambda^2 - 2c_2 - 1}{s_1} & \frac{\lambda^2 - 2c_2 - 1}{s_1} & \frac{-v^2 - 2c_2 - 1}{s_2} & 0 \\ \frac{\lambda^2 + 2c_2 + 1}{s_1} & \frac{\lambda^2 + 2c_2 + 1}{s_1} & \frac{-v^2 + 2c_2 + 1}{s_2} & 0 \\ \frac{\lambda^3 + (1 - 2c_2)\lambda}{s_1} & \frac{-\lambda^3 - (1 - 2c_2)\lambda}{s_1} & 0 & \frac{-v^3 + (1 - 2c_2)v}{s_2} \end{pmatrix},$$

where

$$s_1 = \sqrt{2\lambda((4 + 3c_2)\lambda^2 + 4 + 5c_2 - 6c_2^2)},$$

$$s_2 = \sqrt{v((4 + 3c_2)v^2 - 4 - 5c_2 + 6c_2^2)},$$

that puts the linear terms of the vector field at L_1 into a real Jordan form. This means that the change from position-momenta to new variables $(x_1, y_1, x_2, y_2) \in \mathbb{R}^4$,

$$(5.10) \quad (x, y, p_x, p_y) = C(x_1, y_1, x_2, y_2),$$

casts the quadratic part of the Hamiltonian into

$$(5.11) \quad H_2 = \lambda x_1 y_1 + \frac{\nu}{2}(x_2^2 + y_2^2).$$

The linear equations of motion $(\dot{x}, \dot{y}) = A(x, y)$ associated to (5.11) decouple into a hyperbolic part and a center part,

$$(5.12a) \quad (\dot{x}_1, \dot{y}_1) = A_h(x_1, y_1),$$

$$(5.12b) \quad (\dot{x}_2, \dot{y}_2) = A_c(x_2, y_2),$$

with

$$A_h = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} \quad \text{and} \quad A_c = \begin{pmatrix} 0 & v \\ -v & 0 \end{pmatrix}.$$

Notice that the matrix A of the linear equations (5.12) is in block-diagonal form. It is convenient to diagonalize the matrix A over \mathbb{C} . Consider the *symplectic* change $T^{-1}: \mathbb{R}^4 \rightarrow \mathbb{C}^4$ to complex variables $(q, p) = (q_1, p_1, q_2, p_2) \in \mathbb{C}^4$:

$$(5.13) \quad \begin{aligned} (q_1, p_1, q_2, p_2) &= T^{-1}(x_1, y_1, x_2, y_2) \\ &= \left(x_1, y_1, \frac{1}{\sqrt{2}}(x_2 - iy_2), \frac{1}{\sqrt{2}}(-ix_2 + y_2) \right). \end{aligned}$$

This change casts the quadratic part of the Hamiltonian into

$$(5.14) \quad H_2 = \lambda q_1 p_1 + i\nu q_2 p_2.$$

Equivalently, this change carries A to diagonal form:

$$T^{-1}AT = \Lambda = \text{diag}(\lambda, -\lambda, i\nu, -i\nu).$$

5.2.3. Nonlinear normal form. Assume that the symplectic linear changes of variables (5.10) and (5.13) have been performed in the Hamiltonian expansion (5.8) so that the quadratic part H_2 is already in the form (5.14).

Let us thus write the Hamiltonian as

$$(5.15) \quad H(q, p) = H_2(q, p) + H_3(q, p) + H_4(q, p) + \cdots,$$

where $H_j(q, p)$ are homogeneous polynomials of degree j in the variables $(q, p) \in \mathbb{C}^4$.

As shown in a previous paper [10], we can remove most monomials in the series (5.15) by means of formal coordinate transformations in order to obtain an integrable approximation of the dynamics close to the equilibrium point.

Proposition 5.2 (complete normal form around a saddle×center [10]). *For any integer $N \geq 3$, there exists a neighborhood $\mathcal{U}^{(N)}$ of the origin and a near-identity canonical transformation*

$$(5.16) \quad \mathcal{T}^{(N)}: \mathbb{C}^4 \supset \mathcal{U}^{(N)} \mapsto \mathbb{C}^4$$

that puts the system (5.15) in normal form up to order N , namely,

$$\mathcal{H}^{(N)} := H \circ \mathcal{T}^{(N)} = H_2 + \mathcal{Z}^{(N)} + \mathcal{R}^{(N)},$$

where $\mathcal{Z}^{(N)}$ is a polynomial of degree N that Poisson-commutes with H_2 ,

$$\{\mathcal{Z}^{(N)}, H_2\} \equiv 0,$$

and $\mathcal{R}^{(N)}$ is small:

$$|\mathcal{R}^{(N)}(z)| \leq C_N \|z\|^{N+1} \quad \forall z \in \mathcal{U}^{(N)}.$$

Moreover, the truncated Hamiltonian $H_2 + \mathcal{Z}^{(N)}$ depends only on the basic invariants

$$(5.17a) \quad I_1 = q_1 p_1 = x_1 y_1,$$

$$(5.17b) \quad I_2 = i q_2 p_2 = q_2 \bar{q}_2 = \frac{x_2^2 + y_2^2}{2}.$$

The equations of motion associated to the truncated normal form $H_2 + \mathcal{Z}^{(N)}$ can be integrated exactly.

Remark 5.3. *The reminder $\mathcal{R}^{(N)}$ is very small in a small neighborhood of the origin. Hence, close to the origin, the exact solution of the truncated normal form is a very accurate approximate solution of the original system H .*

Remark 5.4. *In fact, Giorgilli [11] showed that the normal form procedure is convergent; i.e., the power series associated to the canonical transformation $\mathcal{T}^{(N)}$ as $N \rightarrow \infty$ converges in a neighborhood of the origin.*

Remark 5.5. *Let ϕ_1, ϕ_2 be the symplectic conjugate variables to I_1, I_2 , respectively. The basic invariant I_2 is usually called an action variable, and its conjugate variable ϕ_2 is usually called an angle variable. They are given in symplectic polar variables (5.17b).*

We can now write our function ϕ for our change into the well-aligned coordinates (2.3). To do so we compose the inverse transformations given in (5.7), (5.10), (5.13), and (5.16), which gives us

$$(5.18) \quad \phi = \left(\mathcal{T}^{(N)}\right)^{-1} \circ T^{-1} \circ C^{-1} \circ R^{-1}.$$

Remark 5.6. *The above-described method of obtaining normal form coordinates is performed by passing through complex variables. It is possible, though, to arrange the changes so that the combined change of coordinates (5.18) passes from real to real coordinates. The change of coordinates ϕ is a high order polynomial. It is possible to arrange the normal form change of coordinates so that the coefficient of ϕ is real (see [13]). In setting up our change of coordinates for the application of Theorem 2.4 to the RTBP in section 5.4 we have adopted such a procedure.*

Remark 5.7. *In practice, one usually computes a normal form of degree $N = 16$. In our application to the RTBP in section 5.4 we use a normal form of degree $N = 4$. This turns out to be sufficient, since we investigate a relatively close neighborhood of the invariant point, where degree of order four gives us a sufficiently good approximation.*

5.3. Approximating the center manifold in normal form coordinates. In normal form coordinates given by (5.18) the Hamiltonian, by Proposition 5.2, is of the form

$$(5.19) \quad \mathcal{H}^{(N)} = H_2 + \mathcal{Z}^{(N)} + \mathcal{R}^{(N)}, \quad \{\mathcal{Z}^{(N)}, H_2\} \equiv 0.$$

In this section we shall show that when we neglect the reminder term $\mathcal{R}^{(N)}$, and thus consider an approximation of the system, the normal form coordinates given by (5.18) give us a very good understanding of where the center manifold is positioned and of the dynamics on it.

Let U be some small neighborhood of the fixed point (in our discussion for the RTBP this will be L_1), and let $\phi: U \rightarrow \phi(U) \subset \mathbb{R}^4$ be the transformation to normal form coordinates

(5.18). Consider the normal form (5.19) up to order N with associated equations of motion

$$(5.20) \quad \dot{p} = F^\phi(p) := J\nabla\mathcal{H}^{(N)}(p).$$

Consider now the *truncated* normal form up to order N ,

$$\hat{\mathcal{H}}^{(N)} = H_2 + \mathcal{Z}^{(N)},$$

with associated equations of motion

$$(5.21) \quad \dot{p} = \hat{F}^\phi(p) := J\nabla\hat{\mathcal{H}}^{(N)}(p).$$

Recall that the corresponding linearization around the origin is (5.12),

$$(5.22) \quad \dot{p} = Ap, \quad p = (x_1, y_1, x_2, y_2) \in \mathbb{R}^4,$$

where x_1, y_1 are the hyperbolic normal form coordinates (5.17a), and x_2, y_2 are the center normal form coordinates (5.17b). In order to match the notation from section 2, let us denote the center normal form coordinates x_2, y_2 as θ_1, θ_2 , the unstable normal form coordinate x_1 as x , and the stable normal form coordinate y_1 as y . Note that to match the notation we need to swap the order in which the coordinates are written, passing from (x_1, y_1, x_2, y_2) to $(\theta_1, \theta_2, x, y)$.

The truncated system \hat{F}^ϕ has several invariant subspaces. Specifically, the next proposition follows from [16, section 5.1].

Proposition 5.8. *Let*

$$(5.23) \quad E^c = \{(\theta_1, \theta_2, 0, 0) : (\theta_1, \theta_2) \in \mathbb{R}^2\},$$

$$(5.24) \quad E^u = \{(0, 0, x, 0) : x \in \mathbb{R}\},$$

$$(5.25) \quad E^s = \{(0, 0, 0, y) : y \in \mathbb{R}\}.$$

Then, E^c , E^u , and E^s are invariant subspaces of the flow of \hat{F}^ϕ .

Remark 5.9. *These subspaces are invariant under the nonlinear truncated system (5.21) and not just under the linearized system (5.22). It is important to stress here, though, that these subspaces need not be invariant under the full system (5.20).*

Next we claim that E^c is approximately equal to the center manifold W^c of the full system F^ϕ . This is formulated in the next proposition, which follows from [16, section 5.2].

Proposition 5.10. *For each integer r with $N \leq r < \infty$, there exists a (not necessarily unique) local invariant center manifold W^c of F^ϕ of class C^r such that the following hold:*

- W^c is expressible as a graph over E^c ; i.e., there exist a neighborhood $V \subset E^c$ and a map $\chi: V \rightarrow E^u \oplus E^s$ such that

$$W^c = \{(\theta_1, \theta_2, x, y) \in E^c \oplus E^u \oplus E^s : (\theta_1, \theta_2) \in V, (x, y) = \chi(\theta_1, \theta_2)\}.$$

- W^c has N th order contact with E^c ; i.e., χ and its derivatives up to order N vanish at the origin.

Hence, in normal form coordinates, the center manifold W^c of F^ϕ is approximated very accurately (to order N) around the origin by the subspace E^c .

Remark 5.11. *When applying Proposition 5.10 we are faced with a problem for which it is usually very hard to obtain a rigorous bound on the size of the set V . According to the work [11], one can obtain a rigorous bound on the size of the higher order terms of χ on the set V . However, we have not pursued this possibility.*

Let us now briefly discuss the dynamics of the system (5.21) on E^c . To do so we shall use the center normal form coordinates (5.17b) in action angle form; i.e., from now on we will use $(I, \varphi) \in \mathbb{R} \times \mathbb{T}$ for the center part. Proposition 5.2 states that the truncated Hamiltonian $\hat{\mathcal{H}}^{(N)}$ depends only on the action I and not on the angle φ . Thus the restriction of \hat{F}^ϕ to its invariant subspace E^c is

$$(5.26) \quad \dot{I} = 0, \quad \dot{\varphi} = \frac{\partial \hat{\mathcal{H}}^{(N)}}{\partial I} =: \omega(I).$$

The solutions inside E^c with initial conditions $I(0) = I_0$ and $\varphi(0) = \varphi_0$ are $I(t) = I_0$, $\varphi(t) = \omega(I_0)t + \varphi_0$. In the case of the RTBP E^c is two-dimensional, and so the dynamics of the truncated system on E^c is foliated by periodic orbits of increasing action I . Notice from (5.11) that H grows linearly with respect to I (close to the origin), so the periodic orbits also have increasing energy H .

The properties discussed above motivate the use of the normal form coordinates θ_1, θ_2, x, y as the well-aligned coordinates in the sense of section 2.1. They provide a good approximation of the location of the center manifold (locally around the origin). Taking $\bar{B}_c^R = \{(\theta_1, \theta_2) \in \mathbb{R}^2: \|\theta_1, \theta_2\| \leq R\}$, the approximation is given by $\phi^{-1}(\bar{B}_c^R \times \{0\}) \subset \mathbb{R}^4$. Notice also that the center coordinate I is well aligned with the energy (in sense of (2.5)). Let

$$C^R = \{(\theta_1, \theta_2) \in \mathbb{R}^2: \|\theta_1, \theta_2\| = R\}$$

be the invariant circle of radius R for the system (5.21). By (5.26) we have $\hat{\mathcal{H}}^{(N)}(C^{R_1} \times \{0\}) < \hat{\mathcal{H}}^{(N)}(C^{R_2} \times \{0\})$ whenever $R_1 < R_2$. Hence, given an energy h , we can find $R_1, R_2 > 0$ such that

$$\hat{\mathcal{H}}^{(N)}(C^{R_1} \times \{0\}) < h < \hat{\mathcal{H}}^{(N)}(C^{R_2} \times \{0\}).$$

Taking R_1, R_2 sufficiently far (in practice they are still close) from one another and taking sufficiently small $r > 0$, since $\hat{\mathcal{H}}^{(N)}$ and $\mathcal{H}^{(N)}$ are close, we expect also that

$$\mathcal{H}^{(N)}(B_c^{R_1} \times B_u^r \times B_s^r) < h < \mathcal{H}^{(N)}(C^{R_2} \times B_u^r \times B_s^r).$$

Since $\mathcal{H}^{(N)} = H \circ \phi^{-1}$, this will mean that that the bound (2.5) shall be satisfied.

5.4. Application of the main theorem to the center manifold around L_1 . In this section we shall show how to apply Theorem 2.4 in practice.

As described in section 5.2 the change of coordinates to *well-aligned coordinates* can be done using a change to normal coordinates (5.18). We obtain the function ϕ using the algorithm of Jorba [13]. The algorithm allows us to obtain ϕ as a real polynomial, passing from \mathbb{R}^4 to \mathbb{R}^4 .

5.4.1. Methodology. To apply Theorem 2.4 it is enough to derive a rigorous bound on the derivative of F^ϕ . Let us now outline how such a bound can be obtained. Using (2.6), for any $p \in \mathbb{R}^4$ we have

$$(5.27) \quad \begin{aligned} D\left(F^\phi(p)\right) &= D^2\phi(\phi^{-1}(p))D(\phi^{-1})(p)F(\phi^{-1}(p)) \\ &\quad + D\phi(\phi^{-1}(p))DF(\phi^{-1}(p))D(\phi^{-1})(p). \end{aligned}$$

In our computer-assisted proof we apply the above formula using an interval-arithmetic-based software called CAPD (Computer Assisted Proofs in Dynamics¹). This software in particular allows for rigorous interval-enclosure-based computation of high order derivatives of functions on sets. In our application we obtain a global bound for the derivative (2.8) on the entire set D_ϕ . Computing $[DF^\phi(D_\phi)]$ applying (5.27) requires only computing images of functions, derivatives of functions, and a second derivative on a set D_ϕ . All such computations can be performed in CAPD.

Before specifying the size of the set D_ϕ and giving rigorous interval-based numerical results, we have to stress one problem encountered when applying formula (5.27). We take our change to *well-aligned coordinates* ϕ to be a high order polynomial obtained from non-rigorous computations. To apply formula (5.27) directly we would need to know its inverse ϕ^{-1} . Let us stress that one cannot use a numerical approximation of an inverse change and use it as ϕ^{-1} (such a numerical approximate inverse is readily available from algorithms of [13]). To apply (5.27) directly one would have to use a *rigorous* analytic inverse. Since ϕ is a polynomial in high dimension and of high order, its analytic inverse is next to impossible to obtain in practice. To remedy this problem we slightly modify (5.27). Using the fact that $D(\phi^{-1})(p) = (D\phi(\phi^{-1}(p)))^{-1}$, we can rewrite (5.27) as

$$\begin{aligned} DF^\phi(p) &= D^2\phi(\phi^{-1}(p))(D\phi(\phi^{-1}(p)))^{-1}F(\phi^{-1}(p)) \\ &\quad + D\phi(\phi^{-1}(p))DF(\phi^{-1}(p))(D\phi(\phi^{-1}(p)))^{-1}. \end{aligned}$$

This in interval arithmetic notation gives us the following formula for the interval enclosure of DF^ϕ on some set $\mathbf{I} \subset D_\phi$:

$$(5.28) \quad \begin{aligned} [DF^\phi(\mathbf{I})] &\subset [D^2\phi([\phi^{-1}(\mathbf{I})])](D\phi([\phi^{-1}(\mathbf{I})]))^{-1}F([\phi^{-1}(\mathbf{I})]) \\ &\quad + D\phi([\phi^{-1}(\mathbf{I})])DF([\phi^{-1}(\mathbf{I})])(D\phi([\phi^{-1}(\mathbf{I})]))^{-1}. \end{aligned}$$

To compute the right-hand side of the above equation there is no need to invert the function ϕ . It is enough to find a set $[\phi^{-1}(\mathbf{I})]$ which contains the preimage of \mathbf{I} , i.e.,

$$\phi^{-1}(\mathbf{I}) \subset [\phi^{-1}(\mathbf{I})],$$

and for this we do not need to compute the inverse function. For a set $B \subset \mathbb{R}^4$ the following lemma can be used to verify that $\phi^{-1}(\mathbf{I}) \subset B$.

¹<http://capd.ii.uj.edu.pl>

Lemma 5.12. *Let $\phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a homeomorphism, and let $\mathbf{I}, B \subset \mathbb{R}^n$ be two sets homeomorphic to n -dimensional balls. If $\phi(\partial B) \cap \mathbf{I} = \emptyset$ and for some point $p \in B$ we have $\phi(p) \in \mathbf{I}$, then*

$$\phi^{-1}(\mathbf{I}) \subset B.$$

Proof. This follows from elementary topological arguments. ■

To apply the lemma in practice it is convenient to first have a nonrigorous approximation of the inverse function; let us denote it by $\hat{\phi}^{-1}$. This means that

$$\hat{\phi}^{-1}\phi \approx \text{id}.$$

A function $\hat{\phi}^{-1}$ is readily available from algorithms of Jorba [13]. We can then choose $\lambda > 1$ and set $B = \lambda[\hat{\phi}^{-1}(\mathbf{I})]$ (in our application we choose $\lambda = 3$, which we find is large enough for our problem). Then we divide the boundary ∂B into smaller sets and verify that the image by ϕ of each smaller set is disconnected with \mathbf{I} . We also check that for the middle point p in B we have $\phi(p) \in \mathbf{I}$. This by Lemma 5.12 guarantees that $\phi^{-1}(\mathbf{I}) \subset B$.

Remark 5.13. *Once a set B such that $\phi^{-1}(\mathbf{I}) \subset B$ is found, there is a useful trick that can be used to refine this initial guess on the preimage. One can take a very small set $\mathbf{I}_0 \subset \mathbf{I}$ and using Lemma 5.12 find a small set B_0 such that $\phi^{-1}(\mathbf{I}_0) \subset B_0$. The set $[\phi^{-1}(\mathbf{I})]$ can then be chosen as*

$$[\phi^{-1}(\mathbf{I})] = B_0 + [(D\phi(B))^{-1}] [\mathbf{I} - \mathbf{I}_0].$$

Such a choice guarantees that $\phi^{-1}(\mathbf{I}) \subset [\phi^{-1}(\mathbf{I})]$. It is also usually tighter than the initial guess B , which is true especially when the function ϕ is close to identity.

Proof. This follows from the mean value theorem. ■

In a fashion similar to the method from Remark 5.13, to compute the energy for a set $\mathbf{I} \subset D^\phi$, we take some small set $\mathbf{I}_0 \subset \mathbf{I}$ and compute

$$(5.29) \quad [H(\phi^{-1}(\mathbf{I}))] \subset H([\phi^{-1}(\mathbf{I}_0)]) + [DH([\phi^{-1}(\mathbf{I}))]] [[\phi^{-1}(\mathbf{I})] - [\phi^{-1}(\mathbf{I}_0)]] .$$

Remark 5.14. *When applying the above tools to compute $[DF^\phi(\mathbf{I})]$ using (5.28), it pays to use the fact that ϕ is composed of linear changes of coordinates, together with a nonlinear change $\mathcal{T}^{(N)}$ which is close to identity. Keeping track of both linear and nonlinear changes allows us to tighten the interval bounds of computations.*

To prove the existence of a fixed point (in case of the RTBP we take the point L_1) inside of our set D_ϕ we use the interval Newton method.

Theorem 5.15 (see [1]). *Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 function. Let $\mathbf{I} = \Pi_{i=1}^n [a_i, b_i]$, $a_i < b_i$. Assume that the interval enclosure of $DF(\mathbf{I})$, denoted by $[DF(\mathbf{I})]$, is invertible. Let $x_0 \in \mathbf{I}$, and define*

$$N(F, x_0, \mathbf{I}) = - [DF(\mathbf{I})]^{-1} F(x_0) + x_0.$$

If $N(x_0, \mathbf{I}) \subset \mathbf{I}$, then there exists a unique point $x^ \in \mathbf{I}$ such that $F(x^*) = 0$.*

5.4.2. Rigorous interval-based numerical results. For our proof we use a normal form (5.18) of order $N = 4$ as the change of coordinates. At this point we stress once again that ϕ obtained by (5.18) does not need to perfectly align the coordinates of the system. In particular, ϕ was numerically obtained for a single mass parameter from the interval $\mathbf{m} =$

$[3.04041, 3.04043] \times 10^{-6}$ and is applied to all parameters $\mu \in \mathbf{m}$. A numerically obtained polynomial, provided that it aligns its coordinates well enough, is sufficient to prove the existence of a center manifold using our method, provided that the assumptions of Theorem 2.4 can be verified.

We investigate a set

$$D_\phi = \bar{B}_c^R \times \bar{B}_u^r \times \bar{B}_s^r$$

with

$$(5.30) \quad \begin{aligned} R &= \sqrt{2 \cdot 155 \cdot 10^{-4}} \approx 0.176, \\ r &= 5 \cdot 10^{-4}. \end{aligned}$$

Our choice of R by (5.17a) implies that we consider actions $I \in [0, 0.0155]$.

We first prove that we have a fixed point in D^ϕ applying Theorem 5.15. We take $\mathbf{I} = \Pi_{i=1}^4[-25 \cdot 10^{-5}, 25 \cdot 10^{-5}] \subset \text{int}D^\phi$ and compute

$$N(F^\phi, 0, \mathbf{I}) \subset 10^{-5}([-6.420, 6.535] \times [-6.535, 6.418] \times [-6.336, 6.336] \times [-5.524, 5.461]).$$

Clearly $N(F^\phi, 0, \mathbf{I}) \subset \mathbf{I}$, which establishes that for any $\mu \in \mathbf{m}$ the fixed point L_1 is in the interior of D^ϕ .

Next we verify condition (2.5). We take $v = \sqrt{2 \cdot 155 \cdot 10^{-4}} - \sqrt{2 \cdot 150 \cdot 10^{-4}}$, which is equivalent to the ball B_c^{R-v} having actions $I \in [0, 0.015]$. We subdivide $\bar{B}_u^r \times \bar{B}_s^r$ into 9 pieces and cover $\partial \bar{B}_c^R$ by 500 small boxes in \mathbb{R}^2 . Taking the $9 \cdot 500$ sets, using (5.29), we obtain a bound on the energy

$$H(\phi^{-1}(\partial \bar{B}_c^R \times \bar{B}_u^r \times \bar{B}_s^r)) > -1.500445782331.$$

Taking the same type of subdivision, we then show that

$$H(\phi^{-1}(\bar{B}_c^{R-v} \times \bar{B}_u^r \times \bar{B}_s^r)) < -1.500445786588,$$

which establishes (2.5).

To compute $[DF^\phi(D_\phi)]$, we cover \bar{B}_c^R by 31 000 boxes in \mathbb{R}^2 , $\bar{B}_c^R \subset \bigcup_{i=1}^{31000} \mathbf{I}_{c,i}$. Taking $\mathbf{I}_i = \mathbf{I}_{c,i} \times \bar{B}_u^r \times \bar{B}_s^r$, we compute the bound on $[DF^\phi(D_\phi)]$ as an interval hull of all matrices $[DF^\phi(\mathbf{I}_i)]$ (this means that we take an interval matrix $[DF^\phi(D_\phi)]$ so that $[DF^\phi(\mathbf{I}_i)] \subset [DF^\phi(D_\phi)]$ for $i = 1, \dots, 31000$). Each interval matrix $[DF^\phi(\mathbf{I}_i)]$ is computed using (5.28). Thus for all $\mu \in \mathbf{m}$ we obtain a bound for $[DF^\phi(D_\phi)]$ (displayed below with three-digit rough accuracy, rounded up to ensure true enclosure):

$$(5.31) \quad [DF^\phi(D_\phi)] = \begin{pmatrix} [-0.0336, 0.0335] & [2.06, 2.11] & [-0.0526, 0.0521] & [-0.0521, 0.0526] \\ [-2.15, -2.03] & [-0.0422, 0.0422] & [-0.0826, 0.0827] & [-0.0825, 0.0827] \\ [-0.0783, 0.0782] & [-0.0559, 0.0566] & [2.43, 2.64] & [-0.0974, 0.0962] \\ [-0.0782, 0.0783] & [-0.0559, 0.0566] & [-0.0962, 0.0974] & [-2.64, -2.43] \end{pmatrix}.$$

We take $\alpha_h = \beta_v = 2$ and $\alpha_v = \beta_h = \gamma = 1$, which clearly satisfy (2.13). In our application we deal with a single set $N_p = N_0 = D_\phi$, which means that for this set $\rho = R$. With our choice of parameters condition (2.14) clearly holds.

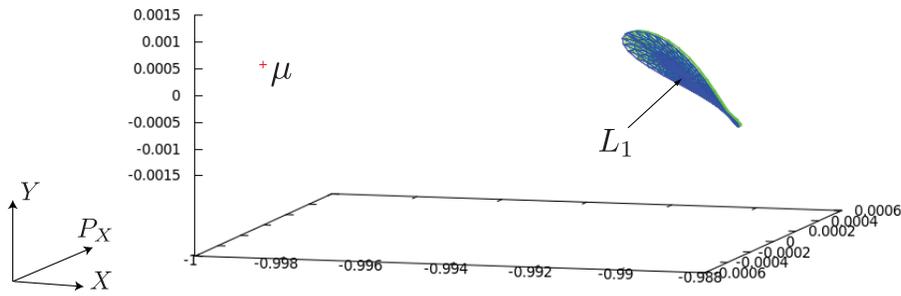


Figure 5. A rough sketch of $\pi_{X,Y,P_X} \phi^{-1}(\bar{B}_c^{R-v} \times \bar{B}_u^r \times \bar{B}_s^r)$, which gives us an idea of the actual size and thickness of the investigated region in which we have proved existence of the center manifold.

Based on (5.31), using (2.9)–(2.12), (2.15), and (2.16), we compute the constants κ_c^{forw} , κ_u^{forw} , κ_s^{forw} , κ_c^{back} , κ_u^{back} , κ_s^{back} , ε_u , ε_s , δ^u , δ^s needed for the verification of the assumptions of Theorem 2.4. The computed constants are written in (7.1) and (7.4) in section 7.

Finally, using the boxes $\mathbf{I}_{c,i}$ we also compute $[\pi_{x,y} F(\bar{B}_c^R \times \{0\} \times \{0\})]$ as the interval hull of all $[\pi_{x,y} F^\phi(\mathbf{I}_{c,i} \times \{0\} \times \{0\})]$ for $i = 1, \dots, 31\,000$ (displayed below with rough accuracy),

$$(5.32) \quad \begin{aligned} &[\pi_{x,y} F(\bar{B}_c^R \times \{0\} \times \{0\})] \\ &= [-0.000960689, 0.000822881] \times [-0.000960693, 0.000822879], \end{aligned}$$

from which E_u , E_s are computed using (2.19) (see (7.1) in section 7). For computation of each $[\pi_{x,y} F^\phi(\mathbf{I}_{c,i} \times \{0\} \times \{0\})]$ we in fact need to further subdivide each box $\mathbf{I}_{c,i}$ into nine parts (this is because E_u and E_s turn out to be our most sensitive estimates). Based on all the computed constants we verify assumptions (2.17)–(2.21) of Theorem 2.4.

The computer-assisted part of the proof has taken 3 hours and 38 minutes of computation on a standard laptop (it is possible to conduct much shorter proofs, but for less accurate enclosures of the manifold than above). Looking at the constants (7.1), (7.4) written in section 7 it is apparent that assumptions (2.17), (2.18) of Theorem 2.4 hold by a large margin. The bottleneck lies in conditions (2.20) and (2.21). This follows from the fact that the bounds computed in (5.32) are large in comparison to r (see (2.19), which binds the two together). This is because far away from the origin the fourth order normal form no longer gives an accurate enough estimate on the position of the manifold, and hence the vector-field in the expansion/contraction direction becomes noticeably nonzero. A simple remedy would be to use a higher order normal form, which would allow for obtaining a tighter enclosure and also a larger domain. This would require longer computations and use of more capable hardware than a standard laptop. Such computations, though, can easily be performed on clusters.

Finally, let us note that the size of the region in which the manifold is found is not negligible. In Figures 5 and 6 we see our region together with the smaller mass (Earth) in the original coordinates of the system. Our set D_ϕ is a four-dimensional “flattened disc,” and in Figure 5 we can see that the disc is not too thick. On our plot the set $\pi_{X,Y,P_X}(\phi^{-1}(D_\phi))$ lies between the two colored flat discs (the blue disc below, and the green disc above; in this resolution they practically merge with one another).

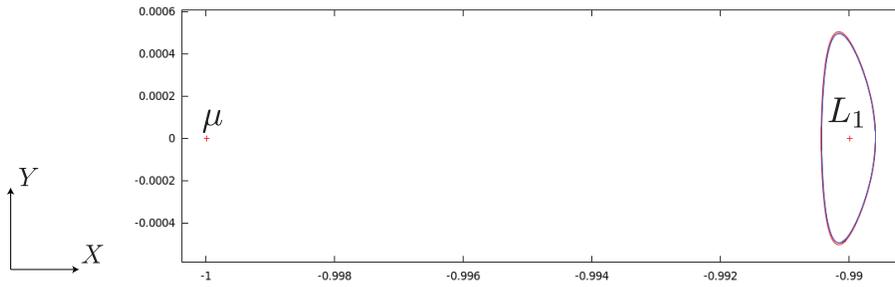


Figure 6. A rough sketch of $\pi_{X,Y}(\phi^{-1}(\partial\bar{B}_c^{R-v} \times \bar{B}_u^r \times \bar{B}_s^r))$ in blue and $\pi_{X,Y}(\phi^{-1}(\partial\bar{B}_c^R \times \bar{B}_u^r \times \bar{B}_s^r))$ in red. We have proved that the manifold is contained in $\phi^{-1}(\bar{B}_c^{R-v} \times \bar{B}_u^r \times \bar{B}_s^r)$ and that orbits starting from it never leave $\phi^{-1}(\bar{B}_c^R \times \bar{B}_u^r \times \bar{B}_s^r)$ when going forward or backward in time.

6. Closing remarks and future work. In this paper we have given a method for detection and proof of existence of center manifolds in a given domain of the system. The method is quite general. It can be applied to any system with an integral of motion which allows for a computation of a normal form around a fixed point. The method also works for arbitrary dimension, which makes it a tool which can be applied to a large family of systems.

The strength of our approach lies in the fact that we can investigate and prove existence of manifolds within large domains, and not only locally around a fixed point. The weakness so far is that the method establishes only Lipschitz-type continuity of the manifold. In forthcoming work we plan to remedy this deficiency. In our view, since we already have established Lipschitz continuity, similar tools combined with standard cohomology equation arguments can be applied to prove higher order smoothness.

We have successfully applied the method to the RTBP. We have not shown, though, that the manifold around L_1 is foliated by periodic orbits. The manifold also is much larger than the domain in which we can prove its existence. We are currently preparing an alternative continuation-type approach which will detect the orbits. This approach will also allow for proofs over a much larger domain.

We would also like to mention that our method allows for rigorous enclosure of the associated stable and unstable manifolds through cone conditions used in the proof. This means that it can be used as a starting point for computation of foliations of stable/unstable manifolds and next as a scattering map associated with splitting of separatrices. In our future work we plan to conduct rigorous computer-assisted computations of the scattering map for the RTBP in the spirit of [10]. Such computations can then be used in the study of structural stability or diffusion as in [9].

7. Appendix. Here we list the bounds needed for the verification of the assumptions of Theorem 2.4. Below constants were computed using (5.32), (5.31) combined with (2.19), (2.9), (2.10), and (2.12):

$$(7.1) \quad \begin{aligned} E_u &= 1.921376952576923, & E_s &= 1.921385550593898, \\ \delta_u &= 2.434891033094052, & \delta_s &= 2.43489806870965, \\ \varepsilon_c &= 0.0979681005281105, & \varepsilon_m &= 0.09657492960501532, \\ \varepsilon_u &= 0.0973872211974159, & \varepsilon_s &= 0.09736755110221817. \end{aligned}$$

Note that $[DF^\phi(D_\phi)]$ and $[\pi_{x,y}F(\bar{B}_c^R \times \{0\} \times \{0\})]$ in (5.31), (5.32) are displayed with very rough accuracy. The above numbers follow from their precise version from the CAPD software.

From (2.8) we have obtained the bounds c^u, c^s (see (2.11)) using the following simple estimates. Our matrix \mathbf{C} from (2.8) is of the form (see (5.31))

$$\mathbf{C} = \begin{pmatrix} \varepsilon_1 & \mathbf{r}_1 \\ \mathbf{r}_2 & \varepsilon_2 \end{pmatrix}.$$

For any matrix $C = \begin{pmatrix} \varepsilon_1 & r_1 \\ r_2 & \varepsilon_2 \end{pmatrix} \in \mathbf{C}$ and any $\theta = (\theta_1, \theta_2)$ for which $\|\theta\| = 1$, using

$$-\frac{1}{2} = -\frac{\theta_1^2 + \theta_2^2}{2} \leq \theta_1\theta_2 \leq \frac{\theta_1^2 + \theta_2^2}{2} = \frac{1}{2},$$

we have

$$(7.2) \quad \begin{aligned} &\theta^T C \theta \\ &= (r_1 + r_2)\theta_1\theta_2 + \varepsilon_1\theta_1^2 + \varepsilon_2\theta_2^2 \\ &\in \left[-\max_{r_1 \in \mathbf{r}_1, r_2 \in \mathbf{r}_2} \frac{|r_1 + r_2|}{2} + \min_{\varepsilon_i \in \boldsymbol{\varepsilon}_i, i=1,2} \varepsilon_i, \max_{r_1 \in \mathbf{r}_1, r_2 \in \mathbf{r}_2} \frac{|r_1 + r_2|}{2} + \max_{\varepsilon_i \in \boldsymbol{\varepsilon}_i, i=1,2} \varepsilon_i \right]. \end{aligned}$$

The bound (7.2) is easily computable using interval arithmetic and (5.31):

$$(7.3) \quad c^u = 0.08051326990530001, \quad c^s = -0.08047205618419317.$$

Here once again the very rough rounding in (5.31) is evident when compared with (7.3).

Estimates (7.1), (7.3) give us

$$(7.4) \quad \begin{aligned} \kappa_c^{\text{forw}} &= 0.3233437648409335, & \kappa_c^{\text{back}} &= -0.3233025511198267, \\ \kappa_u^{\text{forw}} &= 2.289076044785254, & \kappa_u^{\text{back}} &= 2.191558356326563, \\ \kappa_s^{\text{forw}} &= -2.191555556894562, & \kappa_s^{\text{back}} &= -2.289087997924652. \end{aligned}$$

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