SIAM J. APPLIED DYNAMICAL SYSTEMS
Vol. 16, No. 3, pp. 1453–1473

Computer-Assisted Proof of Shil'nikov Homoclinics: With Application to the Lorenz-84 Model*

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Abstract. We present a methodology for computer-assisted proofs of Shil'nikov homoclinic intersections. It is based on geometric bounds on the invariant manifolds using rate conditions, and on propagating the bounds by an interval arithmetic integrator. Our method ensures uniqueness of the parameter for which the homoclinic takes place. We apply the method for the Lorenz-84 atmospheric circulation model, obtaining a sharp bound for the parameter, and also for where the homoclinic intersection of the stable/unstable manifolds takes place.

Key words. Shil'nikov homoclinic, invariant manifolds, nontransversal intersections, computer-assisted proofs

AMS subject classifications. 34C37, 37D05, 37D10, 65G20

DOI. 10.1137/16M1079956

1. Introduction.

A class of three-dimensional systems with a homoclinic orbit for a three-dimensional saddle-focus equilibrium point was studied by Shil'nikov in a series of papers (see, for example, [38, 39, 40]). The homoclinic (usually called the Shil'nikov homoclinic orbit) can bifurcate in a simple as well as in a chaotic way. The type of bifurcation depends on the saddle quantity, a constant derived from the eigenvalues of the linearized vector field at the fixed point. If the saddle quantity is negative, then a unique and stable limit cycle bifurcates from the homoclinic orbit. (This is called the simple Shil'nikov bifurcation.) If it is positive, then there occurs infinitely many periodic orbits of saddle type, and one speaks of the chaotic Shil'nikov bifurcation (see also [27]). Shil'nikov homoclinics are important, since they lead to interesting dynamics. For instance, when combined with the study of the separatrix value, one can infer from them the existence of a Lorenz-type attractor in the system [41].

Detecting Shil'nikov homoclinic intersections analytically is difficult, since in most systems of interest the ODE does not have a closed-form solution. In this paper we present a computer-assisted approach for such proofs. The method is based on computer-assisted estimates on the stable and unstable manifolds and their propagation using a rigorous, interval arithmetic integrator along the flow. To apply our method, one first needs a good numerical understanding of the problem. In particular, one needs to investigate the shape of the manifolds in question and find the approximate value of the parameter for which their intersection exists. Our method can be viewed as an a posteriori validation that the true intersection parameter is

*Received by the editors June 14, 2016; accepted for publication (in revised form) by H. Kokubu May 19, 2017; published electronically August 3, 2017.
http://www.siam.org/journals/siads/16-3/M107995.html
Funding: This research was partially supported by the Polish National Science Center grants 2015/19/B/ST1/01454 and 2016/21/B/ST1/02453.
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contained within an interval enclosure of the numerically established value.

Our estimates for the invariant manifolds are based on the method of *rate conditions* from [9, 10]. These are related to the rate conditions of Fenichel [12, 13, 14, 15]. The difference is that we derive our rate conditions based on the estimates on the derivative at a (large) neighborhood of a normally hyperbolic manifold (in this paper this manifold will be a family of hyperbolic fixed points), and not at the manifold as is done by Fenichel. Since our estimates are more global, we are able to establish existence and obtain explicit bounds on the invariant manifolds within the investigated neighborhood.

The bounds on the manifolds are then propagated along the flow using an interval arithmetic integrator. For the proof of a homoclinic intersection, we use a standard shooting argument, which is based on the Bolzano intermediate value theorem. We also keep track of the dependence of the manifolds on the parameter, which leads to a uniqueness argument for the intersection.

In our approach we use cone-type estimates that follow from rate conditions [9, 10] to establish rigorous bounds on the position of the manifolds, their slope, and their dependence on the parameter. There is another approach for doing this, which is based on the parameterization method [6, 7, 8, 22, 23, 24]. This method provides a powerful tool for numerical computation of stable/unstable manifolds of invariant objects. It can be applied to hyperbolic fixed points (which would fit the setting of our paper) or to normally hyperbolic invariant manifolds. The method is also suitable for computer-assisted proofs and has been successfully applied in the context of hyperbolic fixed points [2, 29, 33, 34], periodic orbits [3, 5, 18], or whiskered tori [16, 17]. (Here we have made just a short selection of related papers; a good overview of the references associated with the method can be found in the monograph [22].)

To demonstrate that our method is applicable we implement it for the Lorenz-84 system [30]. We make a list of conditions that need to be verified in order to obtain the existence and uniqueness of the intersection, and then validate them. The bounds obtained by us are quite sharp. We establish the intersection parameter with 10^{-9} order of accuracy, and the region where the intersection takes place with 10^{-7} order of accuracy. The Lorenz-84 model serves only as an example. Our method is general and can be applied to other systems.

The only other computer-assisted proof of Shil'nikov homoclinics known to us is the work of Wilczak [43]. This method uses a topological shadowing mechanism, which stems from the method of *covering relations* [19, 20] (referred to also in the literature as "correctly aligned windows") and Lyapunov function–type arguments close to the fixed points. Our method is different. We rely on explicit estimates on the manifolds and their slopes, which are derived from rate conditions. Our method implies that the intersection parameter is unique within the given range. The uniqueness was not investigated in [43]. In [43] it is shown that in the investigated system there is an infinite number of Shil'nikov homoclinics, which are derived from symbolic dynamics. We focus on a simpler setting where the intersection is unique.

The paper is organized as follows. Section 2 contains preliminaries. In section 3 we introduce the Lorenz-84 model. Section 4 contains the proof for Shil'nikov-type bifurcations. The proof is based on an assumption that within the investigated neighborhood of the family of hyperbolic fixed points we have estimates on their invariant manifolds. We discuss how to obtain such estimates in section 5. This is based on the *rate conditions* method from [10, 9], adapted to our setting. In section 6 we extend the method to obtain bounds on the dependence of the manifolds on the parameter of the system. Finally, in section 7, we apply our method for the Lorenz-84 system.

2. Preliminaries.

2.1. Notations. Throughout the paper, all norms that appear are standard Euclidean norms. We use the notation $B_k(p,R)$ to denote a ball in $\mathbb{R}^k$ of radius R centered at p. We use the shorthand notation $B_k(R)$ for a ball or radius R in $\mathbb{R}^k$ centered at zero. For a set $A \subset \mathbb{R}^k$, we use $\overline{A}$ to denote its closure and ∂A for its boundary, $\mathrm{int}A$ for its interior, and A^c for the complement. For a point $p = (x, y)$ we use $\pi_x p$ and $\pi_y p$ to denote projections onto x and y coordinates, respectively. We use the notation $(v|w)$ to denote the scalar product between two vectors v and w.

2.2. Interval Newton method. Let X be a subset of $\mathbb{R}^n$. We shall denote by $[X]$ an interval enclosure of the set X, that is, a set

$$[X] = \Pi_{i=1}^n [a_i, b_i] \subset \mathbb{R}^n,$$

such that

$$X \subset [X].$$

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function and $U \subset \mathbb{R}^n$. We shall denote by $[Df(U)]$ the interval enclosure of a Jacobian matrix on the set U. This means that $[Df(U)]$ is an interval matrix defined as

$$[Df(U)] = \left\{ A \in \mathbb{R}^{n\times n} | A_{ij} \in \left[\inf_{x\in U} \frac{\partial f_i}{\partial x_j}(x), \sup_{x\in U} \frac{\partial f_i}{\partial x_j}(x) \right] \text{ for all } i,j = 1,\ldots,n \right\}.$$

Theorem 1 (interval Newton method; see [1]). *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1 function, and let $X = \Pi_{i=1}^n [a_i, b_i]$ with $a_i < b_i$. If $[Df(X)]$ is invertible and there exists an x_0 in X such that*

$$N(x_0, X) := x_0 - [Df(X)]^{-1} f(x_0) \subset X,$$

then there exists a unique point $x^* \in X$ such that $f(x^*) = 0$.

Consider now $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, with the notation $f(x,\theta) \in \mathbb{R}^n$ for $x \in \mathbb{R}^n$ and $\theta \in \mathbb{R}$. For $U \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$ we shall use the notation $[D_x f(U, I)]$ for an $n \times n$ interval matrix of the form

$$[D_x f(U,I)] = \left\{ A \in \mathbb{R}^{n\times n} | A_{ij} \in \left[\inf_{x\in U,\theta\in I} \frac{\partial f_i}{\partial x_j}(x,\theta), \sup_{x\in U,\theta\in I} \frac{\partial f_i}{\partial x_j}(x,\theta) \right] \text{ for } i,j = 1,\ldots,n \right\}.$$

Below theorem is a well-known modification (see, for instance, [36, p. 376]) of the interval Newton method, which includes a parameter.

Theorem 2. *Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a C^1 function, let $X = \Pi_{i=1}^n [a_i, b_i] \subset \mathbb{R}^n$, with $a_i < b_i$, and let $I = [c, d] \subset \mathbb{R}$, with $c < d$. Consider $x_0 \in \mathrm{int}X$ and*

$$N(x_0, X, I) = x_0 - [D_x f(X, I)]^{-1} [f(x_0, I)].$$

If

$$N\left(x_0, X, I\right) \subset \mathrm{int} X,$$

then there exists a function $p : I \to X$ *such that* $F\left(p\left(\theta\right), \theta\right) = 0$.

Remark 3. *By the implicit function theorem,* $p\left(\theta\right)$ *is as smooth as* f.

2.3. Interval arithmetic enclosure for eigenvalues and eigenvectors. The interval Newton method can be applied to find the eigenvalues and eigenvectors of a matrix.

Let A be an $n \times n$ real matrix. In this section we outline how to solve

$$Ax = \lambda x. \tag{1}$$

We consider two cases. In the first, both λ and x will be real, and in the second complex:

$$\begin{aligned} \lambda &= \rho + i\omega, \\ x &= x_{\mathrm{re}} + i x_{\mathrm{im}}. \end{aligned}$$

In the first case, we fix the first coordinate x_1 of $x = (x_1, \tilde{x})$ and treat $\tilde{x} \in \mathbb{R}^{n-1}$ as a variable. (We can also set some other coordinate to be fixed if needed.) We define $f : \mathbb{R}^n \to \mathbb{R}^n$ as

$$f\left(\lambda, \tilde{x}\right) = Ax - \lambda x.$$

We see that solving $f\left(\lambda, \tilde{x}\right) = 0$ is equivalent to (1). A solution of $f\left(\lambda, \tilde{x}\right) = 0$ can be established using the interval Newton method (Theorem 1).

In the second case, we can consider $x_{\mathrm{re}} = (x_{\mathrm{re},1}, \tilde{x}_{\mathrm{re}})$ and $x_{\mathrm{im}} = (x_{\mathrm{im},1}, \tilde{x}_{\mathrm{im}})$, treating $\tilde{x}_{\mathrm{re}}, \tilde{x}_{\mathrm{im}} \in \mathbb{R}^{n-1}$ as variables and $x_{\mathrm{re},1}, x_{\mathrm{im},1}$ as fixed parameters. (We can also fix some other coordinate than the first if needed.) We can consider $f : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined as

$$f\left(\rho, \tilde{x}_{\mathrm{re}}, \omega, \tilde{x}_{\mathrm{im}}\right) = \begin{pmatrix} Ax_{\mathrm{re}} - \rho x_{\mathrm{re}} + \omega x_{\mathrm{im}} \\ Ax_{\mathrm{im}} - \rho x_{\mathrm{im}} - \omega x_{\mathrm{re}} \end{pmatrix}.$$

Clearly $f\left(\rho, \tilde{x}_{\mathrm{re}}, \omega, \tilde{x}_{\mathrm{im}}\right) = 0$ is equivalent to (1), and the solution can again be established using the interval Newton method.

2.4. Linear approximation of solutions of ODEs. In this section we present a technical lemma. Consider an ODE

$$p' = f(p),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Let Φ_t be the flow of the above system.

Lemma 4. *Let* $U \subset \mathbb{R}^n$ *be a convex compact set. Then there exists a constant* $M > 0$ *such that for any* $t > 0$ *and any* $p, q \in \mathbb{R}^n$ *satisfying*

$$\{\Phi_{-s}(p), \Phi_{-s}(q) : s \in [0, t]\} \subset U,$$

we have

$$\Phi_{-t}(p) - \Phi_{-t}(q) = p - q - tC(p-q) + g(t,p,q)$$

for some matrix $C \in [Df(U)]$ (which can depend on p, q, and t)and some g satisfying

$$\|g(t,p,q)\| \leqslant Mt^2 \|p-q\|.$$

Proof. The proof is given in the appendix. ■

Remark 5. *In Lemma 4 we move backwards in time along the flow. We set this up in this way, because later on in our application we will use the lemma in the context of unstable manifolds, where, moving back in time along the flow, we will converge towards a fixed point.*

2.5. Logarithmic norms. Let us begin by defining some matrix functionals that will be used in further proofs. Let $\|\cdot\|$ be a given norm in $\mathbb{R}^n$. Let $A \in \mathbb{R}^{n\times n}$ be a square matrix. By $m(A)$ we will denote the following matrix functional:

$$m(A) = \min_{z\in\mathbb{R}^n, \|z\|=1} \|Az\|.$$

Definition 6. *The* logarithmic norm of A, *denoted by* $l(A)$ *in* [32, 11, 21, 28], *is defined as*

$$l(A) = \lim_{h\to 0^+} \frac{\|I+hA\| - \|I\|}{h}. \tag{2}$$

Moreover

$$m_l(A) = \lim_{h\to 0^+} \frac{m(I+hA) - \|I\|}{h} \tag{3}$$

will be called the logarithmic minimum of A.

Lemma 7. *If $\|\cdot\|$ is the Euclidean norm, then the following equalities hold:*

$$l(A) = \max\{\lambda \in \textit{spectrum of } (A+A^\top)/2\}, \tag{4}$$

$$m_l(A) = \min\{\lambda \in \textit{spectrum of } (A+A^\top)/2\}. \tag{5}$$

Remark 8. *Equality* (4) *is a well-known result (see, for instance,* [21]*). Equation* (5) *is proven in* [9].

Corollary 9. *From Lemma 7, we see that $m_l(-A) = -l(A)$.*

Lemma 10 (see [9]). *Consider the Euclidean norm $\|\cdot\|$. Let $W \subset \mathbb{R}^{n\times n}$ be a compact set, and let $t_0 > 0$. Then for any $t \in (0, t_0]$ and $A \in W$ the following equality holds:*

$$\|I + tA\| = 1 + tl(A) + r(t,A),$$

where

$$\|r(t,A)\| \leqslant Ct^2$$

for some constant $C = C(t_0, W)$.

Lemma 11 (see [9]). *Consider the Euclidean norm $\|\cdot\|$. Let $W \subset \mathbb{R}^{n\times n}$ be a compact set, and let $t_0 > 0$. Then for any $t \in (0, t_0]$ and $A \in W$ the following equality holds:*

$$m(I + tA) = 1 + tm_l(A) + r(t, A),$$

where

$$\|r(t, A)\| \leqslant Ct^2$$

for some constant $C = C(t_0, W)$.

3. Lorenz-84 atmospheric circulation model. The Lorenz-84 model was introduced by Lorenz in [30]. It is a low-order model for the long-term atmospheric circulation. It is considered as the simplest model capable of representing the basic features of the so-called Hadley circulation. Therefore, it has been widely used in meteorological studies. The detailed analysis of this model can be found in [42]. The model equations are

$$\begin{cases} \dot{X} = -Y^2 - Z^2 - aX + aF, \\ \dot{Y} = XY - bXZ - Y + G, \\ \dot{Z} = bXY + XZ - Z, \end{cases} \tag{6}$$

where variable X represents the strength of the globally averaged westerly wind current, and variables Y and Z are the strength of the cosine and sine phases of a chain of superposed waves transporting heat poleward. F and G represent the thermal forcing terms, and the parameter b stands for the advection strength of the waves by the westerly wind current. The coefficient a, if less than 1, allows the westerly wind current to damp less rapidly than the waves. The time unit is equal to the damping time of the waves and is estimated to be five days.

In their paper [37], Shil'nikov, Nicolis, and Nicolis carry out a detailed bifurcation analysis for the Lorenz-84 model with parameters a and b set to classical values $\frac{1}{4}$ and 4, respectively (these values were also considered in many other works; see, for example, [4, 30, 31]). The authors identify the types of the equilibrium points depending on the choice of the domain for the parameters F and G. They show that the problem has either one, two, or three equilibrium points. If parameters F and G are chosen from a proper domain, one of the fixed points, denoted in [37] as O_1, is saddle-focus. The paper [37] presents numerical calculations suggesting the existence of the homoclinic orbit passing through O_1 that is possessed by the system for $F \simeq 4.0$ and $G \simeq 0.08$. The homoclinic is depicted in Figure 1.

Following Shil'nikov et al. [37], we set the parameters $a = \frac{1}{4}$ and $b = 4$. In upcoming sections we will use our method to rigorously enclose the stable and unstable manifolds, and to validate the existence of a homoclinic orbit for saddle-focus fixed point O_1. We prove that such an orbit exists for $F = 4$, and some G, where

$$G \in [0.0752761095, 0.07527611625]\,. \tag{7}$$

Moreover, we show the uniqueness of such G in the interval (7).

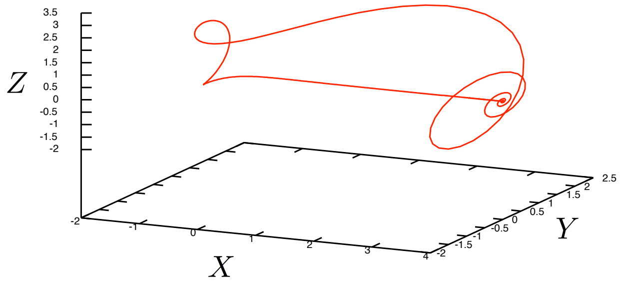

Figure 1. *The Shil'nikov homoclinic in the Lorenz-84 model for $F = 4.0$ and $G \simeq 0.08$.*

4. Establishing Shil'nikov homoclinics. Let us consider the three-dimensional system given by the ODE

$$p' = f(p, \theta), \tag{8}$$

where $f : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is C^1, and $\theta \in \Theta$ is a parameter, with $\Theta = [\theta_l, \theta_r] \subset \mathbb{R}$. Let $\Phi_t(p, \theta)$ be the flow induced by (8).

Suppose that for $\theta \in \Theta$ system (8) has a smooth family of hyperbolic fixed points p_θ^*, with two-dimensional stable and one-dimensional unstable eigenspace.

Below we present a theorem which allows us to prove the existence of a homoclinic orbit in the system. First we need to introduce some notation.

Let $\overline{B}_u(R) = [-R, R] \subset \mathbb{R}$, let $\overline{B}_s(R) \subset \mathbb{R}^2$, and let

$$D = \overline{B}_u(R) \times \overline{B}_s(R) \subset \mathbb{R}^3$$

be a neighborhood of the smooth family of fixed points, meaning that we assume $p_\theta^* \in \mathrm{int} D$ for any $\theta \in \Theta$. The set D will be fixed throughout the discussion. We denote by W_θ^u the local unstable manifold of p_θ^* in D and by W_θ^s the local stable manifold of p_θ^* in D, i.e.,

$$W_\theta^u = \left\{ p \in D : \Phi_t(p, \theta) \in D \text{ for } t \leqslant 0 \text{ and } \lim_{t \to -\infty} \Phi_t(p, \theta) = p_\theta^* \right\}, \tag{9}$$

$$W_\theta^s = \left\{ p \in D : \Phi_t(p, \theta) \in D \text{ for } t \geqslant 0 \text{ and } \lim_{t \to +\infty} \Phi_t(p, \theta) = p_\theta^* \right\}. \tag{10}$$

We assume that W_θ^u and W_θ^s are graphs of the C^1 functions

$$w^u : \overline{B}_u(R) \times \Theta \to \overline{B}_s(R),$$
$$w^s : \overline{B}_s(R) \times \Theta \to \overline{B}_u(R),$$

meaning that (see Figure 2)

$$\begin{aligned} W_\theta^u &= \left\{ (x, w^u(x, \theta)) : x \in \overline{B}_u(R) \right\}, \\ W_\theta^s &= \left\{ (w^s(y, \theta), y) : y \in \overline{B}_s(R) \right\}. \end{aligned} \tag{11}$$

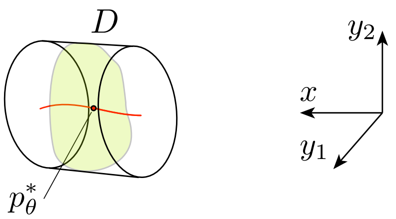

Figure 2. *The local unstable manifold W_θ^u (in red) and the local stable manifold W_θ^s (in green).*

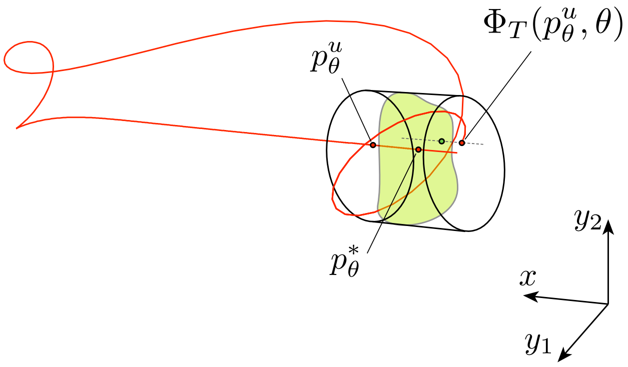

Figure 3. *We have the one-dimensional unstable manifold of p_θ^* (in red) and the two-dimensional local stable manifold W_θ^s in D (in green). The $h(\theta)$ is the signed distance along the x coordinate between W_θ^s and $\Phi_T(p_\theta^u, \theta)$; this is the distance along the dotted line on the plot.*

Let

$$p_\theta^u := (R, w^u(R, \theta)) \in \mathbb{R}^3. \tag{12}$$

Consider $T > 0$ and assume that for all $\theta \in \Theta$, $\Phi_T(p_\theta^u, \theta) \in D$. Let us define

$$h : \Theta \to \mathbb{R}$$

as

$$h(\theta) = \pi_x \Phi_T(p_\theta^u, \theta) - w_\theta^s(\pi_y \Phi_T(p_\theta^u, \theta)). \tag{13}$$

We now state a natural result, that $h(\theta) = 0$ implies an intersection of the stable and unstable manifolds of p_θ^*. (See Figure 3.)

Theorem 12. *If*

$$h(\theta_l) < 0 \qquad \text{and} \qquad h(\theta_r) > 0, \tag{14}$$

then there exists a $\psi \in \Theta$ for which we have a homoclinic orbit to p_ψ^*.

Moreover, if, for all $\theta \in \Theta$, $h'(\theta) > 0$, then ψ is the only parameter for which we have a homoclinic orbit satisfying $\Phi_t(p_\theta^u, \theta) \in D$ for all $t > T$.

Proof. Since w^u, w^s are C^1, so is h. From (14), by the Bolzano intermediate value theorem, it follows that there exists a $\psi \in \Theta$ for which $h(\psi) = 0$. Let $q = \Phi_T(p^u_\psi, \psi)$. Since $h(\psi) = 0$,

$$q = (\pi_x q, \pi_y q) = \left(w^s_\psi(\pi_y q), \pi_y q\right). \tag{15}$$

Since $p^u_\psi \in W^u$, clearly $q = \Phi_T(p^u_\psi, \psi)$ belongs to the unstable manifold of p^*_ψ. All points of the form $(w^s_\psi(y), y)$ belong to the stable manifold of p^*_ψ; hence by (15) so does q, and the stable and unstable manifolds intersect at q.

If $h'(\theta) > 0$ for all $\theta \in \Theta$, then ψ is the only parameter for which h is zero; hence for all $\theta \neq \psi$,

$$\pi_x \Phi_T(p^u_\theta, \theta) \neq w^s_\theta(\pi_y \Phi_T(p^u_\theta, \theta)).$$

By (11), this implies that for $\theta \neq \psi$, $\Phi_T(p^u_\theta, \theta) \notin W^s_\theta$. By (10) this means that for some $t > T$, $\Phi_t(p^u_\theta, \theta) \notin D$, or that we do not have a homoclinic for this parameter. ■

Remark 13. *The inequalities in* (14) *and the sign of* h' *in Theorem* 12 *can be reversed. Then the proof follows from mirror arguments.*

Remark 14. *If the stable and unstable manifolds intersect, then this intersection must be nontransversal. This is because if there is an intersection, then the unstable manifold must lie on the stable manifold. Thus, the tangent vectors to the manifolds do not span* $\mathbb{R}^3$.

To apply Theorem 12, we need to be able to compute estimates on h and its derivative. We note that obtaining a rigorous bound on a time shift map Φ_T along the flow, and on its derivative, can be computed in interval arithmetic using the CAPD[1] package. To compute h and its derivative it is therefore enough to be able to obtain estimates on w^u, w^s, and their derivatives. We discuss how this can be achieved in interval arithmetic in sections 5 and 6. We use these, together with Theorem 12, to provide a computer-assisted proof of a homoclinic intersection in the Lorenz-84 model in section 7.

5. Bounds on unstable manifolds of hyperbolic fixed points.

Consider an ODE

$$q' = f(q), \tag{16}$$

and let

$$D = \overline{B}_u(R) \times \overline{B}_s(R) \subset \mathbb{R}^u \times \mathbb{R}^s.$$

The results of this section are more general than the previously considered ODE in $\mathbb{R}^3$, and here u, s can be any natural numbers. We use a notation $x \in \mathbb{R}^u$ to stand for the unstable coordinate and $y \in \mathbb{R}^s$ for the stable coordinate. For us it will be enough if these coordinates are "roughly" aligned with the eigenspaces of a fixed point. (We do not need to work with precisely linearized local coordinates.) We write $f(x,y) = (f_x(x,y), f_y(x,y))$, where f_x is the projection onto $\mathbb{R}^u$ and f_y is the projection onto $\mathbb{R}^s$.

[1] Computer-assisted proofs in dynamics available as the supplemental file M107995_01.zip [local/web 9.16MB] and also from http://capd.ii.uj.edu.pl/.

Let $L > 0$ be a fixed number. We define

$$\mu_1 = \sup_{z\in D}\left\{ l\left(\frac{\partial f_y}{\partial y}(z)\right) + \frac{1}{L}\left\|\frac{\partial f_y}{\partial x}(z)\right\|\right\},$$
$$\mu_2 = \sup_{z\in D}\left\{ l\left(\frac{\partial f_y}{\partial y}(z)\right) + \frac{1}{L}\left\|\frac{\partial f_x}{\partial y}(z)\right\|\right\},$$
$$\xi = m_l\left(\frac{\partial f_x}{\partial x}(D)\right) - \frac{1}{L}\sup_{z\in D}\left\|\frac{\partial f_x}{\partial y}(z)\right\|.$$

Definition 15. *We say that the vector field f* satisfies rate conditions in D *if*

$$\mu_1 < 0 < \xi, \tag{17}$$

$$\mu_2 < \xi. \tag{18}$$

Definition 16. *We say that $D = \overline{B}_u(R) \times \overline{B}_s(R)$ is* an isolating block for (16) *if the following hold:*

1. *For any $q \in \partial\overline{B}_u(R) \times \overline{B}_s(R)$,*
$$(\pi_x f(q)|\pi_x q) > 0.$$
2. *For any $q \in \overline{B}_u(R) \times \partial\overline{B}_s(R)$,*
$$(\pi_y f(q)|\pi_y q) < 0.$$

Definition 17. *We define the unstable set in D as*

$$W^u = \{z : \ \Phi_t(z) \in D \textit{ for all } t < 0\}.$$

Theorem 18. *Assume that f is C^1 and satisfies rate conditions. Assume also that $D = \overline{B}_u(R) \times \overline{B}_s(R)$ is an isolating block for f. Then the set W^u is a manifold, which is a graph over $\overline{B}_u(R)$. To be more precise, there exists a function*

$$w^u : \overline{B}_u(R) \to \overline{B}_s(R),$$

such that

$$W^u = \left\{(x, w^u(x)) : x \in \overline{B}_u(R)\right\}.$$

Moreover, w^u is Lipschitz with constant L and for $C = 2R(1 + 1/L)$ for any $p_1, p_2 \in W^u$,

$$\|\Phi_{-t}(p_1) - \Phi_{-t}(p_2)\| \leqslant Ce^{-t\xi} \qquad \textit{for all } t > 0. \tag{19}$$

Proof. The result follows directly from Theorem 30 from [9]. Theorem 30 in [9] is written in the context where, apart from x, y, we have an additional "center" coordinate, which is not present here. This is why the number of constants and rate conditions (17)–(18) for Theorem 18 is smaller than the number of constants and associated inequalities needed in [9]. The conditions (17)–(18) imply all the needed assumptions of Theorem 30 from [9] in the absence of the center coordinate. ■

In above theorem we ignore (fix) the parameter. The result can be extended to include the parameter as follows.

Theorem 19. *Consider a parameter-dependent ODE*

$$p' = f(p, \theta)$$

for $\theta \in \Theta$. Assume that the system has a smooth family of hyperbolic fixed points p_θ^*. Assume that for each (fixed) θ, the vector field satisfies the assumptions of Theorem 18. Then the family of unstable manifolds W_θ^u (as defined in (9)) of p_θ^* is given by a graph of a function

$$w^u : \overline{B}_u(R) \times \Theta \to \overline{B}_s(R)$$

(meaning that $W_\theta^u = \left\{(x, w^u(x, \theta)) : x \in \overline{B}_u(R)\right\}$), which is as smooth as f.

Proof. The existence of w^u follows from Theorem 18. We need to justify its smoothness.

From the classical theory (see, for instance, [25, 26, 35]), we know that in a small neighborhood U of $\{(p_\theta^*, \theta)|\theta \in \Theta\}$ (considered in the state space, extended to include the parameter), the family of local unstable manifolds exists and is as smooth as f. Condition (19) ensures that the local manifold is propagated along the flow in the extended space to span the set $D \times \Theta$. Since Φ_t is as smooth as f, this establishes the smoothness of w^u. ■

Remark 20. *In this section we have focused on the unstable manifold. This method can also be applied to obtain bounds on a stable manifold. To do so one can simply change the sign of the vector field.*

6. Dependence of the unstable manifold on parameters. In this section we consider the ODE of the form

$$p' = f(p, \theta) \tag{20}$$

depending on the parameter $\theta \in \Theta$, where $p \in \mathbb{R}^u \times \mathbb{R}^s$ and $f : \mathbb{R}^u \times \mathbb{R}^s \times \Theta \to \mathbb{R}^u \times \mathbb{R}^s$ is a C^1 function, with

$$f(x, y, \theta) = (f_x(x, y, \theta), f_y(x, y, \theta)).$$

Our aim now is to examine the nature of the dependency of function w^u, which parametrizes the unstable manifold in the statement of Theorem 18, on parameter θ.

Let our coordinates be $(x, y, \theta) \in \mathbb{R}^u \times \mathbb{R}^s \times \mathbb{R}$, and let us consider the following sets:

$$\begin{aligned} J_s(q, M) &= \{(x, y, \theta) : \|\pi_{x,\theta} q - (x, \theta)\| \leqslant M \|\pi_y q - y\|\}, \\ J_{cu}(q, M) &= \{(x, y, \theta) : \|\pi_y q - y\| \leqslant M \|\pi_{x,\theta} q - (x, \theta)\|\}, \end{aligned}$$

where $q \in \mathbb{R}^u \times \mathbb{R}^s \times \mathbb{R}$ and $M > 0$. These sets represent cones, depicted in Figure 4. Note that we have

$$(J_{cu}(q, 1/M))^c = \mathrm{int} J_s(q, M). \tag{21}$$

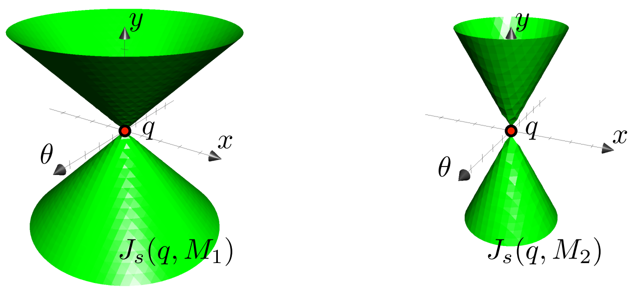

Figure 4. *The cones $J_s(q,M_1)$ and $J_s(q,M_2)$ for $M_1 = 1$ and $M_2 = \frac{1}{2}$.*

Let us consider an ODE given by (20) in the state space extended by parameter, that is,

$$(x', y', \theta') = (f_x(x,y,\theta), f_y(x,y,\theta), f_\theta(x,y,\theta)), \tag{22}$$

where $f_\theta(x,y,\theta) = 0$. Let $\Phi_t(x,y,\theta)$ be the flow induced by (22).

Let $D = \overline{B}_u(R) \times \overline{B}_s(R) \subset \mathbb{R}^u \times \mathbb{R}^s$, and let us define

$$\mathcal{D} = D \times \Theta$$

and the following constants:

$$\mu(M) = l\left(\frac{\partial f_y}{\partial y}(\mathcal{D})\right) + M\left\|\frac{\partial f_y}{\partial(x,\theta)}(\mathcal{D})\right\|, \tag{23}$$

$$\xi(M) = m_l\left(\frac{\partial f_{x,\theta}}{\partial(x,\theta)}(\mathcal{D})\right) - \frac{1}{M}\left\|\frac{\partial f_{x,\theta}}{\partial y}(\mathcal{D})\right\|. \tag{24}$$

Our objective will be to prove the following theorem.

Theorem 21. *Consider that the assumptions of Theorem 19 hold and that $M > 0$ is such that*

$$\mu(M) < 0 \qquad \textit{and} \qquad \xi(M) > \mu(M).$$

Then

$$\left\|\frac{\partial w^u}{\partial\theta}\right\| \leqslant 1/M.$$

The proof of the theorem will be given at the end of the section. To show the result we shall need two technical lemmas.

Lemma 22. *Assume that $M > 0$ is such that*

$$\mu(M) < 0 \qquad \textit{and} \qquad \xi(M) > \mu(M).$$

Then there exist a $c > 0$ and $t_M > 0$ such that for any $q \in \mathcal{D}$ and $p \in J_s(q, M) \cap \mathcal{D}$, $p \neq q$, as long as $\{\Phi_{-t}(p), \Phi_{-t}(q) : t \in [0, t_M]\} \subset \mathcal{D}$, the following inequality holds:

$$\|\pi_y(\Phi_{-t}(p) - \Phi_{-t}(q))\| > (1 + ct)\|\pi_y(p - q)\| \tag{25}$$

for any $t \in (0, t_M)$. Moreover

$$\Phi_{-t}(p) \in J_s(\Phi_{-t}(q), M). \tag{26}$$

Proof. Take any $q \in \mathcal{D}$ and $p \in J_s(q, M) \cap \mathcal{D}$, $p \neq q$, and let $t > 0$ be such that $\{\Phi_{-s}(p), \Phi_{-s}(q) : s \in [0, t]\} \subset \mathcal{D}$.

Since $p \in J_s(q, M)$,

$$\|\pi_{x,\theta}(p - q)\| \leqslant M\|\pi_y(p - q)\|. \tag{27}$$

As a consequence

$$\|p - q\| \leqslant \sqrt{M^2 + 1}\,\|\pi_y(p - q)\|. \tag{28}$$

Therefore since $p \neq q$ we must have

$$\|\pi_y(p - q)\| \neq 0.$$

On the other hand, from Lemma 4 it follows that for some $A \in \left[\frac{\partial f_y}{\partial y}(\mathcal{D})\right]$ and $B \in \left[\frac{\partial f_y}{\partial(x,\theta)}(\mathcal{D})\right]$

$$\begin{aligned}\pi_y(\Phi_{-t}(p) - \Phi_{-t}(q)) &= \pi_y(p - q) - tA\pi_y(p - q) - tB\pi_{x,\theta}(p - q)\\ &\quad + \pi_y g(t, p, q),\end{aligned}$$

where g satisfies $\|g(t, p, q)\| \leqslant \gamma_1 t^2\|p - q\|$ for some constant $\gamma_1 > 0$. Observe that from (28) we have $\|g(t, p, q)\| \leqslant \gamma_1 t^2\|\pi_y(p - q)\|$. From the above, and by using (27) in the second line, Lemma 11 in the third line, Corollary 9 in the fourth line, and (23) in the last line, we obtain

$$\begin{aligned}\|\pi_y(\Phi_{-t}(p) - \Phi_{-t}(q))\| &\geqslant \|(\mathrm{Id} - tA)\pi_y(p - q)\| - t\|B\|\|\pi_{x,\theta}(p - q)\|\\ &\quad - \gamma_1 t^2\|\pi_y(p - q)\|\\ &\geqslant (m(\mathrm{Id} - tA) - tM\|B\|)\|\pi_y(p - q)\|\\ &\quad - \gamma_1 t^2\|\pi_y(p - q)\|\\ &\geqslant (1 + tm_l(-A) - tM\|B\|)\|\pi_y(p - q)\|\\ &\quad - \gamma_2 t^2\|\pi_y(p - q)\|\\ &= (1 + t(-l(A) - M\|B\|))\|\pi_y(p - q)\|\\ &\quad - \gamma_2 t^2\|\pi_y(p - q)\|\\ &\geqslant \left(1 - t\mu(M) - \gamma_2 t^2\right)\|\pi_y(p - q)\|, \end{aligned} \tag{29}$$

where, in light of Lemma 11, the third inequality is satisfied for any $t \in [0, t_0]$, where $t_0 > 0$. Taking a fixed $c \in (0, -\mu(M))$, we see that there exists $t_M > 0$ (independent of p and q) such that for any $t \in (0, t_M)$

$$\left\| \pi_y \left(\Phi_{-t}(p) - \Phi_{-t}(q) \right) \right\| > (1 + tc) \left\| \pi_y (p - q) \right\|,$$

which proves (25).

Again from Lemma 4, we know that for some $A \in \left[\frac{\partial f_{x,\theta}}{\partial (x,\theta)} (\mathcal{D}) \right]$ and $B \in \left[\frac{\partial f_{x,\theta}}{\partial y} (\mathcal{D}) \right]$

$$\begin{aligned} \pi_{x,\theta} \left(\Phi_{-t}(p) - \Phi_{-t}(q) \right) &= \pi_{x,\theta}(p - q) - tA\pi_{x,\theta}(p - q) - tB\pi_y(p - q) \\ &\quad + \pi_{x,\theta} g(t, p, q). \end{aligned}$$

Hence, using (27) in the second line, Lemma 10 in the third line, Corollary 9 in the fourth line, and (24) in the last line,

$$\begin{aligned} \left\| \pi_{x,\theta} \left(\Phi_{-t}(p) - \Phi_{-t}(q) \right) \right\| &\leqslant \left\| \mathrm{Id} - tA \right\| \left\| \pi_{x,\theta}(p - q) \right\| + t \left\| B \right\| \left\| \pi_y (p - q) \right\| \\ &\quad + \gamma_1 t^2 \left\| \pi_y(p - q) \right\| \\ &\leqslant \left(\left\| \mathrm{Id} - tA \right\| M + t \left\| B \right\| \right) \left\| \pi_y(p - q) \right\| \\ &\quad + \gamma_1 t^2 \left\| \pi_y(p - q) \right\| \\ &= M \left((1 + tl(-A)) M + \frac{1}{M} t \left\| B \right\| \right) \left\| \pi_y (p - q) \right\| \\ &\quad + \gamma_2 t^2 \left\| \pi_y(p - q) \right\| \\ &= M \left((1 - tm_l(A)) + \frac{1}{M} t \left\| B \right\| \right) \left\| \pi_y (p - q) \right\| \\ &\quad + \gamma_2 t^2 \left\| \pi_y(p - q) \right\| \\ &\leqslant \left(M - tM\xi(M) + \gamma_2 t^2 \right) \left\| \pi_y (p - q) \right\|, \end{aligned} \tag{30}$$

where, in light of Lemma 10, the third inequality is satisfied for any $t \in [0, t_0]$, where $t_0 > 0$. Since $\xi(M) > \mu(M)$, by combining (29) with (30), we see that for sufficiently small t,

$$\frac{\left\| \pi_{x,\theta} \left(\Phi_{-t}(p) - \Phi_{-t}(q) \right) \right\|}{\left\| \pi_y \left(\Phi_{-t}(p) - \Phi_{-t}(q) \right) \right\|} \leqslant \frac{\left(M - tM\xi(M) + \gamma_2 t^2 \right) \left\| \pi_y(p - q) \right\|}{\left(1 - t\mu(M) - \gamma_2 t^2 \right) \left\| \pi_y(p - q) \right\|} \leqslant M.$$

This means that

$$\left\| \pi_{\theta,x} \left(\Phi_{-t}(p) - \Phi_{-t}(q) \right) \right\| \leqslant M \left\| \pi_y \left(\Phi_{-t}(p) - \Phi_{-t}(q) \right) \right\|,$$

which proves (26). ■

We now return to studying (20). Let us assume that the system has a smooth family of hyperbolic fixed points $p_\theta^* \in \mathrm{int} D$, where $D = \overline{B}_u(R) \times \overline{B}_s(R)$. Let us also assume that for any given $\theta \in \Theta$ the assumptions of Theorem 18 are satisfied. Let w^u be the parameterization from Theorem 19.

Lemma 23. *If the assumptions of Theorem 19 are satisfied and*

$$\mu(M) < 0, \qquad \xi(M) > \mu(M),$$

then for any $x_1, x_2 \in \overline{B}_u(R)$ *and* $\theta_1, \theta_2 \in \Theta$,

$$(x_1, w^u(x_1,\theta_1), \theta_1) \in J_{cu}((x_2, w^u(x_2,\theta_2), \theta_2), 1/M). \tag{31}$$

Proof. Let $q_1 = (x_1, w^u(x_1,\theta_1), \theta_1)$ and $q_2 = (x_2, w^u(x_2,\theta_2), \theta_2)$. If (31) does not hold, then by (21)

$$q_1 \in \mathrm{int} J_s(q_2, M).$$

Note that then

$$0 \leqslant \|\pi_{x,\theta}(q_1 - q_2)\| < M \|\pi_y(q_1 - q_2)\|.$$

By Lemma 22, since $\Phi_{-t}(q_i) \in W^u_{\theta_i} \times \{\theta_i\} \subset D \times \Theta$, we would therefore have

$$\Phi_{-t}(q_1) \in J_s(\Phi_{-t}(q_2), M) \tag{32}$$

for all $t \in \mathbb{R}_+$ (we can apply Lemma 22 with small t several times to obtain (32) for large t). Also by Lemma 22 we would have

$$\|\pi_y(\Phi_{-t}(q_1) - \Phi_{-t}(q_2))\| > (1 + ct)\|\pi_y(q_1 - q_2)\| \to \infty \qquad \text{as } t \to \infty.$$

This contradicts the fact that $\Phi_{-t}(p), \Phi_{-t}(q) \in \mathcal{D}$, and hence (31) must hold true. ■

We are now ready to prove Theorem 21.

Proof of Theorem 21. By Theorem 19, w^u is well defined. By Lemma 23,

$$\|w^u(x,\theta_1) - w^u(x,\theta_2)\| \leqslant 1/M \|(x,\theta_1) - (x,\theta_2)\| = 1/M \|\theta_1 - \theta_2\|,$$

which implies the claim. ■

7. Computer-assisted proof of the Shil'nikov connection in the Lorenz-84 system. To apply our method and conduct a computer-assisted proof we follow these steps:

1. Using Theorem 2, establish an enclosure of the family of hyperbolic fixed points, and following the method from section 2.3, establish bounds on the eigenvalues of the Jacobian at the fixed points to verify hyperbolicity.
2. In local coordinates around the fixed points, using Theorem 19, establish the bounds on the unstable manifolds.
3. By changing the sign of the vector field, using the same procedure as in step 2, establish bounds on the stable manifolds.
4. Using Theorem 21, establish bounds on the dependence of the manifolds on the parameter.

5. Propagate the bounds on the unstable manifold along the flow and establish the homoclinic intersection using Theorem 12.

For our computer-assisted proof we consider the Lorenz-84 system (6) with the parameters $a = \frac{1}{4}$, $b = 4$, $F = 4$, and

$$G \in [G_l, G_r] = [0.0752761095, 0.07527611625]\,. \tag{33}$$

The G will play the role of the parameter θ from earlier sections.

We first use the interval Newton method (Theorem 2) to establish an enclosure of the fixed points:

$$p_G^* \in \begin{pmatrix} [3.9999144633, 3.9999144654] \\ [-0.0008521960, -0.0008521939] \\ [0.0045450712, 0.0045450733] \end{pmatrix} \qquad \text{for all } G \in [G_l, G_r]\,.$$

Next we compute a bound on the derivative of the vector field at the fixed points, and using the method from section 2.3 we establish that for all $G \in [G_l, G_r]$ the eigenvalues are

$$\begin{aligned} \lambda_1 &\in [0.249988, 0.249991]\,, \\ \mathrm{Re}\lambda_2 &\in [-2.999911, -2.999908]\,, \qquad \mathrm{Im}\lambda_2 \in [15.999657, 15.999660]\,, \\ \mathrm{Re}\lambda_3 &\in [-2.999911, -2.999908]\,, \qquad \mathrm{Im}\lambda_3 \in [-15.999660, -15.999657]\,. \end{aligned}$$

This establishes hyperbolicity.

To obtain bounds for the stable/unstable manifolds, we use the local coordinates (x, y_1, y_2),

$$(X, Y, Z) = C\,(x, y_1, y_2) + q_0,$$

with

$$q_0 = (3.9999144643281, -0.00085219497131102, 0.0045450722448356)\,,$$
$$C = \begin{pmatrix} 1 & -0.00016604653053618 & 0.00040407899883959 \\ 0.00016384655297642 & -0.28235213046095 & 0.71764786953905 \\ -0.0011562746220118 & 0.71764798264861 & 0.28235189601999 \end{pmatrix}.$$

The q_0 is close to the fixed points of (6). (Depending on the choice of G the fixed point shifts slightly with the parameter, but we keep q_0 fixed.) Coordinates x, y_1, y_2 align the system so that x is the (rough) unstable direction, and y_1, y_2 are (roughly) stable. Note that we use the same local coordinates for all the parameters G from (33).

In these local coordinates, we use the interval Newton method (Theorem 2) to obtain enclosures of the fixed points for parameters G in (33). In the local coordinates, the fixed points are close to the origin. (See Figure 5; the cones emanate from the fixed points.) We then choose

$$D = \overline{B}_u\,(R) \times \overline{B}_s\,(R)\,,$$

with $R = 10^{-4}$, and use Theorem 19 to obtain an enclosure of the unstable manifold W^u. In our computer-assisted proof, we have a Lipschitz bound $L_u = 10^{-5}$ for the slope of the

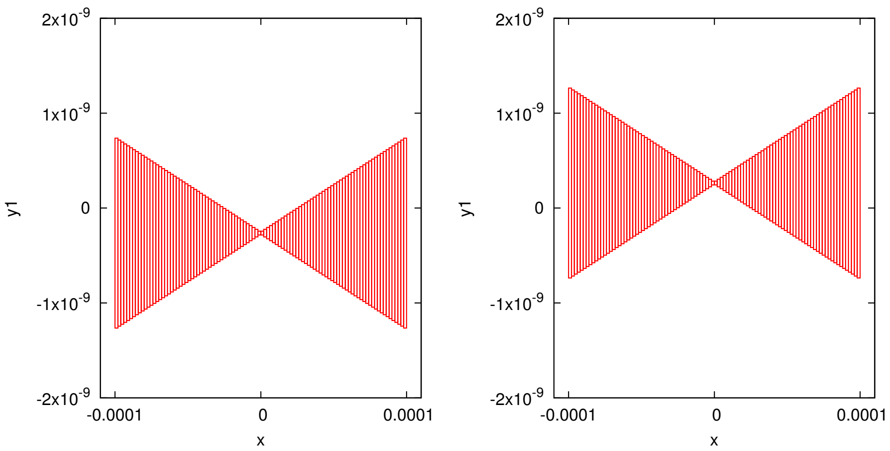

Figure 5. *The projection onto the x, y_1 coordinates of the bounds on W_G^u. On the left we have the bound for $G = G_l$ (the left end of our parameter interval (33)), and on the right for $G = G_r$.*

unstable manifold for all parameters (33). See Figure 5. (Note the scale on the axes. The enclosure is in fact quite sharp.)

To establish the bounds for the stable manifold, we consider the vector field with reversed sign (which makes the stable manifold become unstable; we therefore also swap the roles of the coordinates) and apply Theorem 19 once again. Here we have obtained a Lipschitz bound $L_s = 10^{-3}$. In Figure 6 we see the bound on the enclosure. The two points on the plot are the $\Phi_T(p_G^u, G)$ for $G = G_l$ and $G = G_r$ for the choice of $T = 50$ (see (12) for the definition of p_G^u). We do not plot these as boxes, since our computer-assisted bound gives their size of order 10^{-13}, and such boxes would be invisible on the plot. Note that Figure 6 corresponds to the sketch from Figure 3. In Figure 6 we have the projection onto the x, y_1 coordinates of what happens inside of the set D, without plotting the trajectory along the unstable manifold.

We use the rigorous estimates for $\Phi_T(p_{G_l}^u, G_l)$ and $\Phi_T\left(p_{G_r}^u, G_r\right)$ to compute the following bounds (see (13) for the definition of the function h):

$$h(G_l) \in [1.193520892609e - 07, 1.2017042212622e - 07],$$
$$h(G_r) \in [-1.1920396632516e - 07, -1.1838527119022e - 07].$$

We also make sure that $\Phi_T(p_G^u, G) \in D$ for all $G \in [G_l, G_r]$. We see that assumption (14) of Theorem 12 is satisfied, which means that we have a Shil'nikov homoclinic connection for at least one of the parameters $G \in [G_l, G_r]$.

To establish the bound on $h'(G)$, we first use Theorem 21 to establish an estimate for $\frac{\partial}{\partial G} w^u(x, G)$. In our computer-assisted proof we use Theorem 21 with parameter $M = 2$. We then use Theorem 21 once again (with $M = 2$) to establish bounds for $\frac{\partial}{\partial G} w^s(x, G)$. (We reverse the sign of the vector field to make the manifold unstable, and we swap the roles of stable and unstable coordinates.) We then propagate the bound for $\frac{\partial}{\partial G} w^u(x, G)$ using rigorous, computer-assisted integration, to obtain the bound

$$h'(G) \in [-38.19, -32.49] \qquad \text{for all } G \in [G_l, G_r].$$

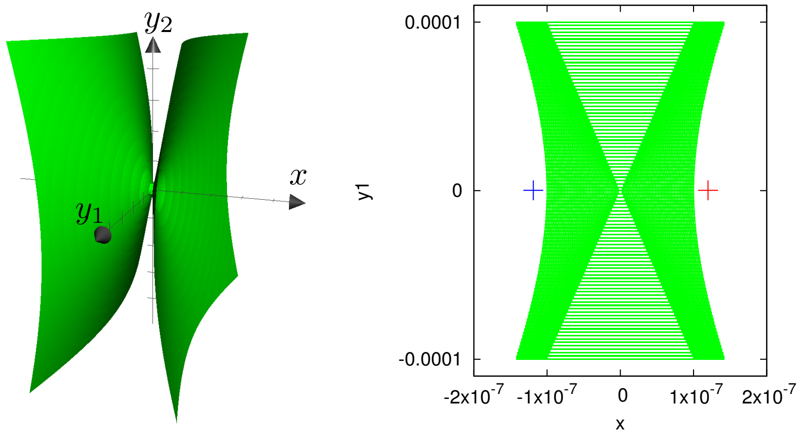

Figure 6. *The bound on W_G^s for all parameters G from* (33)*. On the left we have a nonrigorous plot to illustrate the shape of our bound in three dimensions. On the right, we have a projection onto the x, y_1 coordinates of the rigorous, computer-assisted enclosure. The two points depicted in the right plot are $\Phi_T\left(p^u_{G_l}, G_l\right)$ (on the right, in red) and $\Phi_T\left(p^u_{G_r}, G_r\right)$ (on the left, in blue).*

This, by Theorem 12, establishes the uniqueness of the intersection parameter in $[G_l, G_r]$.

Remark 24. *We do not rule out the possibility that for some parameter $G \in [G_l, G_r]$ the trajectory $\Phi_t\left(p^u_G, G\right)$ could exit D and return again to intersect W^s_G. We have not made such an investigation, which would require a global consideration of the system. What we establish is that we have a single parameter for which the homoclinic orbit behaves as the one in Figure 1.*

The computer-assisted proof was done entirely using the CAPD[2] package and took 4 seconds on a single-core 3GHz Intel i7 processor.

Appendix.

Proof of Lemma 4. Let us take any $t > 0$ and any $p, q \in \mathbb{R}^n$ such that $\{\Phi_{-s}(p), \Phi_{-s}(q) : s \in [0, t]\} \subset U$. Observe that since U is convex,

$$f(p) - f(q) = \int_0^1 Df\left(q + u(p-q)\right) du\, (p-q).$$

Using this, we have

$$\begin{aligned}\Phi_{-t}(p) - \Phi_{-t}(q) &= p - q - \int_0^t f\left(\Phi_{-s}(p)\right) - f\left(\Phi_{-s}(q)\right) ds\\ &= p - q\\ &\quad - \int_0^t \int_0^1 Df\left(\Phi_{-s}(q) + u\left(\Phi_{-s}(p) - \Phi_{-s}(q)\right)\right) du \left(\Phi_{-s}(p) - \Phi_{-s}(q)\right) ds\\ &= p - q - \int_0^t C(s)\left(\Phi_{-s}(p) - \Phi_{-s}(q)\right) ds, \qquad (34)\end{aligned}$$

[2]Computer-assisted proofs in dynamics available as the supplemental file M107995_01.zip [local/web 9.16MB] and also from http://capd.ii.uj.edu.pl/.

where $\{C(s)\}$ is a family of matrices defined as

$$C(s) = \int_0^1 Df\left(\Phi_{-s}(q) + u\left(\Phi_{-s}(p) - \Phi_{-s}(q)\right)\right) du \in [Df(U)].$$

Since f is C^1 in U and U is compact, there exists a constant $L > 0$ such that for any $p \in U$

$$\|Df(p)\| \leqslant L.$$

Using standard Gronwall estimates gives that

$$\Phi_{-s}(p) - \Phi_{-s}(q) = p - q + h(s, p, q), \tag{35}$$

where h satisfies

$$\|h(s, p, q)\| \leqslant \left(e^{sL} - 1\right) \|p - q\|.$$

We can return to (34) and substitute (35) into the term under the integral to obtain

$$\begin{aligned}\Phi_{-t}(p) - \Phi_{-t}(q) &= p - q - \int_0^t C(s)\left(p - q + h(s, p, q)\right) ds \\ &= p - q - tC(p - q) + g(t, p, q)\end{aligned}$$

for

$$C := \frac{1}{t}\int_0^t C(s) ds \in [Df(U)],$$

and

$$g(t, p, q) := -\int_0^t C(s) h(s, p, q)\, ds.$$

Observing that

$$\begin{aligned}\|g(t, p, q)\| &\leqslant \max_{s\in[0,t]} \|C(s)\| \|p - q\| \int_0^t \left(e^{sL} - 1\right) ds \\ &= \max_{s\in[0,t]} \|C(s)\| \|p - q\| \frac{1}{L}\left(e^{Lt} - Lt - 1\right) \\ &\leqslant M \|p - q\| t^2\end{aligned}$$

gives the claim. ■

Acknowledgments. We are grateful to D. Wilczak for his continuous support regarding the use of the CAPD library. We also thank the two anonymous reviewers for their suggestions and comments, which helped us to improve the manuscript.

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