

Characterising blenders via covering relations and cone conditions

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Abstract

A diffeomorphism exhibits a blender if it has an invariant hyperbolic set with the C^1 -robust property that its stable or unstable manifold behaves as a higher-dimensional set than is expected from the underlying hyperbolic splitting. We present a characterisation of a blender based on the topological alignment of certain sets in phase space in combination with the propagation of cones. Importantly, the required conditions can be verified by checking properties of a single iterate of the diffeomorphism, which is achieved by finding finite series of sets that form suitable sequences of alignments. This characterisation is applicable to blenders formed by manifolds of any dimension, in ambient phase spaces of any dimension. Moreover, this approach naturally extends to identifying blenders that are part of heterodimensional cycles, which provides us with the computational tool to establish their robustness. Our setup is flexible and allows for a rigorous, computer-assisted validation based on interval arithmetic. By way of demonstration, we present a computer-assisted proof of the existence of a blender in a three-dimensional Hénon-like family of diffeomorphisms over a considerable range of the relevant parameter.

Keywords: Partial hyperbolicity, diffeomorphism, global invariant manifold, Hénon-like map, heterodimensional cycles

MSC Classification: 37M21 , 37D30 , 65G20 , 37C29 , 37B20

1 Introduction

The notion of a *blender* was introduced by Bonatti and Díaz to construct examples of diffeomorphisms where the existence of a heterodimensional cycle between two saddle fixed or periodic points of different index (dimension of the unstable manifold) is a C^1 -robust property [3]. This robustness is surprising, because the lower-dimensional unstable manifold of one of the saddle points must intersect the stable manifold of the other saddle point in the heterodimensional cycle, which is also the lower dimensional one, meaning that the intersection can only be a non-transverse tangency. In colloquial terms, a blender is a hyperbolic set Λ for which, say, its stable manifold $W^s(\Lambda)$ is embedded in phase space in a way that it behaves as a set of a dimension larger than expected; consequently, if the two saddles from the heterodimensional cycle are contained in Λ then $W^s(\Lambda)$ cannot be avoided by the unstable manifold of the either saddle point in the heterodimensional cycle, even when a C^1 -perturbation is applied. In this paper, we use the technical definition of a blender from [3, 4], which will be introduced formally in Section 2.2. Note that blenders can only exist if the diffeomorphism has a phase space of dimension at least three.

There are only few explicitly known examples of diffeomorphisms with a blender. Perhaps the most studied example is the Hénon-like family that was introduced in [15]; see Section 4 for its formal definition. This example was motivated by the family of endomorphisms from [10], and it represents the lowest-dimensional case of a diffeomorphism in \mathbb{R}^3 . Even though the two saddle fixed points (as well as all saddle periodic points) for this family have the same index, it was proven in [10, 15] that a blender can exist for particular parameter choices. This proof is based on the fact that the underlying Hénon map has a full Smale horseshoe (equivalent to a full shift on two symbols), which implies that the objects of interest are the stable and unstable manifolds of two fixed points: the closure of the homoclinic intersection set for either one of these fixed points is the hyperbolic set. When the central direction is unstable with an expansion rate sufficiently close to 1, the closure of the respective one-dimensional stable manifold of either one of the two fixed points effectively acts as a surface; similar arguments apply for the case of one-dimensional unstable manifolds with a contraction of the central direction that is sufficiently close to 1. In [15–17], this is also referred to as the *carpet property* and it implies that the hyperbolic set is a blender. Since the Hénon-like family maps a three-dimensional box over itself in the form of a three-dimensional horseshoe [17], this type of blender is also referred to as a *blender horseshoe* [11, 12].

In the works [15, 16], the carpet property is verified numerically by approximating the one-dimensional stable (or unstable) manifolds as curves with increasingly larger arclengths, and checking whether the projection of these computed manifolds in a suitable direction densely fill the plane of projection locally near the hyperbolic set. In this way, it has been demonstrated numerically that the Hénon-like family has a blender over an interval of expansion/contraction rates of the central direction, which is a considerably larger range than expected from known analytical results [10, 15]. Such numerical studies of how the carpet property is lost [15–17] also reveal that

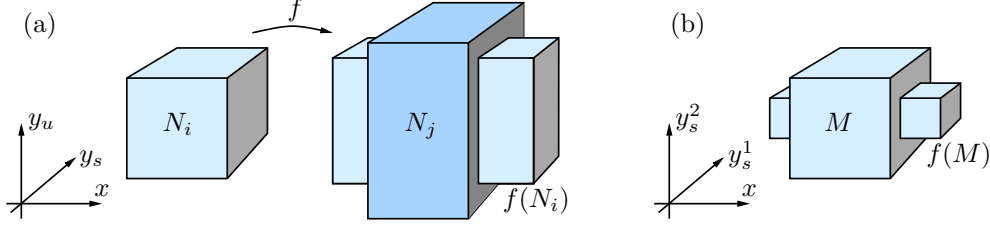


Fig. 1 Illustration of covering relations in a three-dimensional space; the exit set is shaded darker. Panel (a) shows a covering $N_i \xrightarrow{f} N_j$ with exit coordinate x and entry coordinates $y = (y_u, y_s)$, assuming (weak) expansion in the direction y_u and contraction along y_s . Panel (b) shows the case $M \xrightarrow{f} M$ of a box that covers itself, as is typically found near a hyperbolic fixed point with one unstable direction x and two stable directions y_s^1 and y_s^2 .

the three-dimensional horseshoe of the Hénon-like map represents a more complicated mechanism for the creation of a blender than suggested by the affine blender construction from [2], which is discussed in Section 2.1 as a motivating example.

While checking the carpet property via the computation of increasingly longer segments of one-dimensional invariant manifolds is a very useful tool for finding and visualising blenders in specific examples, it cannot be turned into a formal proof. To prove that a given hyperbolic set is actually a blender, the characterisation from [3] requires verification of assumptions on the images of topological (hyper)disks under iterates of the map in a neighbourhood of a horseshoe centred around a hyperbolic fixed point (of an appropriate iterate if necessary). This method was used in [15] to show that the Hénon-like family indeed has a blender at (or rather near) a specific value of central expansion that lies well within the range identified numerically by checking the carpet property.

We present here a new characterisation of a blender that is based on finding a family of sets in phase space with the property that the image of each such set intersects another set in a ‘topologically good way’. This is formalised by the notion of a *covering relation*, as introduced in [28] and defined in Section 2. While the number of sets in the family may be large, the proof of existence requires assumptions on their images under only one single iteration, which means verification can be done in a local, far more efficient manner, with better control over numerical accuracy. More specifically, we consider throughout a diffeomorphism

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n,$$

with $n \geq 3$ and a family of sets $\{N_i\}_{i \in I} \subset \mathbb{R}^n$, where I is a finite index set. Each of the N_i can be thought of as a box with a well-defined selection of faces defined as the ‘exit set’. We say that N_i *covers* N_j , for which we write $N_i \xrightarrow{f} N_j$, if the image set $f(N_i)$ ‘cuts across’ N_j in a topologically good way via its exit set; see also already Definition 5. Figure 1 provides an illustration for the case $n = 3$. The sketch shows the simplest case where the entry and exit coordinates of the boxes are given by the coordinate axes and the exit coordinate x is one dimensional; furthermore, f preserves these coordinates, that is, the image of a box is also aligned with the coordinate axes.

In this simple setting N_i is covering N_j when the exit sets of N_i are mapped to the left and to the right of N_j and the image $f(N_i)$ of N_i can intersect the boundary of N_j only at the exit set of N_j .

Panel (a) show a case where a box N_i maps to a different box N_j . The exit sets for N_i and N_j are their left and right, shaded sides, orthogonal to the exit coordinate x . The other faces comprise the entry set, which is orthogonal to the plane spanned by entry coordinates $y = (y_u, y_s)$. Here, we show the case where x and y_u are expanding directions and the only contraction occurs along y_s . Consequently, the set N_j needs to be ‘taller’ than N_i along y_u , so that $f(N_i)$ does not intersect the top and bottom sides of N_j , because these are not part of the exit set of N_j . Figure 1(b) shows an example of a box M that covers itself. Such a situation is typical if M contains a saddle fixed point; the figure shows the case of a hyperbolic saddle with one unstable and two stable eigendirections, denoted x , y_s^1 , and y_s^2 , respectively.

Observe that both panels (a) and (b) of Figure 1 show cases with a dominant hyperbolic splitting that exhibits strong expansion along x . In panel (a) we have a splitting between the strong expansion x and the coordinates (y_u, y_s) where y_u is a weak expansion and y_s is a contraction. In panel (b) we have a splitting between the expansion along x and the contraction along (y_s^1, y_s^2) . Moreover, the sets N_i , N_j , and M , or more generally, the entire family of sets $\{N_i\}_{i \in I}$ is equipped with a *cone field* and the map f satisfies a cone condition, meaning that the cone field is mapped into itself; see already Definition 6 for the precise statement of cone conditions. We consider a family of topological (hyper)disks that are aligned with this cone field; we refer to them as *horizontal disks* and they are defined formally in Definition 7. With these notions, we are able to state our central result.

In our setting, we also consider a sub-family $\{M_l\}_{l \in L} \subset \{N_i\}_{i \in I}$ of ‘mother sets’ that do not map further to other sets in $\{N_i\}_{i \in I}$.

Theorem 1 (Characterisation of a blender in terms of covering relations). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diffeomorphism, with $n \geq 3$, and assume that there exists a set $U \subset \mathbb{R}^n$ such that its (forward and backward) invariant set Λ is hyperbolic and transitive. Consider a finite family of sets $\{N_i\}_{i \in I} \subset U$ that contains a sub-family $\{M_l\}_{l \in L} \subset \{N_i\}_{i \in I}$, for which the following conditions are satisfied:*

(B1) *For every $l \in L$ and every horizontal disk $[h]$ in M_l , there exist $i \in I \setminus L$ such that $[h] \cap N_i$ is a horizontal disk in N_i .*

(B2) *For every $i \in I \setminus L$, there exists $j \in I$ such that $N_i \xrightarrow{f} N_j$ and f satisfies cone conditions from N_i to N_j (see Definitions 5 and 6 for precise statements).*

If, in addition, the dimension of the exit set of the N_i is larger than the number of contracting directions of Λ , then Λ is a blender.

The two conditions (B1) and (B2) provide us with a flexible framework for establishing the existence of blenders that can be applied in various contexts. They give conditions for a neighbourhood of a hyperbolic set, and specify when such a hyperbolic set is a blender. Note that the simplest case of satisfying (B1) and (B2) is a set M that covers itself, which is a typical setting one encounters, for instance, around a hyperbolic fixed point; see Figure 1(b). However, since the exit set of M is spanned, here, by two stable directions y_s^1 and y_s^2 , this does not imply that there is a blender in M . In general,

to establish the existence of a blender, one should expect to need a possibly large number of covering sets $\{N_i\}_{i \in I}$ and $\{M_l\}_{l \in L} \subset \{N_i\}_{i \in I}$. Nevertheless, verifying (B1) and (B2) only requires checking bounds on first images of the sets N_i (to establish the covering and cone conditions).

As an example of an application of Theorem 1, we consider the Hénon-like family

$$f(X, Y, Z) = (Y, \mu + Y^2 + \beta X, \xi Z + Y) \quad (1)$$

that was studied in [15–17]; see also [10]. An analytical proof of existence was given in [15] for the single parameter point $\mu = -9.5$, $\beta = 0.1$ and $\xi = 1.185$, along with a numerically generated conjecture that a blender exists over the full range of $\xi \in (1, 1.843]$, that is, the stable manifold of the blender acts as though it is two dimensional; see also Definition 3. Similarly, the numerical study in [16] suggests that a blender exists for $\mu = -9.5$, $\beta = 0.3$ and $\xi \in (1, 1.75]$, which we select as an illustration of how to apply Theorem 1. More precisely, in Section 4, we present a proof of the following theorem.

Theorem 2. *Consider the Hénon-like family defined in (1) with $\mu = -9.5$ and $\beta = 0.3$. For every $\xi \in [1.01, 1.125]$, this Hénon-like family has a blender.*

In Section 5, we present a further extension of our approach. We introduce an additional condition (B3) that can be used to establish the existence of robust heterodimensional cycles between two blenders or between a blender and a hyperbolic fixed point; the precise statement is given in Theorem 8. Condition (B3) is similar in spirit to (B1) and (B2) above and also checkable in a computer-assisted way.

There are some differences between our approach and the classical setup by Bonatti and Díaz [3]. Their method is based on appropriate assumptions on the images of topological disks under iterates of the map, in a neighbourhood of a horseshoe at a hyperbolic fixed point. Our method can be applied in this setting, but is not limited to it. More precisely, fixed points or periodic points and their associated invariant manifolds are not used explicitly in our approach. This means that our method can be applied more generally, for example, to hyperbolic sets that arise in the context of hetero/homoclinic tangles between a number of hyperbolic fixed points. In particular, our construction of covering provides a method for verifying the robust existence of heterodimensional cycles.

Another difference is that we do not require conditions on the propagation of disks. Rather, what we require is good topological alignment of images of sets, expressed through covering relations. Covering relations have proven to be a versatile and flexible tool, due to the simplicity of their validation [7–9, 13, 19, 24, 26]. Moreover, we can position our sets in such a way that we have a sequence of coverings, and validation requires only a single iterate of the map instead of considering images of topological disks for several compositions of the map, as in [3]. Therefore, we can use a ‘topological parallel shooting’ approach, that allows us to avoid compositions. In particular, our assumptions are formulated by means of conditions that can be validated by using interval arithmetic and computer-assisted tools. The proof in [15] is for existence of a blender that is a *maximal* (transitive) hyperbolic set. Our approach is, in some sense, more general and, as an example, we present in Section 4 the proof of existence of

a blender in the Hénon-like family for the parameter range as stated in Theorem 2; this blender is contained in a family of covering sequences that are positioned to be disjoint from any fixed point.

In our construction we use a *wall property*, which states that a set in the investigated neighbourhood intersects with every horizontal disk. This is very similar in spirit to the carpet property from [15–17]. The difference is that, instead of projections, we work with horizontal disks. During our construction we ensure that the wall property is satisfied by the stable set of the invariant hyperbolic set, and this implies that the hyperbolic set is a blender.

The paper is organised as follows. Section 2 introduces some necessary background. Specifically we present the affine construction of a blender in Section 2.1 and its formal definition in Section 2.2. We then provide required preliminaries in Section 2.3 by introducing notation and statements regarding cone conditions and covering relations. In Section 3, we prove our main theorem, and we also explain conditions (B1) and (B2) more precisely. In Section 4, we present a computer-assisted proof¹ of Theorem 2 regarding the existence of a 2-blender for the Hénon-like family of diffeomorphisms from [15–17]. This example illustrates how our conditions can be validated in practice. In Section 5, we introduce condition (B3) that can be used to establish the existence of heterodimensional cycles. We end with a discussion and conclusions in Section 6. Furthermore, in the Appendix, we also show how hyperbolicity and transitivity of an invariant set can be established with computer-assisted tools.

2 Background, notation and preliminaries

Our objective is to develop a methodology for establishing the existence of a blender within an explicit domain of the phase space of a given diffeomorphism. To this end, we first discuss the affine example of a blender in Section 2.1, and then give the general definition in Section 2.2. The discussion of our two main tools, cone conditions and covering relations, can be found in Section 2.3.

2.1 The affine blender horseshoe

Bonnati, Crovisier, Díaz, and Wilkinson designed an abstract example in [2] of a diffeomorphism f on \mathbb{R}^3 that features a blender horseshoe. We briefly discuss this example to demonstrate the features of our construction of covering relations, as well as provide intuition behind it. We focus on the geometric features, which are illustrated in Figure 2. The coordinate axes are denoted x , y_u , and y_s , along which f is expanding, weakly expanding, and contracting, respectively. The diffeomorphism f maps two cuboids N_1 and N_2 , which are contained in a larger cube M , tightly back into M as shown in Figure 2(a). Here, the restrictions $f|_{N_1}$ and $f|_{N_2}$ are taken to be affine and the coordinate axes agree with the expanding and contracting directions of f .

A better intuition can be gained by considering the pre-images of these sets, as illustrated in Figure 2(b). This reveals the three-dimensional blender horseshoe and highlights the particular selection of N_1 and N_2 such that $N_1 \cup N_2 = f^{-1}(M) \cap M$.

¹The code for the computer assisted proof is available on the personal web page of the corresponding author.

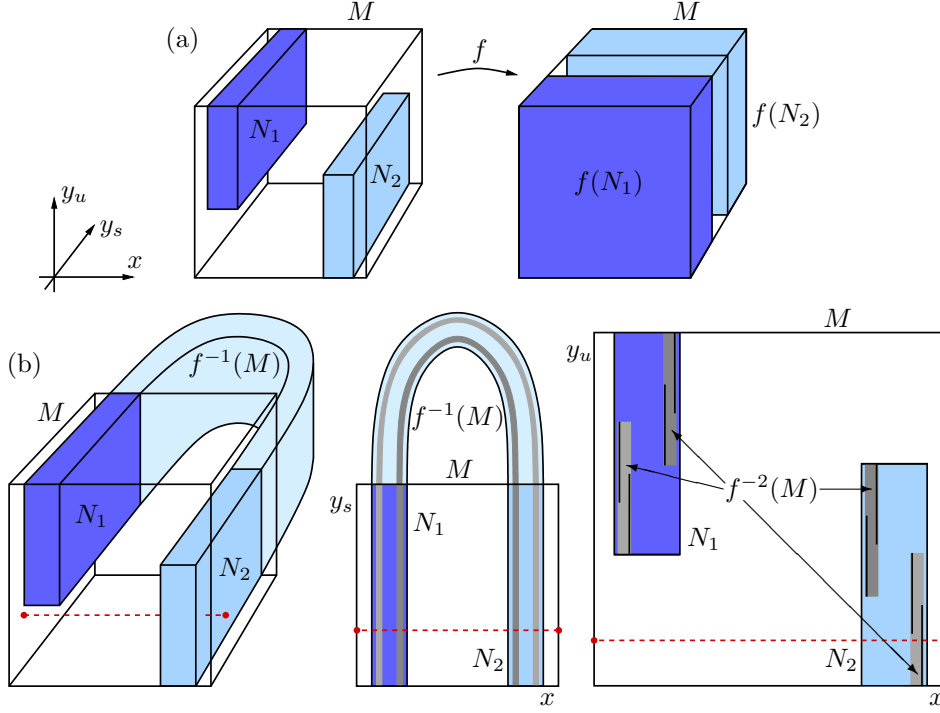


Fig. 2 Sketch of how a blender horseshoe is generated when the map f is affine in a box M defined on the three-dimensional (x, y_s, y_u) -space. Panel (a) illustrates how sub-boxes N_1 (blue) and N_2 (light blue) are contained in the mother set M . Panel (b) shows the three-dimensional horseshoe set $f^{-1}(M)$ that generates $N_1 \cup N_2 = f^{-1}(M) \cap M$, with a top view in the (x, y_s) -plane that also shows $f^{-2}(M)$ (grey set) and a front view in the (x, y_u) -plane that additionally shows $f^{-3}(M)$ (black set). Note that any horizontal segment, such as the dashed red line, that traverses M in the x -direction must intersect $f^{-j}(M) \cap M$ for any $j \geq 1$.

Indeed, the top view of the (x, y_s) -plane shows the well-known affine Smale horseshoe construction inside a square [22]. The (forward and backward) invariant set of f in M is a hyperbolic Cantor set, which we denote Λ . Further pre-images $f^{-j}(M) \cap M$ form a nested sequence of cuboids in M , with a limit equal to $W^s(\Lambda) \cap M$, that is, the closure of the stable manifold of Λ inside M ; due to the affine nature of f , the set $W^s(\Lambda)$ locally near Λ is a Cantor set of line segments that stretch all the way across in the y_s -direction.

Notice from the projections onto the (x, y_u) -plane in Figure 2(b) that the cuboids N_1 and N_2 overlap with respect to the weakly expanding y_u -direction. As a result, any ‘horizontal’ segment in the x -direction that joins the left and right faces of M must intersect at least one of the preimages in $f^{-j}(M) \cap M$, for any given j ; see the (red) dashed line as an example of such a horizontal segment in panel (b). It follows from taking the limit as $j \rightarrow \infty$ that the given horizontal segment must intersect the stable manifold $W^s(\Lambda) \cap M$. This is the surprising and characterising feature of the blender, which can be interpreted as $W^s(\Lambda)$ behaving effectively like an object of a higher dimension, a surface in this case, when viewed along the x -direction. Importantly, the

existence of such an intersection is a C^1 -robust property: if one slightly perturbs the horizontal segment to a nearby smooth curve, or smoothly perturbs the underlying blender horseshoe construction itself, the intersection found for the affine case persists.

We capture these characterising properties of a blender in terms of a covering relation in conjunction with cone conditions. To this end, we first provide the formal definition of a blender of any dimension and in an ambient space of any dimension.

2.2 Formal definition of a blender

We consider the definition from [3, 4] of a blender for a diffeomorphism f on \mathbb{R}^n , with $n \geq 3$, that has a hyperbolic compact invariant set Λ . We remark that there are a number of different but related definitions of blenders [3–6] in a variety of different contexts; see also the discussion in [15].

Definition 1. *The set $\Lambda \subset \mathbb{R}^n$ is hyperbolic if for every $p \in \Lambda$, the n -dimensional tangent space $T_p\Lambda$ splits into a direct sum $T_p\Lambda = E_p^u \oplus E_p^s$ of (possibly trivial) vector spaces (fibre bundles) E_p^u and E_p^s , respectively, that are invariant under the derivative Df of f . More precisely, for nonzero vectors $v_u \in E_p^u$ and $v_s \in E_p^s$, we have*

$$\begin{aligned} Df(p) v_u &\in E_{f(p)}^u, \\ Df(p) v_s &\in E_{f(p)}^s, \end{aligned}$$

and there exist $c > 0$ and $\lambda \in (0, 1)$, independent of p , such that, for all $j \geq 1$,

$$\begin{aligned} \|Df^{-j}(p) v_u\| &< c \lambda^j \|v_u\|, \\ \|Df^j(p) v_s\| &< c \lambda^j \|v_s\|. \end{aligned}$$

Definition 2 (Unstable and stable manifolds of Λ). *The unstable and stable manifolds of a hyperbolic set Λ are given by*

$$W^u(\Lambda) = \bigcup_{p \in \Lambda} W_p^u \quad \text{and} \quad W^s(\Lambda) = \bigcup_{p \in \Lambda} W_p^s,$$

where W_p^u and W_p^s are the pointwise unstable and stable fibres, respectively, defined as

$$\begin{aligned} W_p^u &= \{q \in \mathbb{R}^n \mid \|f^{-j}(p) - f^{-j}(q)\| \rightarrow 0 \text{ as } j \rightarrow \infty\} \quad \text{and} \\ W_p^s &= \{q \in \mathbb{R}^n \mid \|f^j(p) - f^j(q)\| \rightarrow 0 \text{ as } j \rightarrow \infty\}. \end{aligned}$$

Definition 3 (k -blender). *Let $\Lambda \subset \mathbb{R}^n$ be an invariant transitive hyperbolic set with d_s -dimensional pointwise stable fibres. For $k \in \mathbb{N}$ with $k > d_s$, we say that Λ is a k -blender if there exists a set \mathcal{S} of $(n - k)$ -dimensional surfaces, which is open in the C^1 -topology, such that for every $S \in \mathcal{S}$,*

$$S \cap W^s(\Lambda) \neq \emptyset.$$

Remark 1. *The only known explicit examples are for $k = d_s + 1$ (and $d_s = 1$), which is the particular case studied in detail in the literature; for example, see [3, 4]. The existence of k -blenders for larger k is a topic of active research; for example, see [1], where such a blender is called a superblender.*

The intuition behind this definition is that $W^s(\Lambda)$, which is of dimension d_s less than k , behaves as if it was a k -dimensional manifold, because it has robust intersections with a family of $(n - k)$ -dimensional surfaces. The hyperbolic set of the affine example in Section 2.1 is, hence, a 2-blender; for this case, the one-dimensional ‘surfaces’ in \mathcal{S} are curve segments in the x -direction.

2.3 Cone conditions, covering relations and horizontal disks

Our objective is to develop a methodology for establishing the existence of blenders within an explicit domain and providing an explicit estimate for the set of surfaces \mathcal{S} that intersect $W^s(\Lambda)$. Our main two tools for doing so are cone conditions and covering relations.

More specifically, we wish to establish the existence of a blender which intersects a finite union

$$\bigcup_{i \in I} N_i := \bigcup_{i \in I} N_i \subset \mathbb{R}^n$$

of compact sets $N_i \subset \mathbb{R}^n$ with index set I . We equip each N_i with a local coordinate change $\gamma_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that maps points in N_i to pairs $(x, y) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} = \mathbb{R}^n$, with the components $x \in \mathbb{R}^{d_x}$ and $y \in \mathbb{R}^{d_y}$ representing ‘topological exit’ and ‘topological entry’ coordinates, and corresponding projections π_x and π_y , respectively. We refer to N_i as an *h-set* (which stands for hyperbolic-type set) that is formally defined as follows; see also Figure 1.

Definition 4 (h-set [14, 28]). *A compact subset $N \subset \mathbb{R}^n$ is an h-set if there exists a homeomorphism $\gamma : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that*

$$N_\gamma := \gamma(N) = \overline{B}_{d_x} \times \overline{B}_{d_y}, \quad (2)$$

where \overline{B}_{d_x} and \overline{B}_{d_y} are closed unit balls of dimension d_x and d_y , respectively. We define the exit set N^- and the entry set N^+ as

$$N^- := \gamma^{-1}(\partial \overline{B}_{d_x} \times \overline{B}_{d_y}) \quad \text{and} \quad N^+ := \gamma^{-1}(\overline{B}_{d_x} \times \partial \overline{B}_{d_y}).$$

Remark 2. *To define the closed balls \overline{B}_{d_x} and \overline{B}_{d_y} in (2) we can use different norms on \mathbb{R}^{d_x} and \mathbb{R}^{d_y} , respectively. For applications in computer-assisted proofs, a natural choice is to use the maximum norm, in which case the balls are Cartesian products of intervals, or simply (hyper)cubes.*

From now on, we assume that the N_i are h-sets for all $i \in I$. The dimensions d_x and d_y do not depend on the index $i \in I$. For simplicity we refer to the local coordinates as (x, y) regardless of the choice of N_i , keeping in mind that they are associated with a local coordinate change γ_i .

Remark 3. We do not need to assume that the dimensions d_x and d_y agree with the dimensions d_u and d_s of the unstable and stable fibres of Λ from Definition 1. In our methodology the exit coordinates x are associated with a strong hyperbolic expansion, and the entry coordinates y could include weak hyperbolic expansion.

When the diffeomorphism f is expressed in local coordinates of sets N_i and N_j for $i, j \in I$ then we refer to it as

$$f_{ji} = \gamma_j \circ f \circ \gamma_i^{-1}.$$

This representation is useful for the definition of a covering relation between two h-sets.

Definition 5 (Covering between h-sets [14, 28]). We say that the h-set N_i covers the h-set N_j , which we denote

$$N_i \xrightarrow{f} N_j,$$

if there exist:

- (I) a homotopy $\varsigma : [0, 1] \times N_{\gamma_i} \rightarrow \mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$, where $N_{\gamma_i} := \gamma_i(N_i)$, such that for any $(x, y) \in N_{\gamma_i}$ and any $t \in [0, 1]$,

$$\begin{aligned} \varsigma(0, (x, y)) &= f_{ji}(x, y), \\ \varsigma(t, \gamma_i(N_i^-)) \cap N_{\gamma_j} &= \emptyset, \\ \varsigma(t, N_{\gamma_i}) \cap \gamma_j(N_j^+) &= \emptyset, \end{aligned}$$

and

- (II) a linear map $A : \mathbb{R}^{d_x} \rightarrow \mathbb{R}^{d_x}$ such that

$$\begin{aligned} \varsigma(1, (x, y)) &= (Ax, 0), \\ A(\partial \overline{B}_{d_x}) &\subset \mathbb{R}^{d_x} \setminus \overline{B}_{d_x}. \end{aligned}$$

For our purpose of proving the existence of a blender, a covering $N_i \xrightarrow{f} N_j$ corresponds to a strong expansion for the x -component, while the y -component includes contraction as well as weak expansion. Hence, the picture in this case is as shown in Figure 1(a). In particular, the situation in Figure 1(b) of a set covering itself is excluded, because the invariant set contained in $\bigcup N_i$ could then be a mere saddle fixed point. Such a saddle fixed point would have a $(d_s = 2)$ -dimensional stable manifold, so the only possible choice of $k > d_s$ is $k = 3$, and it is not possible to have a 3-blender in \mathbb{R}^3 .

We quantify the level of expansion and contraction induced by a covering relation in terms of a cone condition defined for the respective h-sets. In the local coordinates given by γ_i , we define a cone at $z_{\gamma_i} = (\pi_x(z_{\gamma_i}), \pi_y(z_{\gamma_i})) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$ as

$$\mathcal{C}_{\gamma_i}(z_{\gamma_i}) := \left\{ (q_x, q_y) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \mid \|q_x - \pi_x(z_{\gamma_i})\|_{d_x} > \|q_y - \pi_y(z_{\gamma_i})\|_{d_y} \right\}, \quad (3)$$

where $\|\cdot\|_{d_x}$ and $\|\cdot\|_{d_y}$ are norms on \mathbb{R}^{d_x} and \mathbb{R}^{d_y} , respectively. Figure 3 shows an example of a cone in local coordinates defined on \mathbb{R}^3 with $d_x = 1$ and $d_y = 2$, where $\|\cdot\|_{d_x}$ and $\|\cdot\|_{d_y}$ are given by the respective (suitably scaled) L_1 -norm.

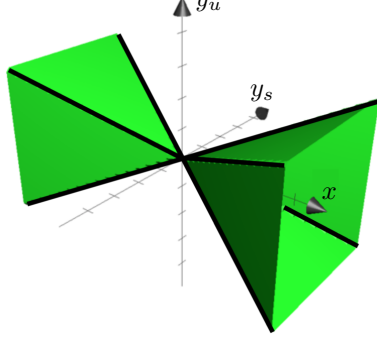


Fig. 3 Example of a cone $C(0)$ defined for local coordinates of a three-dimensional space $\mathbb{R}^{d_x} \times \mathbb{R}^{d_y} = \mathbb{R}^1 \times \mathbb{R}^2$ with chosen norms $\|x\|_{d_x} = \frac{1}{2}|x|$ and $\|y\|_{d_y} = \|(y_u, y_s)\|_{d_y} = \max(|y_u|, |y_s|)$.

Remark 4. The choice of the norms $\|\cdot\|_{d_x}$ and $\|\cdot\|_{d_y}$ can be independent of the choice of the norms that determine the closed balls \overline{B}_{d_x} and \overline{B}_{d_y} in Definition 4.

In the state space of the map f the cone at $z \in N_i$ is defined as

$$\mathcal{C}_i(z) := \gamma_i^{-1}(\mathcal{C}_{\gamma_i}(\gamma_i(z))). \quad (4)$$

Remark 5. We do not rule out the situation that $z \in N_i \cap N_j$ for $i \neq j$, in which case typically $\mathcal{C}_i(z) \neq \mathcal{C}_j(z)$. However, this will not present a problem in our methodology.

Definition 6 (Cone conditions). Let f be a diffeomorphism and let N_i, N_j be two h -sets. We say that f satisfies cone conditions from N_i to N_j if, for any $z \in N_i$ such that $f(z) \in N_j$, we have

$$f(\mathcal{C}_i(z)) \subset \mathcal{C}_j(f(z)). \quad (5)$$

Importantly, one can have cone conditions in the setting when there is no contraction along the entry coordinates y . What is needed is sufficiently strong expansion of the exit coordinates x , as is demonstrated with the following toy example.

Example 1. Consider the linear map $f(x, y_u, y_s) = (4x, 2y_u, \frac{1}{2}y_s)$ on \mathbb{R}^3 , and let $\|\cdot\|$ be the standard Euclidean norm. We have $d_x = 1$ and $d_y = 2$ with $y = (y_u, y_s)$, where y_s is a stable and y_u an unstable coordinate. Despite the expansion along the y_u -coordinate, cone conditions hold for any choice of $N_i, N_j \subseteq \mathbb{R}^3$. Indeed, let $z \in \mathbb{R}^3$ and let $q \in \mathbb{R}^3$ be such that $\|\pi_x(q - z)\| - \|\pi_y(q - z)\| = \|x\| - \|(y_u, y_s)\| > 0$, where we use the notation $q - z = (x, y_u, y_s)$. Since f is linear, we have

$$\begin{aligned} \|\pi_x(f(q) - f(z))\| - \|\pi_y(f(q) - f(z))\| &= \|\pi_x f(q - z)\| - \|\pi_y f(q - z)\| \\ &= \|4x\| - \|(2y_u, \frac{1}{2}y_s)\| \\ &> 2(\|x\| - \|(y_u, \frac{1}{4}y_s)\|) \\ &\geq 2(\|x\| - \|(y_u, y_s)\|) \\ &> 0. \end{aligned}$$

The cone condition is used to define topological disks of dimension d_x that in local coordinates are ‘well aligned’ with the strongly expanding x -component of $\gamma_i(N_i)$; we refer to such d_x -dimensional disks as horizontal disks.

Definition 7 (Horizontal disk). *The d_x -dimensional surface $[h] \subset \mathbb{R}^n$ is a horizontal disk in N_i if in local coordinates it is a graph over the x -component and it satisfies a cone condition. More formally, the following two requirements hold:*

(I) *there exists a continuous function $h : \overline{B}_{d_x} \rightarrow \overline{B}_{d_y}$ such that*

$$\gamma_i([h]) = (x, h(x)),$$

and

(II) *for every $x_1, x_2 \in \overline{B}_{d_x}$ with $x_1 \neq x_2$ we have*

$$(x_1, h(x_1)) \in \mathcal{C}_{\gamma_i}(x_2, h(x_2)).$$

We say that $[h]$ is a horizontal disk in a family of h -sets $\{N_i\}_{i \in I}$ if there exists $k \in I$ such that $[h]$ is a horizontal disk in N_k .

The key tool needed to prove Theorem 1 is a result from [27, Thm 7], which we state here using our notation.

Theorem 3 ([27, Thm 7]). *Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with $n \geq 3$, is a diffeomorphism and we have a finite sequence*

$$N_{i_0} \xrightarrow{f} N_{i_1} \xrightarrow{f} \dots \xrightarrow{f} N_{i_k}$$

of covering relations $\{N_{i_m}\}_{i_m \in I} \subset \mathbb{R}^n$, such that f satisfies cone conditions from $N_{i_{m-1}}$ to N_{i_m} for $m = 1, \dots, k$. Then, if $[h_{i_0}]$ is a horizontal disk in N_{i_0} , there exists a horizontal disk $[h_{i_k}]$ in N_{i_k} , such that

$$[h_{i_k}] = \{f^k(z) \mid z \in [h_{i_0}] \text{ and } f^m(z) \in N_{i_m} \text{ for } m = 1, \dots, k\}.$$

3 Proof of Theorem 1

To recap the setting, we consider a diffeomorphism f on \mathbb{R}^n , a set $U \subset \mathbb{R}^n$ whose invariant set

$$\Lambda = \text{Inv}(f, U) := \{z : f^k(z) \in U \text{ for all } k \in \mathbb{Z}\}$$

is hyperbolic and transitive, and a finite family of h -sets $\{N_i\}_{i \in I} \subset U$, each of which with topological exit coordinates $x \in \mathbb{R}^{d_x}$ and topological entry coordinates $y = (y_u, y_s) \in \mathbb{R}^{d_y}$ as given by the respective local coordinate changes γ_i . Moreover, each h -set N_i is equipped with cones, as given by (3) in local coordinates and by (4) in the original coordinates. We also consider a sub-family $\{M_l\}_{l \in L} \subset \{N_i\}_{i \in I}$ of ‘mother sets’, which play the role of initial sets for our methodology. The claim of Theorem 1 is that Λ is a d_y -blender, provided conditions (B1) and (B2) are satisfied.

Before we present the proof of Theorem 1, we provide some intuition behind conditions (B1) and (B2). The local coordinates for the coverings and cone conditions

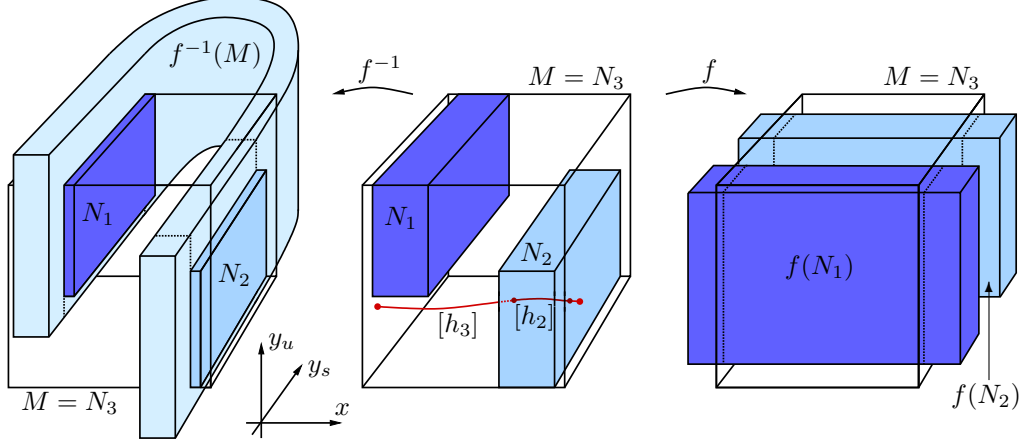


Fig. 4 A modification of the affine blender from Figure 2 so that $N_1 \xrightarrow{f} M$ and $N_2 \xrightarrow{f} M$; also shown is a horizontal disk $[h_3]$ (red curve) of $M = N_3$, which generates a horizontal disk $[h_2]$ (dark red curve) of N_2 .

in (B1) and (B2) are in terms of a splitting $x \in \mathbb{R}^{d_x}$ and $y \in \mathbb{R}^{d_y}$. The hyperbolic nature of the invariant set, on the other hand, is in terms of the associated unstable and stable fibres of dimensions d_u and d_s , respectively. We will sometimes use the phrase that conditions (B1) and (B2) hold with (d_x, d_y, d_u, d_s) to state these dimensions specifically. We require $d_x < d_u$, which implies that the y -component in local coordinates can be split into $y = (y_u, y_s)$, such that the unstable/expanding coordinates are $(x, y_u) \in \mathbb{R}^{d_u}$ and the stable/contracting coordinates are $y_s \in \mathbb{R}^{d_s}$.

Remark 6. *The horizontal disks $[h]$ from condition (B1) play the role of the set \mathcal{S} of surfaces from Definition 3. More specifically, for the hyperbolic set Λ , we show that each horizontal disk $[h]$ in $\{N_i\}_{i \in I}$ intersects the stable manifold $W^s(\Lambda)$ of Λ . Note that dimensions of the horizontal disks $[h]$ and the stable manifold of Λ do not add up to the dimensions of the phase space, which is the characterising feature of a blender.*

In \mathbb{R}^3 , the horizontal disks are (one-dimensional) curve segments and an example segment is shown in Figure 2(b). On the other hand, for $d_x = d_u$, conditions (B1) and (B2) are trivially satisfied by the general statement that $\Lambda = W^u(\Lambda) \cap W^s(\Lambda)$; as Figure 1(b) illustrates, Λ could be a fixed point in this case, but to have a blender we must have $d_x < d_u$, which means that this case is ruled out and not part of our considerations. Figure 4 shows how the example of the affine blender from Section 2.1 and Figure 2 can be modified to fit into the framework of Theorem 1. A suitable small reduction of the set M achieves that the h-sets N_1 and N_2 cover M , as is shown in Figure 4 with sketches of images and preimages. Hence, with $N_3 = M$ we can take $\{N_i\}_{i \in I} = \{N_1, N_2, M\}$ and $\{M_l\}_{l \in L} = \{M\}$. As already stated, $N_1 \xrightarrow{f} M$ and $N_2 \xrightarrow{f} M$, which ensures (B2). Condition (B1) is also satisfied, since every horizontal disk $[h_3]$ in $M = N_3$, which is a curve segment aligned with the x -direction here, must intersect N_1 or N_2 . The intersection set is, hence, a horizontal disk in N_1 or N_2 , provided the cones in M are at least as sharp as the cones in the sets N_i . As is

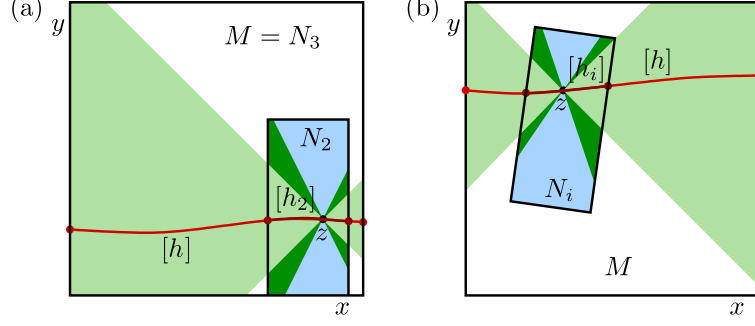


Fig. 5 To satisfy conditions (B1) and (B2), the cones in M (light green), need to be contained in the cones in N_i (dark green) at every point of $z \in [h_i]$; this is illustrated in panel (a) for $N_2 \subset M$ of the affine example from Figure 4, and in panel (a) for a general h-set $N_i \subset M$.

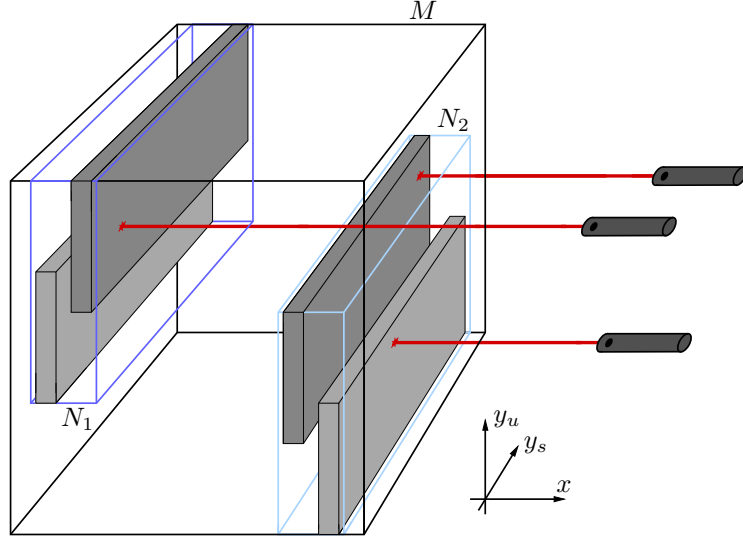


Fig. 6 Illustration of a wall in $\{N_1, N_2, M\}$ formed by $f^{-2}(M) \cap M$ (light- and dark-grey sets). Any laser beam aligned with the x -direction, even if it were to take a slightly curvy path, cannot pass through the wall; compare with Figure 2(b).

sketched in Figure 5(a) for $[h_2]$ in N_2 , this can be ensured by an appropriate choice of local coordinates; Figure 5(b) shows an example for a more general h-set N_i .

The following notion of a wall is used as a building block for the proof of Theorem 1, and it is illustrated in Figure 6 for the affine case from Section 2.1.

Definition 8 (Wall for a family of h-sets). *A set $A \subset \mathbb{R}^n$ is a wall for the family of h-sets $\{N_i\}_{i \in I}$ if every horizontal disk $[h]$ in $\{N_i\}_{i \in I}$ intersects A .*

Figure 6 shows a mother set M with two subsets, denoted N_1 and N_2 ; see also Figure 4. Any horizontal disk $[h]$ in $\{N_1, N_2, M\}$, which are curve segments aligned with the x -direction, intersects at least one of the light- and dark-grey boxes in $f^{-2}(M) \cap M$.

Hence, $f^{-2}(M) \cap M$ is a wall for $\{N_1, N_2, M\}$. The following two lemmas emphasise useful consequences of having a wall for a family of h-sets $\{N_i\}_{i \in I}$, and how to construct a wall if conditions (B1) and (B2) from Theorem 1 are satisfied.

Lemma 4. *If A is a wall for the family of h-sets $\{N_i\}_{i \in I}$ and conditions (B1) and (B2) from Theorem 1 are satisfied then*

$$\mathcal{G}(A) := f^{-1}(A) \cap \bigcup N_i$$

is also a wall for $\{N_i\}_{i \in I}$.

Proof. First, let us take a horizontal disk $[h_i]$ in N_i , for any $i \in I \setminus L$ (that is, $N_i \notin \{M_l\}_{l \in L}$). Our objective is to show that $[h_i] \cap \mathcal{G}(A) \neq \emptyset$. By (B2) we have $N_i \xrightarrow{f} N_j$ for some $j \in I$, and f satisfies cone conditions from N_i to N_j . By Theorem 3 (applied for $k = 1$) we obtain a horizontal disk $[h_j]$ in N_j such that

$$[h_j] = \{f(z) \mid z \in [h_i] \text{ and } f(z) \in N_j\}.$$

Since A is a wall, there exists $z \in [h_j] \cap A \neq \emptyset$, which implies that $f^{-1}(z) \in f^{-1}(A) \cap N_i \subset \mathcal{G}(A)$; hence, $[h_i] \cap \mathcal{G}(A) \neq \emptyset$, as required.

Now consider a horizontal disk $[h]$ in M_l for some $l \in L$. Our objective is to show that $[h] \cap \mathcal{G}(A) \neq \emptyset$. From condition (B1) we can choose $i \in I \setminus L$ and a horizontal disk $[h_i]$ in N_i such that $[h] \cap N_i = [h_i]$. By condition (B2) there exists a $j \in I$ such that $N_i \xrightarrow{f} N_j$, with f satisfying cone conditions from N_i to N_j . As in the first part of the proof, we apply Theorem 3 to conclude $[h_i] \cap \mathcal{G}(A) \neq \emptyset$; since $[h_i] \subset [h]$, this implies that $[h] \cap \mathcal{G}(A) \neq \emptyset$, as required. \square

Lemma 5. *Let $\{N_i\}_{i \in I}$ be a family of h-sets in \mathbb{R}^n for which conditions (B1) and (B2) from Theorem 1 are satisfied. Then the set*

$$A = \left\{ z \in \mathbb{R}^n \mid f^k(z) \in \bigcup N_i \text{ for all } k \in \mathbb{N} \right\}$$

is a wall for $\{N_i\}_{i \in I}$.

Proof. By definition, $\bigcup N_i$ is trivially a wall in $\{N_i\}_{i \in I}$. We now apply Lemma 4 inductively to obtain a sequence of walls A_ℓ for $\ell \in \mathbb{N}$, with

$$A_0 := \bigcup N_i, \quad \text{and} \\ A_{\ell+1} := \mathcal{G}(A_\ell).$$

Since I is finite, the union $\bigcup N_i$ is compact and the limit $A_\infty = \lim_{\ell \rightarrow \infty} A_\ell$ exists and is not empty. More precisely, every sequence $\{z_\ell\}_{\ell \in [0, \infty)}$ with $z_\ell \in A_\ell$ for all $\ell \in [0, \infty)$

has a convergent subsequence

$$\{z_{\ell_m}\}_{m=0}^{\infty} \text{ with } z_{\ell_m} \in A_{\ell_m} \text{ and } \lim_{m \rightarrow \infty} z_{\ell_m} = z_{\infty} \in A_{\infty}.$$

We proceed by showing that A_{∞} is a wall. Take a horizontal disk $[h]$ in $\bigcup_{i \in I} N_i$. Since the A_{ℓ} are walls, we have a sequence of points

$$z_{\ell} \in [h] \cap A_{\ell} \subset \bigcup N_i.$$

Again, the sets $\bigcup N_i$ and $[h]$ are compact, so we can choose a convergent subsequence z_{ℓ_m} and its limit $\lim_{m \rightarrow \infty} z_{\ell_m} \in [h] \cap A_{\infty}$, as required.

Finally, by construction, for every point $z \in A_{\infty}$ we have

$$f^k(z) \in \bigcup N_i \text{ for every } k \geq 0,$$

which means that $A_{\infty} \subset A$ and, hence, A is a wall as well. \square

We now proceed with the actual proof of Theorem 1. We assume that there exists a finite family of sets $\{N_i\}_{i \in I} \subset U \subset \mathbb{R}^n$, and a sub-family $\{M_l\}_{l \in L} \subset \{N_i\}_{i \in I}$ of mother sets for a diffeomorphism $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with $n \geq 3$, such that conditions (B1) and (B2) hold with (d_x, d_y, d_u, d_s) and $d_y > d_s$. The goal is to show that the transitive hyperbolic invariant set $\Lambda = \text{inv}(f, U)$ is a d_y -blender. Clearly

$$\begin{aligned} A &= \left\{ z \in \mathbb{R}^n \mid f^k(z) \in \bigcup N_i \text{ for all } k \in \mathbb{N} \right\} \\ &\subset \left\{ z \in \mathbb{R}^n \mid f^k(z) \in U \text{ for all } k \in \mathbb{N} \right\} \subset W^s(\Lambda). \end{aligned} \tag{6}$$

Hence $W^s(\Lambda)$ is a wall, because A is already a wall, by Lemma 5. This means that for every horizontal disk $[h]$ in $\{N_i\}_{i \in I}$, we have

$$[h] \cap W_{\Lambda}^s \neq \emptyset.$$

In particular, this is also true for all C^1 horizontal disks. Therefore, we can take the set of surfaces \mathcal{S} from Definition 3 to be the family of all C^1 horizontal disks $[h]$ in $\{N_i\}_{i \in I}$. Horizontal disks are d_y -dimensional, and $d_y > d_s$, so it follows that Λ is a d_y -blender, which concludes our proof. \square

Remark 7. Conditions (B1) and (B2) ensure that Λ is non-empty; this is guaranteed by (6), i.e., the fact that $A \subset W^s(\Lambda)$, and $A \neq \emptyset$.

3.1 Finding blenders in practice

We can apply Theorem 1 to establish the existence of a blender for a diffeomorphism $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, with $n \geq 3$. Notice that conditions (B1) and (B2) are formulated in terms of

a finite family of sets $\{N_i\}_{i \in I} \subset \mathbb{R}^n$, with the required sub-family $\{M_l\}_{l \in L} \subset \{N_i\}_{i \in I}$ of mother sets; more precisely, it suffices to construct finite sequences of coverings

$$N_{i_0} \xrightarrow{f} N_{i_1} \xrightarrow{f} \dots \xrightarrow{f} N_{i_k} = M_{l'}, \quad (7)$$

for some $l' \in L$, such that conditions (B1) and (B2) are satisfied. Considering an arbitrary horizontal disk $[h]$ of some mother set M_l , we take $N_{i_0} \subseteq M_l$ as the initial h-set that intersects with $[h]$ and construct a covering sequence that satisfies cone conditions. The length and the choice of the h-sets involved in this covering sequence and the choice of $l' \in L$ generally depends on the choice of M_l and the choice of $[h]$. Intuitively, a sequence of the form (7) allows us to establish a ‘link’ between a horizontal disk $[h]$ in a mother set M_l and a possibly different mother set $M_{l'}$. For each such sequence, the set $M_{l'}$ is placed at the end. Such a sequence can be constructed as a covering sequence of h-sets with cone conditions precisely if conditions (B1) and (B2) hold for some (d_x, d_y, d_u, d_s) with $d_y > d_s$.

Remark 8. *In fact, conditions (B1) and (B2) ensure that we are able to find sequences of the form (7). By (B1) we can always start from some h-set N_{i_0} that, in a good way, intersects a horizontal disk $[h]$ from a mother set M_l . Condition (B2) ensures for any given h-set N_i that, if there is no $l' \in L$ such that $N_i = M_{l'}$, then N_i covers some other h-set. Hence, a covering starting from N_{i_0} either covers some $M_{l'}$ in one step, or it can be continued to a longer sequence of coverings of the form (7). Condition (B2) also ensures that each such sequence ends with a mother set $M_{l'}$ for some $l' \in L$. (We cannot have infinite sequences since for each covering, the sets need to be enlarged along the y_u -coordinate; an infinite sequence of coverings would lead to a blow-up.) Thus, we start in a horizontal disk in a mother set M_l and finish in this or another mother set $M_{l'}$.*

As an example, Figure 7 illustrates how such a covering can be found near a homoclinic orbit to a hyperbolic fixed point p . The mother set M_l in panel (a) is chosen such that it contains p and part of its stable and unstable manifolds, denoted $W^u(p)$ and $W^s(p)$, respectively. Near a homoclinic orbit, the sequence of h-sets N_i , with $N_{i_0} \in \{N_1, N_2\} \subset M_l$, can be chosen such that, after a sequence of coverings, the h-set N_{i_k} , for some $i_k \in I$, returns to M_l ; hence, $M_{l'} = M_l$ in this example. Figure 7(a) shows two sequences of h-sets: one covering starts with $N_{i_0} = N_1$ (dark blue) and the other covering starts with $N_{i_0} = N_2$ (light blue). The h-sets are positioned along the intersections of the stable and unstable manifold of p . Successive h-sets N_{i_m} in the covering sequence must be increasingly elongated along their corresponding y_u -coordinates, which also means that the final mother set $M_{l'}$ in the sequence is chosen to be large along this coordinate; consequently, it will not cover any of the sets N_{i_k} since these are smaller along y_u .

In practice, to establish cone conditions as required for condition (B2), it is convenient to choose small initial h-sets N_{i_0} , as in Figure 7(b), instead of the large initial h-sets like N_1 and N_2 in panel (a). These initial h-sets need to satisfy condition (B1). Thus, they need to overlap to ensure that any horizontal disk $[h]$ from M_l leads to a horizontal disk in some initial N_{i_0} . Choosing small initial h-sets allows to use smaller

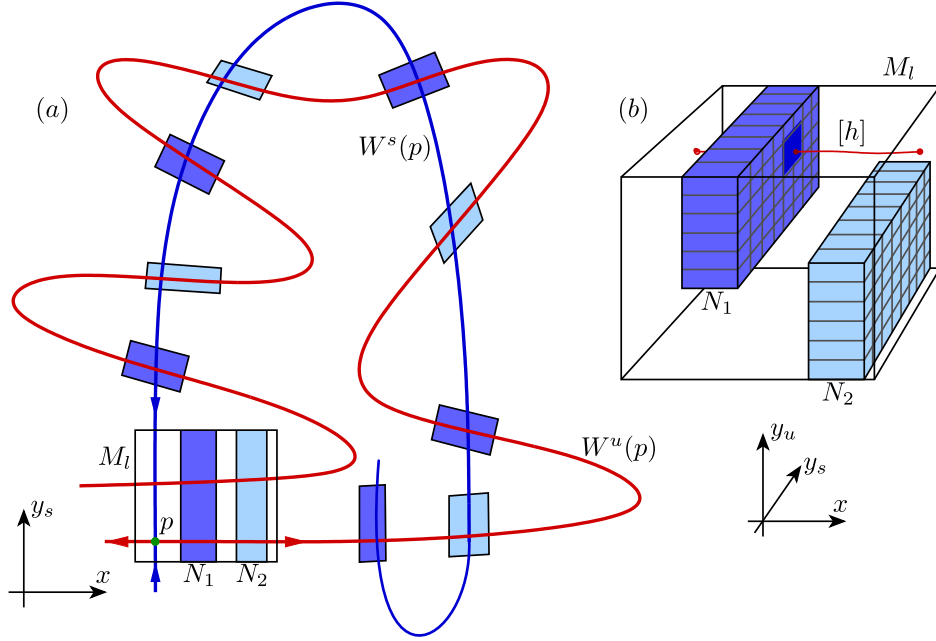


Fig. 7 Construction of a covering sequence. Panel (a) shows an example where an h-set M_l near a hyperbolic fixed point is covered by an appropriate iterate of an h-set $N_1 \subset M_l$. As panel (b) illustrates, the h-set N_1 itself is divided into overlapping smaller h-sets; such a subset is shown in darker blue with a horizontal disk $[h]$ through it.

intermediate h-sets in the sequences of coverings, which improves the accuracy of computations.

4 A 2-blender in the Hénon-like family

We are finally ready to prove Theorem 2, where we claim that the Hénon-like family (1) has a 2-blender for every $\xi \in [1.01, 1.125]$ when $\mu = -9.5$ and $\beta = 0.3$. The proof precisely follows the approach described above. We construct explicit covering relations and verify the assumptions of Theorem 1 with computer-assisted methods; we then apply Theorem 1, which completes the proof. The constructed covering relations also provide an explicit bound on the set in phase space that contains the surfaces that intersect the stable manifold of the 2-blender.

Recall that the Hénon-like family is defined by the function $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ with

$$f(X, Y, Z) = (Y, \mu + Y^2 + \beta X, \xi Z + Y),$$

and we fix $\mu = -9.5$ and $\beta = 0.3$ as in Theorem 2. We assume $\xi > 1$, so that we have $d_x = 1$, and the skew-product structure of f ensures that the weakly expanding local coordinate y_u is the Z -direction, provided ξ is sufficiently small. Here, we use phase variables X , Y and Z to distinguish them from the local coordinates $x \in \mathbb{R}^{d_x}$ and

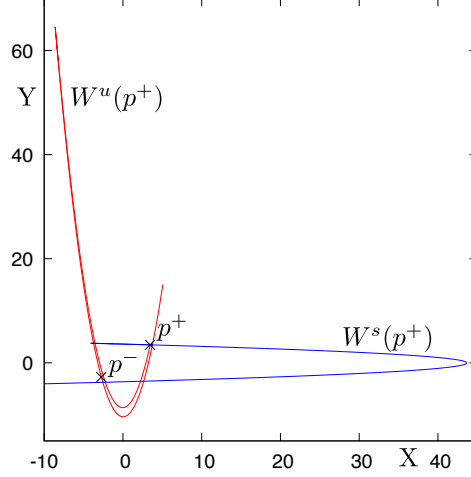


Fig. 8 Projection onto the (X, Y) -plane of the fixed points p^\pm (crosses) of the Hénon-like map (1) with $\mu = -9.5$ and $\beta = 0.3$, with computed initial parts of the unstable and stable manifolds of p^+ , labeled $W^u(p^+)$ (red curve) and $W^s(p^+)$ (blue curve), respectively.

$y \in \mathbb{R}^{d_y}$. As can easily be checked, f has the two fixed points

$$p^\pm = \left(\rho^\pm, \rho^\pm, \frac{\rho^\pm}{1 - \xi} \right),$$

with

$$\rho^\pm = \frac{1}{2} (1 - \beta) \pm \frac{1}{2} \sqrt{(1 - \beta)^2 - 4\mu},$$

which are hyperbolic saddles for $\mu = -9.5$, $\beta = 0.3$ and $\xi > 1$, with two-dimensional unstable and one-dimensional stable manifolds that intersect transversally. This follows from the fact that the restriction of f to the (X, Y) -plane is the standard Hénon map, which is known to feature a full Smale horseshoe for $\mu = -9.5$ and $\beta = 0.3$. Moreover, this skew-product structure of f implies that the Z -direction is an f -invariant (weakly unstable) fibre bundle for points in \mathbb{R}^3 . This means, in particular, that the projection of the one-dimensional stable manifold $W^s(p^+)$ of p^+ is exactly the stable manifold of the fixed point (ρ^+, ρ^+) of the standard Hénon map, while the two-dimensional unstable manifold $W^u(p^+)$ projects to a one-dimensional curve in the (X, Y) -plane, namely, exactly the one-dimensional unstable manifold of (ρ^+, ρ^+) for the standard Hénon map; in other words, $W^u(p^+)$ is a ruled manifold given by the direct product of this one-dimensional unstable manifold and vertical straight lines. A top view of the fixed points p^\pm with the manifolds $W^u(p^+)$ and $W^s(p^+)$ is shown in Figure 8; the manifolds of p^- are not shown in this figure.

4.1 Construction of a family of h-sets

Before we can apply Theorem 1, we must construct a family of the h-sets $\{N_i\}_{i \in I}$, with a selected subset of mother sets $\{M_l\}_{l \in L}$ such that conditions (B1) and (B2) are satisfied. To this end, it suffices to consider only two covering sequences, namely,

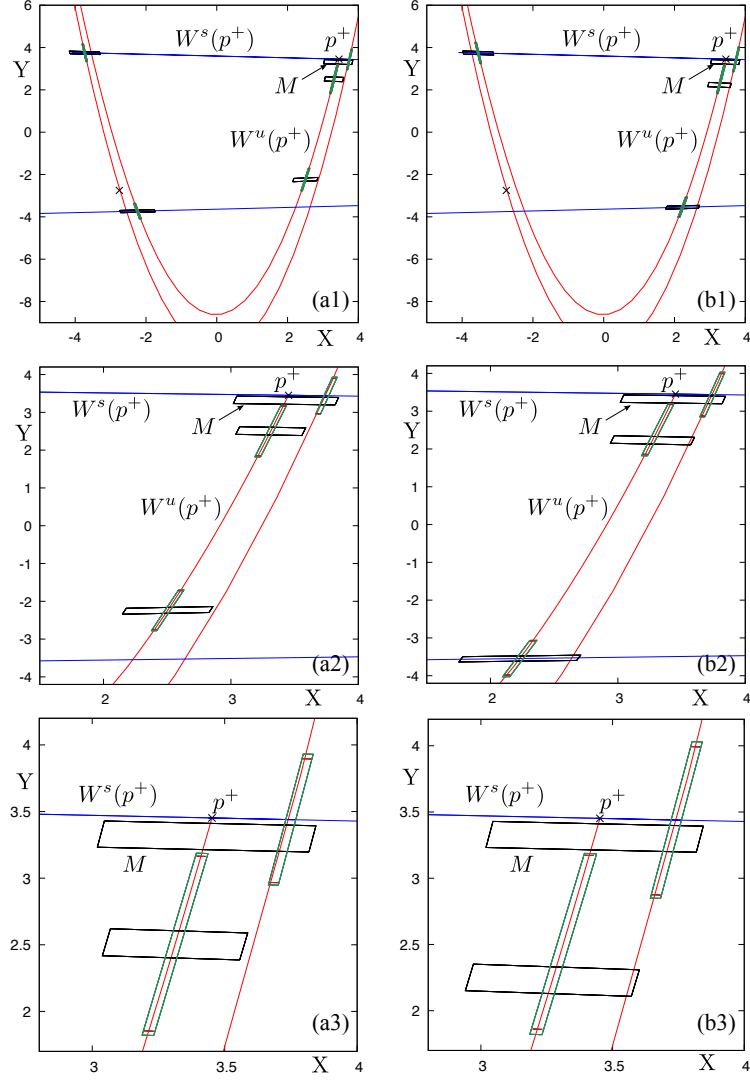


Fig. 9 Top view of the two covering sequences used in the proof, shown by three successive enlargements of the (X, Y) -plane in panels (a1)–(a3) and (b1)–(b3), respectively. The first sequence consists of five and the second of four h-sets (black ‘horizontal rectangular’ sets) parallel to $W^s(p^+)$. The images of these h-sets (dark green ‘vertical rectangular’ sets) are elongated along $W^u(p^+)$; notice the images of the exit sets (brown horizontal lines) in panels (a3) and (b3).

h-sets that are aligned with the manifolds of p^+ and positioned along two different homoclinic orbits to p^+ . Figure 9 gives an idea of their specific locations in projection onto the (X, Y) -plane; compare also with Figure 7. The first column of Figure 9 shows four h-sets (black boxes) chosen such that they cover one of these homoclinic orbits; panel (a1) shows five h-sets and the successive enlargements in panels (a2) and (a3) provide more detail near p^+ . The second column of Figure 9 shows the selection of

four h-sets that cover another homoclinic orbit; again, panel (b1) shows the complete covering sequence and panels (b2) and (b3) are successive enlargements near p^+ . The dark-green sets in Figure 9 are the images of the nine h-sets and the horizontal brown lines in the bottom four panels indicate the images of the exis sets. Note that each rectangle is, in fact, a three-dimensional box with an appropriate range of the Z-coordinate. In particular, the images of the h-sets are also straight boxes with respect to the Z-coordinate, but the actual Z-range for these images, as well as that of their exit sets, depends on the Z-range of the corresponding h-set.

To be more precise, for both coverings, we use the same mother set, which is the h-set labeled M that lies closest to p^+ in Figure 9; this box is positioned slightly away from p^+ along $W^u(p^+)$; it is best seen in the two bottom enlargements in panels (a3) and (b3) of Figure 9. The sides of the mother set, as shown in the (X, Y)-plane, are roughly parallel to the directions of strong expansion (or the direction of $W^u(p^+)$ in Figure 9), and contraction (the direction of $W^s(p^+)$). These h-sets are boxes, that is, they also have sides parallel to the Z-direction, but this is not shown in Figure 9. Following Definition 4, we wish to represent M in local coordinates given by the splitting $(x, y) := (x, y_u, y_s)$ with $d_x = 1$ and $d_y = 2$, as the product $\overline{B}_{d_x} \times \overline{B}_{d_y}$ of two balls. As mentioned in Remark 2, a natural choice is to use the maximum norm on both \overline{B}_{d_x} and \overline{B}_{d_y} , so that M is represented by a Cartesian product of intervals. We define the local coordinate transformation $\gamma_M : M \rightarrow \overline{B}_{d_x} \times \overline{B}_{d_y}$, such that

$$\gamma_M(M) := [-0.1, 0.1] \times \left([-2, 2] \times [-0.4, 0.4] \right). \quad (8)$$

Here, γ_M is defined explicitly as the affine map

$$\gamma_M(z) := A_M^{-1} [z - p_M(\xi)], \text{ for } z \in M, \quad (9)$$

where the columns of the matrix A_M are formed by the (normalized) eigenvectors for $\xi = 1.1$, that is,

$$A_M := \begin{bmatrix} 0.131936 & 0 & -0.998261 \\ 0.984126 & 0 & 0.0447916 \\ 0.118698 & 1 & -0.0383312 \end{bmatrix},$$

and the origin is shifted to the center of the box, which we choose as the point

$$p_M = p_M(\xi) := (3.4319, 3.4319, z_M(\xi)),$$

close to p^+ ; here, the X- and Y-components are explicit, but more work is needed to decide on the Z-component $z_M(\xi)$, which is explained further below. We remark that A_M can be kept fixed over the entire parameter range $\xi \in [1.01, 1.125]$.

The other h-sets in the sequences of covering relations are chosen similarly in an iterative manner, with the X- and Y-coordinates of their center points approximately positioned at the two homoclinic orbits; compare with Figure 9. Leaving the specific choice of Z-coordinate again for later, we select the following two sequences:

$$p_{10}(\xi) := p_M(\xi),$$

$$\begin{aligned}
p_{11}(\xi) &:= (3.3127, 2.5032, z_{11}(\xi)), \\
p_{12}(\xi) &:= (2.5032, -2.2401, z_{12}(\xi)), \\
p_{13}(\xi) &:= (-2.2401, -3.7312, z_{13}(\xi)), \\
p_{14}(\xi) &:= (-3.7312, 3.7495, z_{14}(\xi)).
\end{aligned}$$

and

$$\begin{aligned}
p_{20}(\xi) &:= p_M(\xi), \\
p_{21}(\xi) &:= (3.2714, 2.2300, z_{21}(\xi)), \\
p_{22}(\xi) &:= (2.2300, -3.5459, z_{22}(\xi)), \\
p_{23}(\xi) &:= (-3.5459, 3.7421, z_{23}(\xi));
\end{aligned}$$

Observe that, in projection onto the (X, Y) -plane, $p_{11} \mapsto p_{12} \mapsto p_{13} \mapsto p_{14}$ and $p_{21} \mapsto p_{22} \mapsto p_{23}$ under the action of f . Furthermore, to good approximation, we have $p_{10} \mapsto p_{11}$ and $p_{20} \mapsto p_{21}$, while the (X, Y) -projection of the images of both p_{14} and p_{23} is approximately that of p_M as well. The linearization about each of the sequences $\{p_{1b}\}_{b=0,\dots,k_1}$, with $k_1 = 4$, and $\{p_{2b}\}_{b=0,\dots,k_2}$, with $k_2 = 3$, gives rise to appropriate affine changes of coordinates of the form

$$\gamma_{1b}(z) = A_{1b}^{-1} [z - p_{1b}(\xi)] \quad \text{and} \quad \gamma_{2b}(z) = A_{2b}^{-1} [z - p_{2b}(\xi)], \quad (10)$$

with $A_{10} = A_{20} = A_M$ and

$$\begin{aligned}
A_{1(b+1)} &\approx Df(p_{1b}) A_{1b} \text{ for } b = 0, \dots, k_1, \\
A_{2(b+1)} &\approx Df(p_{2b}) A_{2b} \text{ for } b = 0, \dots, k_2,
\end{aligned}$$

that define the h-sets N_{1b} , for $b = 0, \dots, k_1$ and N_{2b} , for $b = 0, \dots, k_2$. Note that $\gamma_{10} = \gamma_{20} = \gamma_M$, since $A_{10} = A_{20} = A_M$ and $p_{10}(\xi) = p_{20}(\xi) = p_M(\xi)$. The local coordinates of the other h-sets are defined recursively based on successive iterates of f . Specifically, the matrices for coordinate changes at points p_{1b} along the first homoclinic orbit were chosen as

$$\begin{aligned}
A_{11} &= \begin{bmatrix} 0.15187 & 0 & -0.64804 \\ 1.0123 & 0 & 0.039354 \\ 0.17202 & 1 & -0.038011 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0.15622 & 0 & 0.85005 \\ 0.78914 & 0 & 0.056412 \\ 0.18542 & 1 & -0.053097 \end{bmatrix}, \\
A_{13} &= \begin{bmatrix} 0.12178 & 0 & 1.2185 \\ -0.53836 & 0 & 0.04927 \\ 0.15326 & 1 & -0.043093 \end{bmatrix}, \quad A_{14} = \begin{bmatrix} -0.08308 & 0 & 1.0643 \\ 0.62561 & 0 & -0.046216 \\ -0.057064 & 1 & 0.040401 \end{bmatrix};
\end{aligned}$$

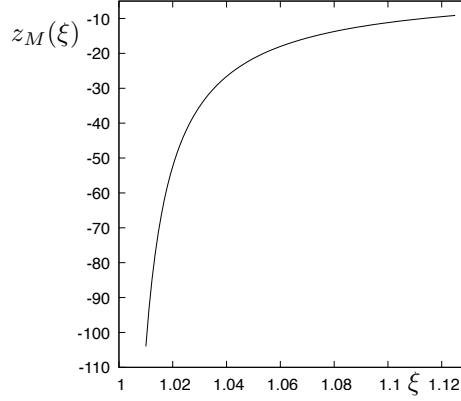


Fig. 10 Graph of $z_M(\xi)$ for $\xi \in [1.01, 1.125]$ that satisfy Equations (11) and (12).

and those for coordinate changes at points p_{2b} along the second homoclinic orbit were chosen as

$$A_{21} = \begin{bmatrix} -0.15209 & 0 & -0.78781 \\ -1.0012 & 0 & 0.053503 \\ -0.17003 & 1 & -0.050450 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} -0.15451 & 0 & 1.1557 \\ -0.69615 & 0 & 0.049190 \\ -0.18337 & 1 & -0.043018 \end{bmatrix},$$

$$A_{23} = \begin{bmatrix} -0.10743 & 0 & 1.0625 \\ 0.75471 & 0 & -0.046214 \\ -0.13856 & 1 & 0.040391 \end{bmatrix}.$$

The lengths of the column vectors for the matrices A_{1b} and A_{2b} are chosen so that the expansion and contraction in each local map is (roughly) the same for each covering relation involved in a given homoclinic excursion.

Let us now return to the questions how to choose the Z-ranges for the h-sets M , N_{1b} , for $b = 0, \dots, k_1$ and N_{2b} , for $b = 0, \dots, k_2$. Recall that the Z-coordinate represents the weakly expanding coordinate y_u . Our goal is to achieve a covering relation as sketched in Figure 4. To this end, we define the Z-coordinates of the center points in each h-set N_{1b} and N_{2b} recursively as follows. We formally define

$$z_{1(k+1)}(\xi) = \pi_Z f(p_{1k}(\xi)) = \xi z_{1k}(\xi) + \pi_Y p_{1k}(\xi),$$

$$z_{2(k+1)}(\xi) = \pi_Z f(p_{2k}(\xi)) = \xi z_{2k}(\xi) + \pi_Y p_{2k}(\xi),$$

where

$$z_{10}(\xi) = z_{20}(\xi) := z_M(\xi),$$

and π_Y and π_Z are the projections onto the Y- and Z-coordinates, respectively. In our implementation, we choose different values of $z_M(\xi)$ for different ξ to ensure that,

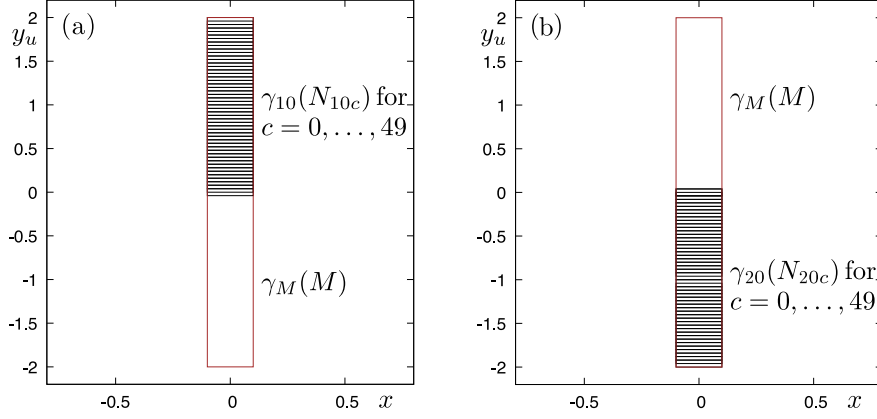


Fig. 11 Projection onto the local (x, y_u) -coordinate plane of the mother set M , together with, in panel (a), the initial h-sets N_{10c} along the first homoclinic orbit from Figure 9(a) and, in panel (b), the initial h-sets N_{20c} along the second homoclinic orbit from Figure 9(b), both for $c = 0, \dots, 49$.

after an excursion along the first homoclinic orbit given by the points p_{1b} , the change in the Z -coordinate is negative, meaning that

$$l_1(z_M(\xi), \xi) := \pi_Z f(p_{14}(\xi)) = \xi z_{14}(\xi) + \pi_Y p_{14}(\xi) < z_M(\xi). \quad (11)$$

Simultaneously, we require that, after an excursion along the second homoclinic orbit given by p_{2b} , the change in the Z -coordinate is positive, meaning that

$$l_2(z_M(\xi), \xi) := \pi_Z f(p_{23}(\xi)) = \xi z_{23}(\xi) + \pi_Y p_{23}(\xi) > z_M(\xi). \quad (12)$$

By combining Equations (11) and (12), we find $z_M(\xi) = z$ numerically as the ξ -parametrised solutions to the scalar equation $\frac{1}{2}(l_1(z, \xi) + l_2(z, \xi)) - z = 0$. Figure 10 shows the result of this approach and represents the choice of $z_M(\xi)$ we used for the computer-assisted proof.

As mentioned just before the start of Section 4, to check condition (B2) in practice, it is convenient to subdivide the resulting two families of h-sets $\{N_{1b}\}_{b=0, \dots, k_1}$ and $\{N_{2b}\}_{b=0, \dots, k_2}$ into smaller overlapping h-sets; in local coordinates, this overlap only needs to be present in the y_u -direction. We use a total of 100 subdivisions, 50 each for N_{1b} and N_{2b} , identified by the index $c \in \{0, 1, \dots, 49\}$. The entire family $\{N_i\}_{i \in I}$ of h-sets is then defined as

$$\gamma_{ab}^{-1}(N_{abc}) := [-0.1, 0.1] \times (\mathcal{I}_{abc} \times [-0.4, 0.4]),$$

where the set of indices for the h-sets $N_i = N_{abc}$ is

$$I := \{abc \mid ab \in \{10, 11, 12, 13, 14, 20, 21, 22, 23\}, \text{ and } c \in \{0, 1, \dots, 49\}\} \cup \{0\};$$

here, the index 0 is added so that we can choose $N_0 := M$. The intervals \mathcal{I}_{abc} for the local coordinate y_u are chosen as follows. The intervals \mathcal{I}_{10c} and \mathcal{I}_{20c} for $c \in$

$\{0, 1, \dots, 49\}$ are defined as

$$\mathcal{I}_{10c} := c\Delta + [-\Delta, \Delta] \quad \text{and} \quad \mathcal{I}_{20c} := -c\Delta + [-\Delta, \Delta],$$

with $\Delta = \frac{2}{50} = 0.04$. Note that

$$\mathcal{I}_{100} = \mathcal{I}_{200} = [-\Delta, \Delta].$$

Moreover, for $c = 49$, the right edge of \mathcal{I}_{10c} is 2 and the left edge of \mathcal{I}_{20c} is -2 . Hence, the union of the 50 intervals \mathcal{I}_{10c} form the interval $[-\Delta, 2]$ and the union of the 50 intervals \mathcal{I}_{20c} forms the interval $[-2, \Delta]$. In total, we have 100 overlapping intervals that range over $[-2, 2]$. Therefore, since $\gamma_M = \gamma_{10c} = \gamma_{20c}$, by construction, we have

$$\gamma_M(M) = \bigcup_{c=0}^{49} \gamma_{10}(N_{10c}) \cup \bigcup_{c=0}^{49} \gamma_{20}(N_{20c}). \quad (13)$$

The h-sets N_{10c} and N_{20c} , for $c \in \{0, \dots, 49\}$ are shown in Figure 11 in projection onto the local (x, y_u) -coordinate plane. They are the initial h-sets for the sequences of coverings (7). We intentionally arrange these h-sets to overlap so that they enclose every horizontal disk in M , as is required for condition (B1); this is discussed in more detail at the end of Section 4.2.2.

The remainder of the intervals \mathcal{I}_{abc} , for $abc \in I$ with $b > 0$, are chosen recursively by computing the images under $\gamma_{ab} \circ f \circ \gamma_{a(b-1)}^{-1}$ of the boxes $[-0.1, 0.1] \times (\mathcal{I}_{a(b-1)c} \times [-0.4, 0.4])$, and choosing local coordinates for \mathcal{I}_{abc} such that it contains the y_u -range of the resulting image. This is done automatically by our implementation and ensures that we have topological entry along y_u .

4.2 Validation of conditions (B1) and (B2)

We claim that the family of h-sets $\{N_{abc}\}_{abc \in I}$, with $ab \in \{10, 11, 12, 13, 14, 20, 21, 22, 23\}$ and $c \in \{0, \dots, 49\}$ satisfies conditions (B1) and (B2) of Theorem 1. In this section, we explain how we prove this in a computer-assisted manner. The additional requirement of hyperbolicity and transitivity of the invariant set in a suitable set $U \subset \mathbb{R}^3$ is addressed in the Appendix. There are various methods for the validation of covering relations and cone conditions; we direct the reader to [8, 14, 23, 25, 27, 28], which deal with this topic in various contexts. When the topological exit set for the coverings and cone conditions is of dimension $d_x = 1$, which is the specific case for the Hénon family, we can use the approach outlined in the following sections.

4.2.1 Validating covering relations when $d_x = 1$

If $d_x = 1$ then $\overline{B}_{d_x} = [-1, 1]$ and the exit set N_i^- consists of two parts for any $i \in I$; in the coordinates given by the homeomorphism γ_i they are

$$N_{\gamma_i, l}^- = \{-1\} \times \overline{B}_{d_y} \quad \text{and} \quad N_{\gamma_i, r}^- = \{1\} \times \overline{B}_{d_y},$$

where the subscript ‘ l ’ stands for ‘left’, and ‘ r ’ for ‘right’. This notation allows us to formulate the following useful result.

Lemma 6. *Let $N_i, N_j \subset \mathbb{R}^n$, with $n \geq 3$, be two h -sets with associated transformations γ_i and γ_j that each map to a product of closed balls in local coordinates (x, y) ; here, the component x is one dimensional with corresponding projection operator π_x . Furthermore, let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a diffeomorphism. Recall that the action of f in local coordinates is denoted by $f_{ji} = \gamma_j \circ f \circ \gamma_i^{-1}$. Assume that either*

$$\pi_x [f_{ji}(N_{\gamma_i, l}^-)] < -1 \quad \text{and} \quad \pi_x [f_{ji}(N_{\gamma_i, r}^-)] > 1, \quad (14)$$

or

$$\pi_x [f_{ji}(N_{\gamma_i, l}^-)] > 1 \quad \text{and} \quad \pi_x [f_{ji}(N_{\gamma_i, r}^-)] < -1. \quad (15)$$

Then $N_i \xrightarrow{f} N_j$, provided

$$f_{ji}(N_{\gamma_i}) \cap \overline{B}_{d_x} \times (\mathbb{R}^{d_y} \setminus B_{d_y}) = \emptyset. \quad (16)$$

Proof. We define a homotopy $\varsigma : [0, 1] \times N_{\gamma_i} \rightarrow \mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$ as

$$\varsigma(t, (x, y)) = \begin{cases} (\pi_x f_{ji}(x, y), (1 - 2t) \pi_y f_{ji}(x, y)), & \text{if } t \in [0, \frac{1}{2}], \\ ((2t - 1)Ax + 2(1 - t)\pi_x f_{ji}(x, y), 0), & \text{if } t \in (\frac{1}{2}, 1], \end{cases}$$

where A is the trivial 1×1 matrix $A = [2]$ for the case of assumption (14), and $A = [-2]$ for assumption (15). This homotopy fulfils the conditions from Definition 5. \square

We use Lemma 6 to check whether the c -dependent families of sequences $\{N_{abc}\}_{abc \in I}$ for $a = 0$ and $a = 1$ are, in fact, covering sequences. More precisely, for each $c \in \{0, \dots, 49\}$, we validate that either assumptions (14) and (16) or assumptions (15) and (16) hold for the h -sets

$$N_{1(b-1)c} \xrightarrow{f} N_{1bc} \quad \text{for } b = 1, 2, 3, 4, \quad (17)$$

$$N_{2(b-1)c} \xrightarrow{f} N_{2bc} \quad \text{for } b = 1, 2, 3, \quad (18)$$

We also validate that the final pairs in these two sequences are covering relations, namely, that we have

$$N_{14c} \xrightarrow{f} M \quad \text{for } c = 0, \dots, 49, \quad (19)$$

$$N_{23c} \xrightarrow{f} M \quad \text{for } c = 0, \dots, 49. \quad (20)$$

We refer again to Figure 9, which also shows the images of the sets involved in the two covering sequences; those associated with the covering relations in (17) and (19) are depicted in the left column, and those associated with (18) and (20) in the right column. Figure 12 shows for each of the two homoclinic excursions, the last covering

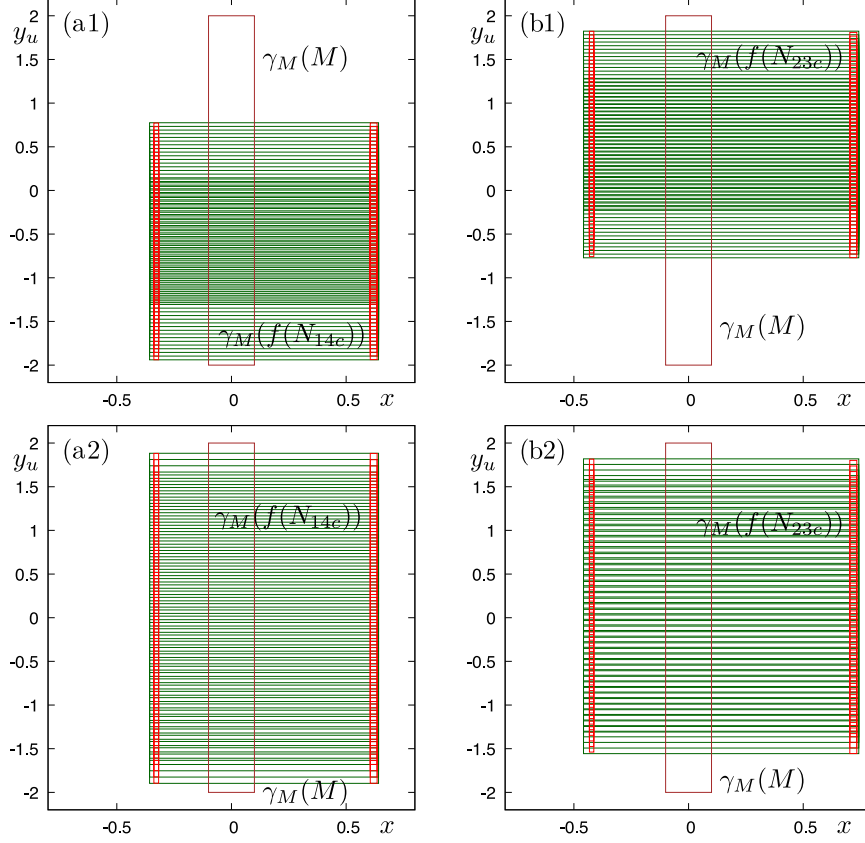


Fig. 12 The covering in local coordinates of the set M (dark red vertical rectangle) by final sets, shown in projection onto the (x, y_u) -plane. Panels (a1) and (b1) show the coverings $N_{14c} \xrightarrow{f} M$ and $N_{23c} \xrightarrow{f} M$, for $\xi \in [1.01, 1.011]$, respectively, and panels (a2) and (b2) show these coverings for $\xi \in [1.124, 1.125]$; here $c = 0, \dots, 49$ and the bounds on the images of the exit sets are shown in red.

relation in the sequence in projection onto the (x, y_u) -plane of local coordinates; see also the green vertical boxes $f(N_{14c})$ and $f(N_{23c})$ in the top-right corners of panels (a3) and (b3) of Figure 9, respectively. In Figure 12, we also show the projections of the coverings (19)–(20) onto the local coordinates (x, y_u) induced by γ_M . The covering relations (17)–(20) form the covering sequences that must satisfy cone conditions, as required for condition (B2).

4.2.2 Validating cone conditions when $d_x = 1$

To prove cone conditions for $d_x = 1$ we make use of the notion of an *interval enclosure of a derivative*. This means that each element in the Jacobian matrix is an interval given by the range of possible values achieved by the corresponding partial derivatives. For $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and a set $U \subset \mathbb{R}^n$ we write the interval enclosure of the $n \times n$

Jacobian matrix Dg in U as

$$[Dg(U)] := \left[(Dg(U))_{k,m} \right]_{k,m \in \{1, \dots, n\}} \subset \mathbb{R}^{n \times n},$$

where,

$$(Dg(U))_{k,m} = \left[\inf_{z \in U} \frac{\partial \pi_{x_k} g}{\partial x_m}(z), \sup_{z \in U} \frac{\partial \pi_{x_k} g}{\partial x_m}(z) \right].$$

For a vector $v \in \mathbb{R}^n$, the matrix product $[Dg(U)] v$ is given by

$$[Dg(U)] v := \{ Dv \mid D \in [Dg(U)] \} \subset \mathbb{R}^n.$$

We also introduce the following notation for a particular choice of cone. Recall from Equation (3) that a cone in local coordinates at $z_{\gamma_i} = \gamma_i(z) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_y}$ is defined as

$$\mathcal{C}_{\gamma_i}(z_{\gamma_i}) := \left\{ (q_x, q_y) \in \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \mid \|q_x - \pi_x(z_{\gamma_i})\|_{d_x} > \|q_y - \pi_y(z_{\gamma_i})\|_{d_y} \right\}.$$

In our special case with $n = 3$ and local coordinates (x, y_u, y_s) , we choose the L_∞ -norm for both projections, that is,

$$\|\pi_x(x, y_u, y_s)\|_{d_x} = \|x\|_1 = |x|,$$

and

$$\|\pi_y(x, y_u, y_s)\|_{d_y} = \|y\|_2 = \|(y_u, y_s)\| := \max \left\{ \frac{|y_u|}{\kappa_u}, \frac{|y_s|}{\kappa_s} \right\},$$

for some positive constants $\kappa_u, \kappa_s > 0$. Using these norms, we define a cone at $z \in N_i$ as

$$\mathcal{C}_i(z) = \gamma_i(\mathcal{C}_{\gamma_i}(z_{\gamma_i}; \kappa_u, \kappa_s)),$$

where

$$\begin{aligned} \mathcal{C}_{\gamma_i}(z_{\gamma_i}; \kappa_u, \kappa_s) = \{ q \in \mathbb{R}^n \mid & q - \gamma_i(z) = x(1, y_u, y_s) \\ & \text{with } |y_u| < \kappa_u, |y_s| < \kappa_s \text{ and } x \in \mathbb{R} \}. \end{aligned}$$

With the above definition of a cone, we validate that the family of h-sets $\{N_{abc}\}_{abc \in I}$, with $ab \in \{10, 11, 12, 13, 14, 20, 21, 22, 23\}$ and $c \in \{0, \dots, 49\}$ satisfies cone conditions; this can be done by using only local coordinates with the following lemma.

Lemma 7. *Let $i, j \in I$, and let the h-sets N_i and N_j be equipped with the cones*

$$\begin{aligned} \mathcal{C}_i(z) &= \gamma_i(\mathcal{C}_{\gamma_i}(z_{\gamma_i}; \kappa_u^i, \kappa_s^i)) \text{ and} \\ \mathcal{C}_j(z) &= \gamma_j(\mathcal{C}_{\gamma_j}(z_{\gamma_j}; \kappa_u^j, \kappa_s^j)), \end{aligned}$$

for fixed constants $\kappa_u^i, \kappa_s^i, \kappa_u^j, \kappa_s^j > 0$, respectively. Suppose that for every $v \in \{1\} \times [-\kappa_u^i, \kappa_u^i] \times [-\kappa_s^i, \kappa_s^i]$ and for every vector

$$w = (w_x, w_{y_u}, w_{y_s}) \in [Df_{ji}(N_{\gamma_i})] v$$

we have $w_x \neq 0$ and

$$\left(1, \frac{w_{y_u}}{w_x}, \frac{w_{y_s}}{w_x}\right) \in \{1\} \times [-\kappa_u^j, \kappa_u^j] \times [-\kappa_s^j, \kappa_s^j] \quad (21)$$

Then f satisfies cone conditions from N_i to N_j .

Proof. Let $z \in N_i$ and take $q = (q_x, q_y) \in \mathcal{C}_{\gamma_i}(z_{\gamma_i}; \kappa_u^i, \kappa_s^i)$. Since $\gamma_j(f(z)) = f_{ji}(z_{\gamma_i})$, we must show that $f_{ji}(q) \in \mathcal{C}_{\gamma_j}(\gamma_j(f(z)); \kappa_u^j, \kappa_s^j)$. From the Mean Value Theorem, we have

$$f_{ji}(q) - f_{ji}(z_{\gamma_i}) \in [Df_{ji}(N_{\gamma_i})] (q - z_{\gamma_i}),$$

which implies that

$$\begin{aligned} f_{ji}(q) - f_{ji}(z_{\gamma_i}) &= |\pi_x(q - z_{\gamma_i})| [Df_{ji}(N_{\gamma_i})] \frac{q - z_{\gamma_i}}{|\pi_x(q - z_{\gamma_i})|} \\ &= |\pi_x(q - z_{\gamma_i})| w, \end{aligned}$$

for some $w = (w_x, w_{y_u}, w_{y_s}) \in [Df_{ji}(N_{\gamma_i})] v$, with $v = (q - z_{\gamma_i}) / |\pi_x(q - z_{\gamma_i})|$. Our choice of norms combined with assumption (21) guarantees that $w \in \mathcal{C}_{\gamma_j}(0; \kappa_u^j, \kappa_s^j)$. Hence, $f_{ji}(q) - f_{ji}(z_{\gamma_i}) \in \mathcal{C}_{\gamma_j}(0; \kappa_u^j, \kappa_s^j)$ and, thus, $f_{ji}(q) \in \mathcal{C}_{\gamma_j}(f_{ji}(z); \kappa_u^j, \kappa_s^j)$, as required. \square

We apply Lemma 7 as follows. Recall that the h-sets in our family are defined in local coordinates as

$$\gamma_{ab}^{-1}(N_{abc}) := [-0.1, 0.1] \times (\mathcal{I}_{abc} \times [-0.4, 0.4]),$$

and the intervals \mathcal{I}_{abc} are defined recursively based on iterates of the intervals

$$\mathcal{I}_{10c} := c\Delta + [-\Delta, \Delta] \quad \text{and} \quad \mathcal{I}_{20c} := -c\Delta + [-\Delta, \Delta],$$

with $\Delta = \frac{2}{50} = 0.04$. We choose $\kappa_u = \kappa_s = \frac{1}{2}\Delta = 0.02$ as the positive constants associated with cones for the h-sets N_{10c} and N_{20c} for $c = 0, \dots, 49$. Hence, for $i \in \{10c, 20c\}_{c \in \{0, \dots, 49\}}$, the cones are defined in local coordinates as

$$\mathcal{C}_{\gamma_i}(z_{\gamma_i}; \tfrac{1}{2}\Delta, \tfrac{1}{2}\Delta) = \left\{ q \in \mathbb{R}^n \mid \begin{aligned} &\tfrac{1}{2}\Delta |\pi_x(q - \gamma_i(z))| > |\pi_{y_u}(q - \gamma_i(z))| \text{ and} \\ &\tfrac{1}{2}\Delta |\pi_x(q - \gamma_i(z))| > |\pi_{y_s}(q - \gamma_i(z))| \end{aligned} \right\}.$$

We note that, due to our choice $\kappa_u = \kappa_s = \frac{1}{2}\Delta$ and the fact that the sets $\{N_{10c}, N_{20c}\}_{c=0, \dots, 49}$ overlap and (13), for every horizontal disk $[h]$ in M we can choose some $N_i \in \{N_{10c}, N_{20c}\}_{c=0, \dots, 49}$ so that $[h] \cap N_i = [h_i]$, for some horizontal disk $[h_i]$ in N_i . This ensures (B1). We propagate the cones by using Lemma 7 with appropriate choices for the constants κ_u and κ_s to ensure that cone conditions are satisfied for each of the coverings. In this computer-assisted way, we show that condition (B2) is satisfied.

4.3 Final steps in proving Theorem 2

The validation of the conditions (B1) and (B2) described in Section 4.2 for the family of h -sets $\{N_i\}_{i \in I}$ constructed in Section 4.1 is performed for any choice $\xi \in [1.01, 1.125]$ for the Hénon-like family (1). The proof that there exists a set $U \subset \mathbb{R}^3$ such that $\bigcup N_i \subset U$ and the invariant set of f in U is hyperbolic and transitive uses standard, earlier established methods and can be found in the Appendix. Since the sets involved in our construction depend on the choice of the parameter ξ , we apply a form of interval enclosure on the parameter as well. More precisely, we subdivide the parameter interval $[1.01, 1.125]$ into 115 sub-intervals of width 10^{-3} and conduct our construction separately on each of these sub-intervals; the computer-assisted validation over the entire range of ξ -values takes under a second on a standard laptop. Taken together, we obtain a proof of the existence of a 2-blender for the ξ -family of Hénon-like maps with $\xi \in [1.01, 1.125]$.

We have used the CAPD library² [18] as the tool for our interval arithmetic computations.

5 Robust heterodimensional cycles

We foresee that our method can be particularly useful to locate and compute (robust) heterodimensional cycles, which are key elements for generating higher-dimensional (wild) chaotic dynamics [6, 20, 21]. In our context, their definition is as follows; for example, see [3].

Definition 9 (Heterodimensional cycle). *Let Λ and $\tilde{\Lambda}$ be two hyperbolic sets of a diffeomorphism $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ with stable dimensions $d_s > 0$ and $\tilde{d}_s > 0$, and with unstable dimensions $d_u > 0$ and $\tilde{d}_u > 0$, respectively. We say that f has a heterodimensional cycle if $d_u \neq \tilde{d}_u$, while both $W^u(\Lambda) \cap W^s(\tilde{\Lambda}) \neq \emptyset$ and $W^s(\Lambda) \cap W^u(\tilde{\Lambda}) \neq \emptyset$.*

Note that this definition implies that $d_u + \tilde{d}_s < n$ or $d_s + \tilde{d}_u < n$, which means that a heterodimensional cycle is an object of codimension 1 or higher. Nevertheless, the existence of a heteroclinic cycle can be a robust property due to the existence of a blender [5, 6].

As an example, we consider a heterodimensional cycle in \mathbb{R}^3 , between a 2-blender Λ , with $d_s = 1$ and $d_u = 2$, and a hyperbolic fixed point $\tilde{\Lambda} = p$ with $\tilde{d}_s = 2$ and $\tilde{d}_u = 1$. Hence, p has a two-dimensional stable manifold $W^s(p)$ and a one-dimensional unstable manifold $W^u(p)$, which interact with the global manifolds of the blender Λ . This situation is illustrated in Figure 13, where the blender is the invariant set inside M and the hyperbolic fixed point p lies in \tilde{M} . Figure 13 also shows an example of a *connecting sequence*, which is a notion adapted from [8] that we require for our construction.

Definition 10 (Connecting sequence [8]). *Consider two h -sets M and \tilde{M} , and a family of h -sets $\{K_j\}_{j \in J}$ of a diffeomorphism $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Let $[h]$ be a horizontal disk in M . We say that $K_{j_0}, \dots, K_{j_r} \in \{K_j\}_{j \in J}$ is a connecting sequence from $[h]$ to \tilde{M} if*

1. $[h] \cap K_{j_0} = [h_{j_0}]$ for some horizontal disk $[h_{j_0}]$ in K_{j_0} ;

²Computer Assisted Proofs in Dynamics: <http://capd.ii.uj.edu.pl>.

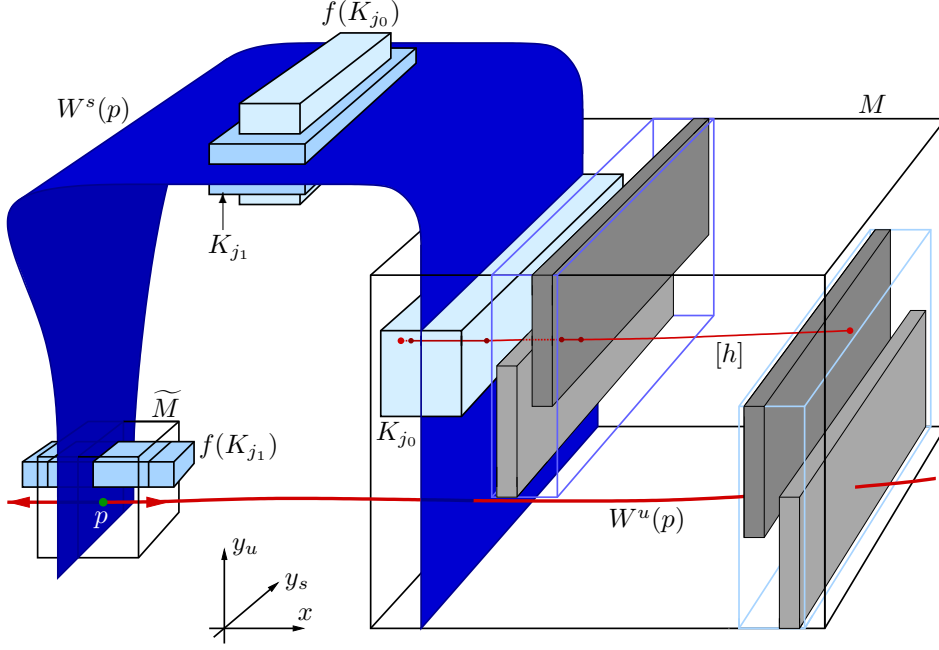


Fig. 13 Sketch of a heterodimensional cycle between a 2-blender in M and a hyperbolic fixed point p in \widetilde{M} , featuring a connecting sequence K_{j_0}, K_{j_1} from a horizontal disk $[h]$ in M to \widetilde{M} .

2. We have a sequence of coverings

$$K_{j_0} \xrightarrow{f} K_{j_1} \xrightarrow{f} \dots \xrightarrow{f} K_{j_r} \xrightarrow{f} \widetilde{M};$$

3. The diffeomorphism f satisfies cone conditions from $K_{j_{m-1}}$ to K_{j_m} , for $m = 1, \dots, r$, as well as cone conditions from K_{j_r} to \widetilde{M} .

The connecting sequence between a horizontal disk $[h]$ in M and \widetilde{M} in Figure 13 is given by the sets K_{j_0} and K_{j_1} . This connecting sequence, combined with the covering relations associated with the mother sets M and \widetilde{M} guarantee the existence of a heterodimensional cycle, provided the assumptions for Theorem 1 hold together with one additional condition, as formulated in the following theorem.

Theorem 8 (Existence of a heterodimensional cycle). *Consider two sets $U, \widetilde{U} \subset \mathbb{R}^n$ with two hyperbolic invariant sets*

$$\Lambda = \text{Inv}(f, U) \quad \text{and} \quad \widetilde{\Lambda} = \text{Inv}(f, \widetilde{U}).$$

Let $\{M_l\}_{l \in L} \subset \{N_i\}_{i \in I} \subset U$ and $\{\widetilde{M}_l\}_{l \in \widetilde{L}} \subset \{\widetilde{N}_i\}_{i \in \widetilde{I}} \subset \widetilde{U}$ be families of h -sets for a diffeomorphism $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$. Assume that conditions (B1) and (B2) hold for $\{N_i\}_{i \in I}$ with (d_x, d_y, d_u, d_s) and for $\{\widetilde{N}_i\}_{i \in \widetilde{I}}$ with $(d_x, d_y, \widetilde{d}_u, \widetilde{d}_s)$, but $d_u \neq \widetilde{d}_u$. Suppose further that there exists a family of h -sets $\{K_j\}_{j \in J}$ that satisfies the following condition.

(B3) For every $l \in L$ and for every horizontal disk $[h]$ in M_l there exists $l' \in \tilde{L}$ and a connecting sequence in $\{K_j\}_{j \in J}$ from $[h]$ to $\tilde{M}_{l'}$; likewise, for every $l \in \tilde{L}$ and for every horizontal disk $[h]$ in \tilde{M}_l there exists $l' \in L$ and a connecting sequence $\{K_j\}_{j \in J}$ from $[h]$ to $M_{l'}$; these connecting sequences are finite, meaning that their lengths are bounded by the same constant.

Then

$$W^u(\Lambda) \cap W^s(\tilde{\Lambda}) \neq \emptyset \quad \text{and} \quad W_\Lambda^s(\Lambda) \cap W^u(\tilde{\Lambda}) \neq \emptyset.$$

In other words, f has a heterodimensional cycle, because $d_u \neq \tilde{d}_u$.

Proof. We only prove that $W^u(\Lambda) \cap W^s(\tilde{\Lambda}) \neq \emptyset$; the proof of $W_\Lambda^s(\Lambda) \cap W^u(\tilde{\Lambda}) \neq \emptyset$ follows from mirror arguments. Observe that the backward invariant set of f in $\bigcup N_i$ is a subset of $W^u(\Lambda)$, because $\bigcup N_i \subset U$ and $\Lambda = \text{Inv}(f, U)$ is hyperbolic. Similarly, since $\bigcup \tilde{N}_i \subset \tilde{U}$ and $\tilde{\Lambda} = \text{Inv}(f, \tilde{U})$ is hyperbolic, the forward invariant set of f in $\bigcup \tilde{N}_i$ is a subset of $W^s(\tilde{\Lambda})$.

Since conditions (B1) and (B2) hold, we have that for every $l \in L$ and every horizontal disk $[h]$ in M_l there exists a finite sequence of covering relations in $\{N_i\}_{i \in I}$ from $[h]$ to $M_{l'}$ for some $l' \in L$; see Remark 8. We denote such a covering sequence by

$$M_l \supset N_{i_0} \xrightarrow{f} N_{i_1} \xrightarrow{f} \dots \xrightarrow{f} N_{i_{k[h]}} \xrightarrow{f} M_{l'},$$

where the choice of the h-sets $N_{i_0}, N_{i_1}, \dots, N_{i_{k[h]}}$ and the choice of $k[h] \geq 1$ may depend on $[h]$. By Theorem 3 there exists a horizontal disk $[h'] \in M_{l'}$ such that

$$[h'] = \left\{ f^{k[h]}(z) \mid z \in [h] \text{ and } f^m(z) \in N_{i_m} \text{ for } m = 1, \dots, k[h] \right\}.$$

We wish to emphasise the dependence of the horizontal disk $[h'] \in M_{l'}$ on $[h] \in M_l$ and introduce the notation

$$\mathcal{N}([h]) := [h'].$$

Hence, for $z \in \mathcal{N}([h])$ we have $f^{-m}(z) \in \bigcup N_i$ for $m = 1, \dots, k[h]$, for some $k[h] \geq 1$.

Similarly, conditions (B1) and (B2) and Theorem 3 imply that for every $l \in \tilde{L}$ and every horizontal disk $[\tilde{h}] \in \tilde{M}_l$, there exists $l' \in \tilde{L}$ and a horizontal disk $\tilde{\mathcal{N}}(\tilde{h}) \in \tilde{M}_{l'}$, constructed from forward images of $[\tilde{h}]$.

Finally, condition (B3) and Theorem 3 imply that for every $l \in L$ and every horizontal disk $[h]$ in $M_l \subset \bigcup N_i$ there exists $l' \in \tilde{L}$ and a finite connecting sequence

$$M_l \supset K_{j_0} \xrightarrow{f} K_{j_1} \xrightarrow{f} \dots \xrightarrow{f} K_{j_{r[h]}} \xrightarrow{f} \tilde{M}_{l'},$$

with $r[h] \geq 1$, such that a horizontal disk $\mathcal{K}([h])$ in $\tilde{M}_{l'} \subset \bigcup \tilde{N}_i$ can be constructed from forward images of $[h]$.

We now take an arbitrary $l \in L$ and an arbitrary horizontal disk $[h_0]$ in M_l . Then the triple

$$[h_\ell] := \mathcal{N}^{(\ell)}([h_0]), \quad [\tilde{h}_\ell] := \mathcal{K}([h_\ell]), \quad [\tilde{h}'_\ell] := \tilde{\mathcal{N}}^{(\ell)}([\tilde{h}_\ell]), \quad \text{for } \ell \in \mathbb{N},$$

forms a connecting sequence from $[h_0]$ to a horizontal disk in $\bigcup \tilde{N}_i$, of arbitrary length controlled by iterating $\mathcal{N}(\cdot)$ in $\{N_i\}_{i \in I}$ and $\tilde{\mathcal{N}}(\cdot)$ in $\{\tilde{N}_i\}_{i \in \tilde{I}}$ an arbitrary $\ell \in \mathbb{N}$ times. Consequently, we obtain a point $\tilde{z}'_\ell \in [\tilde{h}'_\ell]$ such that

1. there exists $n_1(\ell) \geq \ell$ such that $\tilde{z}_\ell := f^{-n_1(\ell)}(\tilde{z}'_\ell) \in [\tilde{h}_\ell]$ and $f^{-m}(\tilde{z}_\ell) \in \bigcup \tilde{N}_i$ for $m = 0, \dots, n_1(\ell)$;
2. there exists $n_2(\ell) \geq 1$ such that $z_\ell := f^{-n_2(\ell)}(\tilde{z}_\ell) \in [h_\ell]$;
3. there exists $n_3(\ell) \geq \ell$ such that $f^{-n_3(\ell)}(z_\ell) \in [h_0]$ and $f^{-m}(z_\ell) \in \bigcup N_i$ for $m = 0, \dots, n_3(\ell)$.

Since the length of each connecting sequence in $\{K_j\}_{j \in J}$ is bounded by the same constant, the total number of possible coverings in the connecting sequences is finite. Therefore, we can choose $n_2 \geq 1$ fixed and select only a subsequence of triples $[h_{\ell_m}]$, $[\tilde{h}_{\ell_m}]$ and $[\tilde{h}'_{\ell_m}]$ with $n_2(\ell_m) = n_2$, independently of ℓ_m .

Recall that $\bigcup N_i$ and $\bigcup \tilde{N}_i$ are compact, which implies that both sequences of iterates $\tilde{z}_{\ell_m} = f^{-n_1(\ell_m)}(\tilde{z}'_{\ell_m})$ and $f^{-n_3(\ell_m)}(z_{\ell_m}) = f^{-(n_3(\ell_m)+n_2)}(\tilde{z}_{\ell_m})$ contain a convergent subsequence. Hence, we can choose a subsequence ℓ_{m_k} such that

$$\lim_{k \rightarrow \infty} \tilde{z}_{\ell_{m_k}} = \tilde{z}^* \in \bigcup \tilde{N}_i \quad \text{and} \quad \lim_{k \rightarrow \infty} z_{\ell_{m_k}} = z^* \in \bigcup N_i,$$

where $f^{n_2}(z^*) = \tilde{z}^*$. Hence, we have established the existence of a trajectory passing through z^* that belongs to $W^u(\Lambda) \cap W^s(\tilde{\Lambda})$. Therefore, $W^u_\Lambda \cap W^s_{\tilde{\Lambda}} \neq \emptyset$, as required. \square

Theorem 8 can be used, in particular, to establish a heterodimensional cycle between a blender Λ and a hyperbolic fixed point p , as suggested in Figure 13. The hyperbolic fixed point can be enclosed in a single h-set \tilde{M} with $\tilde{M} \xrightarrow{f} \tilde{M}$, and we can simply define the family of h-sets as $\{\tilde{M}_l\}_{l \in \tilde{L}} = \{\tilde{N}_i\}_{i \in \tilde{I}} = \{\tilde{M}\}$. Since for the blender Λ we have $d_y > d_s$, the cycle is indeed heterodimensional.

Remark 9. *Since the construction of the covering sequences in Theorem 8 is robust under C^1 -perturbations, our approach constitutes a new proof that the existence of a heterodimensional cycle is a C^1 -robust property; see also [5, 6].*

6 Discussion and conclusions

We presented a computer-assisted method for proving the existence of a blender for a given family of diffeomorphisms. The algorithm requires construction of a finite number of covering sequences with h-sets that intersect a transitive hyperbolic set. We select the necessary covering sequences by following different homoclinic excursions, but this is only a convenient approach and the algorithm itself does not depend on knowledge of homoclinic or heteroclinic connecting orbits. The method is very flexible in that it can be applied to ranges of a specified parameter. We successfully applied the algorithm in Section 4 to demonstrate existence of a 2-blender for the Hénon-like family (1) from [15, 16] over the range $\xi \in [1.01, 1.125]$ of the weak expansion rate ξ . Our construction involves two different homoclinic orbits and, notably, we chose not to include the two fixed points of the Hénon-like family in the family of h-sets. Hence,

the blender we identified here is an invariant subset of the (maximal) blender that has been considered in previous work [15–17]. Indeed, our construction and computer-assisted method of proof of existence of a blender do not require first to identify the underlying *maximal* transitive hyperbolic set.

Our objective here was to provide a proof of concept for an example previously considered in the literature, and we have been able to do so without difficulty and in a computationally efficient manner. Numerical evidence in [16] suggests that the blender exists well beyond the range proven with our algorithm (up to $\xi \approx 1.6$). We believe that the ξ -range of validity of our computer-assisted proof could be extended via a combination of the following three considerations.

- When using the h-sets introduced above, the bottle neck for the proof with $\xi > 1.125$ is that the final coverings $N_{14c} \xrightarrow{f} M$ for $c = 0, \dots, 49$ fail, because the sets do not align in the local weakly expanding coordinate y_u . This is due to the large expansion by ξ^5 that results from taking five steps along a sequence of coverings. A remedy could be to consider larger h-sets, which require fewer steps in the sequences of coverings. However, this would complicate validation of the required conditions. For example, in our current setting, to validate covering and cone conditions we compute interval enclosures of $Df_{ji}(N_{\gamma_i})$, $f_{ji}(N_{\gamma_i})$, $f_{ji}(N_{\gamma_i,l}^-)$ and $f_{ji}(N_{\gamma_i,r}^-)$ without any subdivisions of the sets N_{γ_i} , $N_{\gamma_i,l}^-$ and $N_{\gamma_i,r}^-$, and our bounds turn out to be sharp enough. Enlarging the h-sets would likely require subdivisions, which would slow down the computations.
- We are only using the homoclinic connections associated with the hyperbolic fixed point p^+ . One could perform a similar construction by using p^- , which could lead to a different, extended parameter range for ξ .
- We could place one mother h-set M^+ close to p^+ and another mother h-set M^- close to p^- and, instead of using homoclinic orbits, position the h-sets along heteroclinic orbits between p^+ and p^- . Such heteroclinic connections will require fewer iterates.

It is beyond the scope of this paper to consider further refinements with such more complicated constructions of covering relations and to obtain a more comprehensive coverage of the known ξ -range with a blender of the Hénon-like family. Again, our goal was to introduce and demonstrate a general method. Indeed, proving the existence of blenders, and also of heterodimensional cycles, in other dynamical systems remains an interesting and challenging task for future research.

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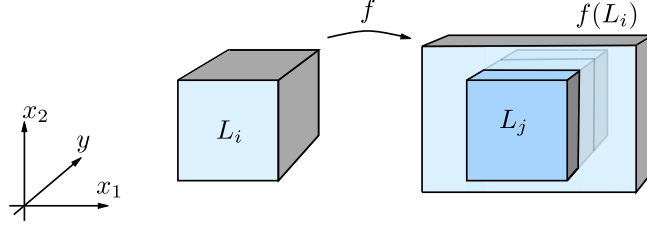


Fig. 14 Illustration of a covering relation $L_i \xrightarrow{f} L_j$ with $d_u = 2$ and $d_s = 1$.

A Establishing hyperbolicity and transitivity

In Section 4, we applied Theorem 1 to prove the existence of a 2-blender for the Hénon-like family (1). For completeness, we explain how we validate with computer-assisted tools that there exists a set U whose invariant set is hyperbolic and transitive, and which contains $\bigcup_{i \in I} N_i$, as required for Theorem 1.

Consider a set of indexes $J := \{0, 11, 12, 13, 14, 21, 22, 23\}$ and assume that we have a sequence of h-sets L_j for $j \in J$. These h-sets will have $d_x = 2$ and $d_y = 1$. This means that the dimensions of the exit set matches the dimension of the unstable bundle, and the dimension of the entry set matches the dimension of the stable bundle. We will choose these sets to ensure that

$$L_0 \xrightarrow{f} L_{11} \xrightarrow{f} L_{12} \xrightarrow{f} L_{13} \xrightarrow{f} L_{14} \xrightarrow{f} L_0, \quad (22)$$

$$L_0 \xrightarrow{f} L_{21} \xrightarrow{f} L_{22} \xrightarrow{f} L_{23} \xrightarrow{f} L_0. \quad (23)$$

We emphasise that in (22–23) the topological alignment for the coverings agrees with the contraction and expansion of the system. The topological alignment involved in the coverings from the sequences (22–23) is depicted in Figure 14. Due to the good topological alignment of the exit and entry coordinates with the coordinates of hyperbolic contraction and expansion we can ensure that the sequences of coverings (22–23) start and finish with the same set L_0 .

For our h-sets L_j we use the same local coordinates as for the h-sets N_j . Namely, we use γ_M defined in (9) for the local coordinates of L_0 and γ_{1b}, γ_{2b} defined in (10) for the local coordinates of L_j for $j \in J \setminus \{0\}$. (The local coordinates (x_1, x_2, y) of the sets L_0, L_{1b} and L_{2b} correspond to the local coordinates (x, y_u, y_s) of the sets M, N_{1b} and N_{2b} , respectively; namely, $x_1 = x, x_2 = y_u$ and $y = y_s$.)

We have chosen our sets so that

$$\begin{aligned} M &\subset L_0 \\ N_{1bc} &\subset L_{1b} \quad \text{for } c \in \{0, \dots, 49\} \text{ and } b = 1, 2, 3, 4, \\ N_{2bc} &\subset L_{2b} \quad \text{for } c \in \{0, \dots, 49\} \text{ and } b = 1, 2, 3. \end{aligned} \quad (24)$$

Because of (24) we have

$$\bigcup_{i \in I} N_i \subset U := \bigcup_{j \in J} L_j.$$

In more detail, we have chosen the sets L_0, L_{1b} and L_{2b} so that their projections onto the local coordinates x_1, y are the same as for M, N_{1b} and N_{2b} , respectively. (This also means that the projections of L_0, L_{1b} and L_{2b} onto the coordinates X, Y are the same as the projections of M, N_{1b} and N_{2b} , respectively.) Moreover, we have chosen the sets L_0, L_{1b} and L_{2b} to be larger along the local coordinate x_2 than the sets M, N_{1b} and N_{2b} , respectively. In our computer program we have chosen the set L_0 in the local coordinates given by γ_M to be of size $\pi_{x_2} L_{\gamma_M} = [-100, 100]$. This means that $M \subset L_0$ since $\pi_{x_2} L_{\gamma_M}$ is much larger than $\pi_{x_2} M = [-2, 2]$; see (8). We have chosen such a large interval since it works well for all parameters $\xi \in [1.01, 1.125]$. We choose the sizes of the intermediate sets in the covering sequences (24) so that for a covering $L_i \xrightarrow{f} L_j$ we have $\pi_{x_2} f_{ji}(L_{\gamma_i}) \subset \pi_{x_2} L_{\gamma_j}$ and so that $N_{jc} \subset L_j$ for $c = 0, \dots, 49$ and $j \in J \setminus \{0\}$; this is illustrated in Figure 14. (This choice is performed automatically by our computer program.) We then use an analogue of Lemma 6 to validate the coverings (22)–(23).

A.1 Establishing hyperbolicity

To validate hyperbolicity we can use the notion from [23] of a strongly hyperbolic covering relation. Recall that our h-sets $\{L_j\}_{j \in J}$ are equipped with the local coordinates $x = (x_1, x_2) \in \mathbb{R}^{d_x} = \mathbb{R}^2$ and $y \in \mathbb{R}^{d_y} = \mathbb{R}$. The coordinates (x_1, x_2) are $x_x = d_u = 2$ dimensional and expanding, and y is $d_y = d_s = 1$ dimensional and contracting. However, even though $d_u + d_s = n$, we do not assume or require that the coordinates (x_1, x_2) and y are perfectly aligned with the unstable and stable fibre bundles E^u and E^s , respectively.

Definition 11 (Strong hyperbolicity of h-sets [23]). *Let $\{L_j\}_{j \in J}$ be a family of h-sets associated with diffeomorphisms $f_{ji} : \mathbb{R}^{d_x} \times \mathbb{R}^{d_y} \rightarrow \mathbb{R}^n$. Define the $n \times n$ diagonal matrix*

$$Q = \text{diag}(\text{Id}_{d_u}, -\text{Id}_{d_s}), \quad (25)$$

where Id_{d_u} and Id_{d_s} are the identity matrices of dimensions d_u and d_s , respectively. We say that $\bigcup_{j \in J} L_j$ is strongly hyperbolic if for every $i, j \in J$ such that $f(L_i) \cap L_j \neq \emptyset$, the matrix

$$[Df_{ji}(\gamma_i(z))]^\top Q [Df_{ji}(\gamma_i(z))] - Q, \quad (26)$$

is positive definite for every $z \in L_i \cap f^{-1}(L_j)$.

To establish that an invariant set in $U = \bigcup_{j \in J} L_j$ is hyperbolic we can now invoke a known result from [23], namely,

Theorem 9. [23] *If $U = \bigcup L_j$ is strongly hyperbolic then the (forward and backward) invariant set*

$$\Lambda = \text{Inv}(f, U)$$

is a hyperbolic set.

For every local map f_{ji} involved in the coverings (22–23), we follow Definition 11 and validate that every matrix from the interval enclosure

$$[Df_{ji}(L_{\gamma_i})]^\top Q [Df_{ji}(L_{\gamma_i})] - Q \quad (27)$$

is positive definite; this proves hyperbolicity of the invariant set that is contained in U for the Hénon-like family (1). Theorem 9 does not guarantee that Λ is non empty. However, this is ensured by conditions (B1) and (B2), as pointed out in Remark 7.

A.2 Establishing transitivity

To establish transitivity we show that the invariant set Λ in U for f is conjugate to a transitive set $\Lambda_\Sigma \subset \Sigma = \{0, 1\}^{\mathbb{Z}}$ for the shift map. For this, we use the following proposition.

Proposition 10. *Consider an infinite sequence of covering relations*

$$\cdots \xrightarrow{f} L_{j_{-1}} \xrightarrow{f} L_{j_0} \xrightarrow{f} L_{j_1} \xrightarrow{f} L_{j_2} \xrightarrow{f} \cdots,$$

where L_{j_k} is selected from the finite family of h-sets $\{L_j\}_{j \in J}$ for all $k \in \mathbb{Z}$. For each covering $L_{j_k} \xrightarrow{f} L_{j_{k+1}}$, define the interval enclosure $[Df_k(L_{\gamma_{j_k}})]$, where $f_k := \gamma_{j_{k+1}} \circ f \circ \gamma_{j_k}$ represents f in local coordinates. Suppose for every f_k , we have that every $n \times n$ matrix in the interval enclosure

$$[Df_k(L_{\gamma_{j_k}})]^\top Q [Df_k(L_{\gamma_{j_k}})] - Q \quad (28)$$

is strictly positive definite; here, the matrix Q is the same matrix as defined in (25). Then, for every $\varepsilon > 0$ there exists $m = m(\varepsilon) > 0$, such that for $z_1, z_2 \in L_{j_0}$ with

$$f^k(z_1), f^k(z_2) \in L_{j_k} \quad \text{for all } k \in \{-m, \dots, m\},$$

we have $\|z - z'\| < \varepsilon$. (Here we use the standard Euclidean norm $\|\cdot\|$ in \mathbb{R}^n .)

Proof. Let $z_1, z_2 \in L_{j_0}$ and define $z_\ell^\gamma := \gamma_{j_0}(z_\ell)$ for $\ell = 1, 2$. We will consider two cases, namely, $(z_1^\gamma - z_2^\gamma)^\top Q (z_1^\gamma - z_2^\gamma) \geq 0$ and $(z_1^\gamma - z_2^\gamma)^\top Q (z_1^\gamma - z_2^\gamma) < 0$.

Assume first that $(z_1^\gamma - z_2^\gamma)^\top Q (z_1^\gamma - z_2^\gamma) \geq 0$. Since the index set J is finite and the h-sets L_j , for $j \in J$, are compact, the family of interval enclosures (28) for all $k \in \mathbb{Z}$ is compact. This means that for suitably small $\delta > 0$ and for $\lambda > 1$ suitably close to 1, every matrix in the interval enclosure

$$[Df_k(L_{\gamma_{j_k}})]^\top Q [Df_k(L_{\gamma_{j_k}})] - \lambda Q - \delta \text{Id}_n$$

is strictly positive definite. Let us introduce the auxiliary notation $q(z) := z^\top Q z$. Taking $z = (x, y) = (x_1, x_2, y)$ from (25), we observe that

$$q(z) = q(x, y) = (x, y)^\top Q (x, y) = \|x\|^2 - \|y\|^2 \leq \|x\|^2 + \|y\|^2 = \|z\|^2.$$

By the Mean Value Theorem, for any $p_1, p_2 \in L_{\gamma_{j_k}}$, we have

$$f_k(p_1) - f_k(p_2) = \int_0^1 \frac{d}{ds} f_k(p_2 + s(p_1 - p_2)) ds$$

$$\begin{aligned}
&= \left(\int_0^1 Df_k(p_2 + s(p_1 - p_2)) \, ds \right) (p_1 - p_2) \\
&=: D(p_1 - p_2),
\end{aligned}$$

where $D \in [Df_k(L_{\gamma_{j_k}})]$ depends on p_1 and p_2 . Consequently,

$$\begin{aligned}
&q(f_k(p_1) - f_k(p_2)) - \lambda q(p_1 - p_2) - \delta \|p_1 - p_2\|^2 \\
&= (p_1 - p_2)^\top (D^\top Q D - \lambda Q - \delta \text{Id}_n) (p_1 - p_2) > 0.
\end{aligned}$$

This means that, since $q(z_1^\gamma - z_2^\gamma) \geq 0$,

$$\begin{aligned}
&\|f_m \circ \dots \circ f_0(z_1^\gamma) - f_m \circ \dots \circ f_0(z_2^\gamma)\|^2 \\
&\geq q(f_m \circ \dots \circ f_0(z_1^\gamma) - f_m \circ \dots \circ f_0(z_2^\gamma)) \\
&\geq \lambda q(f_{m-1} \circ \dots \circ f_0(z_1^\gamma) - f_{m-1} \circ \dots \circ f_0(z_2^\gamma)) \\
&\vdots \\
&\geq \lambda^m q(f_0(z_1^\gamma) - f_0(z_2^\gamma)) \\
&\geq \lambda^m (\lambda q(z_1^\gamma - z_2^\gamma) + \delta \|z_1^\gamma - z_2^\gamma\|^2) \\
&\geq \lambda^m \delta \|z_1^\gamma - z_2^\gamma\|^2.
\end{aligned}$$

Since $\lambda > 1$ and $\delta > 0$, we will have $\|z_1^\gamma - z_2^\gamma\| < \varepsilon$ as required, provided we choose m sufficiently large.

The proof for the case $(z_1^\gamma - z_2^\gamma)^\top Q (z_1^\gamma - z_2^\gamma) < 0$ follows from mirror arguments, using the inverse map. \square

We are now ready to prove that the invariant set Λ in $U = \bigcup L_j$ for the ξ -dependent Hénon-like family (1) with $\mu = -9.5$ and $\beta = 0.3$ is transitive. Let us equip $\Sigma = \{0, 1\}^\mathbb{Z}$ with the usual metric $d : \Sigma \times \Sigma \rightarrow \mathbb{R}_+$, where

$$d(a, b) = \sum_{k \in \mathbb{Z}} \frac{1}{2^{|k|}} |a_k - b_k|,$$

and consider the usual shift map $\sigma : \Sigma \rightarrow \Sigma$, with $\sigma((a_k)_{k \in \mathbb{Z}}) = (a_{k-1})_{k \in \mathbb{Z}}$. Let us consider $\alpha = 10000$ and $\beta = 1000$, and the set Λ_Σ that consists of (infinite) sequences in Σ constructed from the finite sequences α and β , and from shifts of such sequences. Note that $\sigma(\Lambda_\Sigma) = \Lambda_\Sigma$ and Λ_Σ is transitive for the map σ . We show that Λ is transitive by constructing a conjugacy between the dynamics on Λ and on Λ_Σ .

Consider the transformation $\phi : \Lambda \rightarrow \Lambda_\Sigma$ that assigns to a point $z \in \Lambda$ a sequence $a = \phi(z) \in \Lambda_\Sigma$ according to the following rule

$$a_k = 1 \iff f^k(z) \in L_0.$$

The intuition behind this definition of ϕ is based on the two covering sequences (22) and (23). If $z \in L_0 \cap \Lambda$ then the trajectory starting from z will pass through either the first or the second sequence of coverings (22) or (23), respectively. Hence, we either have α or β as the first entries in $\phi(z)$. The remaining entries are also determined by the order in which the trajectory follows these two different sequences of coverings.

The two sequences of coverings (22) and (23) can be glued in arbitrary order to produce infinite sequences, and through any such infinite sequence there exists an orbit of f passing through it (see [28]). Hence, it follows that ϕ is surjective, and we have the conjugacy

$$\phi \circ f = \sigma \circ \phi.$$

Moreover, continuity of f implies continuity of ϕ (if we choose two sufficiently close initial points in Λ , their trajectories pass through the same sequences of coverings and, hence, follow the same symbols, for a prescribed length). Continuity of ϕ^{-1} follows from Proposition 10. Therefore, there is a conjugacy between Λ and Λ_Σ , and the fact that Λ_Σ is transitive implies that Λ is transitive.

We finish by noting that the interval enclosure defined in (28) is the same as the one from (27) that we used to validate hyperbolicity.

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