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Arnold diffusion in the planar restricted elliptic three body problem

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Abstract

In the planar restricted circular three body problem, for the values C of the Jacobi constant smaller but close to the value C_2 associated with the critical point L_2, there exists a family of the so called Lapunov periodic orbits around the equilibrium point. We will show that when the planar restricted elliptic three body problem is considered as a perturbation of the circular problem most of the Lapunov orbits persist and are perturbed into a Cantor set of invariant tori. We will show that there exist transition chains between the tori, which arise from transversal intersections of the corresponding invariant manifolds. In the elliptic three body problem these intersections are not restricted to a constant energy manifold. The intersections are transversal in the full phase space and each transition involves a change of energy.

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Motivation and preliminaries

1.1 Introduction

Arnold diffusion is the occurrence of a loss of stability of invariant tori of integrable Hamiltonian systems under small perturbations. When in 1964 in his work [3] Arnold introduced the concept, he conjectured that this phenomenon appears in the three body problem. The proof of the existence of Arnold diffusion though for any given physical system has been quite elusive. In fact up to this date the number of explored examples of this phenomenon is very small. Arnold's conjecture for the three body problem has finally been proven in the case of the planar restricted three body problem in 1993 by Xia in [35] and later in the case of the planar three body problem in [36]. Xia has shown the occurrence of Arnold diffusion close to a transversal homoclinic orbit to a periodic orbit at infinity. The aim of our work is to prove a similar result, but in the case of the family of Lapunov periodic orbits at the libration point L_2 in the planar restricted elliptic three body problem.

The starting point of our discussion will be the planar restricted circular three body problem, where two large masses μ and $1-\mu$ rotate around each other on circular orbits and the equations describe the motion of a third massless particle. Such a case was considered by Llibre, Martinez and Simo in [21] for energies of solutions close to the energy of the libration point L_2. There it has been shown that there exists a family of parameters $\{\mu_k\}_{k=2}^{\infty}$ for which we have a homoclinic orbit to the libration point L_2. This orbit circles once around the larger mass $1-\mu_k$ and returns to the point L_2. What is more it has been shown that for μ close to any of the values μ_k, for a Lapunov orbit around L_2 with an energy sufficiently close to the energy of L_2, the stable and unstable manifold of the Lapunov orbit intersect transversally; this dynamics is restricted to a constant energy level and leads to a homoclinic tangle. Later the problem has been investigated by Koon, Lo, Marsden and Ross in [17] where smaller energies were considered. In such a case the chaotic dynamics is extended to include the Lapunov orbits around the libration point L_1. This has been later proven by Wilczak and Zgliczyński using the method of covering relations and computer assisted methods in [34], for the case of the Jupiter-Sun system and the energy of the comet Oterma.

All of the above mentioned results have a common feature: the transversality of the intersections and the chaotic dynamics of the system is always restricted to a constant energy manifold. This is because the above mentioned problems come from autonomous Hamiltonian equations. We however are going to consider the planar restricted elliptic three body problem, where the equations are no longer autonomous and therefore a change of energy of the solutions is possible. We will consider the circular case discussed by Llibre, Martinez and Simo in [21] and generalize the problem to allow the orbits of the two larger masses μ and $1-\mu$ to be elliptic with small eccentricities e. We will treat this as a perturbation of the circular case. We will show that most of the Lapunov orbits around L_2 persist under the perturbation. What is more, we will show that the rich dynamics associated with these orbits and obtained in [21] also survives. In addition to that we will show that not only will we have chaotic oscillations of the solutions, but at the same time the energies of these solutions will also diffuse chaotically. In effect the dynamics of the elliptic problem is by one dimension richer than the dynamics of the circular problem, where all solutions are restricted to a constant energy manifold. To be more precise, we will prove the following Theorem

Theorem 1.1 (Main Theorem)

For sufficiently small mass $\mu \in \{\mu_k\}_{k=2}^{\infty}$ and for energies close to the energy of the libration point L_2, for sufficiently small eccentricities e of the elliptic problem, most of the Lapunov orbits around L_2 survive and are perturbed to invariant tori. What is more, there exist a homoclinic and a heteroclinic tangle between the surviving tori which involves diffusion in energy. (Such a heteroclinic tangle between invariant tori is the mechanism of the so called Arnold diffusion).

Throughout some of the so far explored examples a certain pattern can be observed in the methods with which Arnold diffusion is detected in the apriori unstable systems (that is for systems which prior to a perturbation already have low dimensional normally hyperbolic invariant tori). First the normally hyperbolic invariant manifold of the Hamiltonian system foliated by invariant tori is found. The tori are required to have hyperbolic stable and unstable manifolds and a transversal intersection of these manifolds or an existence of a homoclinic orbit to at least one of the tori needs to be established. Secondly a perturbation of the system is considered. By perturbation theory ([12], [33]) of normally hyperbolic manifolds, the normally hyperbolic invariant manifold and its stable and unstable manifolds persist under the perturbation. The third step is to show that on the perturbed invariant manifold most of the invariant tori survive. This under appropriate nondegeneracy conditions is a result of the celebrated Kolmogorov Arnold Moser Theorem (KAM) [4],[16]. Using more recent versions of the theorem (for example [11], [12] or [37]) it can be shown that most of the invariant tori persist and form a Cantor set having a positive measure in the invariant manifold. The last step is to show that the stable and unstable manifolds of the surviving tori intersect transversally. This is usually done by the use of a Melnikov type method along a homoclinic orbit of the unperturbed problem. The transversal intersections between the invariant manifolds of the perturbed tori lead to homoclinic tangles for each of the surviving tori. In addition to this we also have a chaotic diffusion along the Cantor set of homoclinic tangles between the tori. Such behavior is given throughout the literature the name of Arnold diffusion.

The above mentioned procedure has been extensively developed by Wiggins [31], [32] in the case of perturbations of completely integrable Hamiltonian systems. It has also been used by Moeckel [23] to detect transition tori in the case of the planar five body problem. The same pattern is followed by Delshams, Llave and Seara [9] to show unbounded diffusion of energy for perturbations of geodesic flows on a two dimensional torus. In our work we will also follow the above described method. In the case of the planar restricted elliptic three body problem considered earlier by Xia in [35], [36] a similar method was used but in a setting which allowed him to omit having to calculate the Melnikov integral.

When applying the method to prove the existence of Arnold diffusion for a given physical problem the stpdf of the above procedure which present the biggest obstacles are usually the checking of the assumptions of the KAM theorem and the computation of the Melnikov integral, the rest of the argument being usually a standard procedure. In particular the Melnikov integral for a given equation can easily prove to be impossible to compute. The coordinates of Xia ([35], [36]) in his examples for the elliptic three body problem allowed him to omit this problem. We will not have this benefit. In our case we will compute the Melnikov integral by substituting the equations of the elliptic three body problem with the simpler equations of the Hill's problem, which for sufficiently small masses μ prove to be an adequate approximation. The use of the Hill's problem will also be a useful tool when checking the assumptions of the KAM theorem. Such an approach has one serious drawback. The results obtained only hold for sufficiently small μ. Let us note though that even with this simplified approach the computations are quite laborious.

The paper is organized as follows. Chapter one contains the introduction and preliminaries. In Chapter two we recall the earlier results on the planar restricted three body problem of [21]. In particular we recall the result that there exist transversal homoclinic intersections of the stable and unstable manifolds of the family of Lapunov orbits around the Libration point L_2. In the third chapter we apply a version of the Lapunov Theorem of Moser [25] to prove the existence and the twist property of the family of Lapunov orbits at L_2. The twist property will play a crucial role in the application of the KAM theorem later on in chapter six. The twist property is obtained by approximating the elliptic problem with the equations of the Hill's problem. In chapter four we derive the equations of the planar restricted elliptic three body problem in the rotating coordinates and show that these equations can be viewed as a perturbation of the equations of the circular case. Having established all the necessary notations and preliminary results in the first four chapters, in chapter five we present the method and intuition behind the proof of Theorem 1.1. In chapter six we apply the theory of normally hyperbolic invariant manifolds and the KAM Theorem to show that most of the Lapunov orbits around L_2 persist under the perturbation from the circular problem to the elliptic problem. In chapter seven we use a Melnikov type argument for detecting the transversal intersections between the stable and unstable manifolds of the perturbed Lapunov orbits. In chapter eight we compute the Melnikov integral for the elliptic three body problem. We use simple symmetry arguments to show that there exists a point at which the Melnikov function is equal to zero. Showing that at this point the derivative of the Melnikov function is nonzero boils down to computing a particular integral over an appropriate orbit of the Hill's problem. This is done numerically. It is quite likely that this could be done using analytical estimations, but the technical aspects of such estimations are rather lengthy and tedious. In chapter nine we gather all the results to prove the main Theorem 1.1.

On the whole the paper obtains two new results. The first is the persistence result of the Libration point L_2 and the orbits which surround it. The second is the survival of the dynamics observed in the circular case and the existence Arnold diffusion between the perturbed orbits.

Let us finish the introduction by outlining the limitations of the obtained result and also with a motivation why such a result could be interesting. Let us start with the limitations. Our results are proved for sufficiently small mass μ, for energies sufficiently close to the energy of the Libration point L_2 and for a sufficiently small eccentricity (perturbation) e. The method does not give us however any estimations on the values for which such dynamics actually occurs. The result is purely a perturbation type statement. This is similar to the result in the circular problem of Llibre, Martinez and Simo [21] which is the starting point of our discussion. This can also be said about Xia's results for Arnold diffusion [35], [36]. The question whether for a given astronomical problem, such as the Jupiter-Sun system, where both the mass μ and the eccentricity (perturbation) e is given, would have similar dynamics is an open problem. The Melnikov type argument presented in this thesis could potentially be applied for these types of problems but other major obstacles (the application of the KAM theorem for a given perturbation being the chief among them) would need to be overcome.

The reason why the result of this paper could be of interest is that the Lapunov orbits around Libration points and the manifolds associated with them in the planar restricted circular three body problem find applications in actual space mission planning. Let us list some examples. In the 1991 Hiten mission a spacecraft reached the moon using a new type of transfer using this methodology [6]. In 1999 a in the paper of Koon, Lo, Marsden and Ross [18] the methodology was used to plan a space mission to one of the Jupiter's moons Europa. In 2002 this approach was generalized to include control theory [29]. All of the above mentioned applications have considered the circular three body problem as the approximation of the real life elliptic case. In this dissertation we investigate what happens with the dynamic structure when the circular problem is perturbed into an elliptic one. It turns out that most of the structures considered in the circular problem survive the perturbation, which justifies in some way the use of the circular problem as the approximation of the elliptic problem. One point of interest though, which is shown in our paper, is the potential diffusion in energy which is not taken into account in the circular case. We have to stress that the results obtained by us are for a very particular problem which does not have direct applications in space travel. On the other hand the result can be viewed as a first step to further investigation and provide insight as to what might happen in the real life problems of space mission planning. Such problems have not yet been investigated.

1.2 Preliminaries

Let us start by introducing some notations and facts which will be used throughout the paper.

1.2.1 Hamiltonian systems

Hamiltonian equations are well known and used throughout physics entities. The equations are given as

$$\dot{x} = J\nabla_x H(t,x), \tag{1.1}$$

where $x = (q,p) \in \mathbb{R}^n \times \mathbb{R}^n$, the matrix J is given as

$$J = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}, \tag{1.2}$$

where id is the $n \times n$ identity matrix. The function $H : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is called the Hamiltonian of the system (1.1), the integer n is referred to as the number of degrees of freedom, the vector q is the vector of coordinates and the vector p is the vector of momenta. The equation (1.1) can be rewritten in the q, p coordinates as

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(t,q,p) \\ \dot{p} &= -\frac{\partial H}{\partial q}(t,q,p). \end{aligned} \tag{1.3}$$

For autonomous Hamiltonian equations we have the well known property that the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is conserved along a solution (q,p) of (1.3)

$$\frac{d}{dt}H(q,p) = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} = \frac{\partial H}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial H}{\partial q} = 0. \tag{1.4}$$

We will refer to this property as the conservation of energy for autonomous Hamiltonians.

1.2.2 Poisson bracket

Numerous properties of Hamiltonian systems can be expressed using the Poisson bracket. In our case we will look at the Poisson bracket in terms of the fact that it measures the change of a given function along a solution of a Hamiltonian equation. To be more specific let us first introduce the definition.

Definition 1.2

Let H, G be smooth functions from an open set $U \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$. The Poisson bracket is defined by

$$\{G, H\} = \nabla G^T J \nabla H = \frac{\partial G}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial G}{\partial p}\frac{\partial H}{\partial q}. \tag{1.5}$$

Let us consider an autonomous Hamiltonian system

$$\dot{x} = J\nabla H, \tag{1.6}$$

and its solution $\phi(t, x_0)$. Using the chain rule we can compute

$$\frac{d}{dt}G(t, \phi(t, x_0)) = \frac{\partial}{\partial t}G(t, \phi(t, x_0)) + \{G, H\}(t, \phi(t, x_0)). \tag{1.7}$$

From the above equation we can see that the evolution of G along the solution of (1.6) is expressed through the Poisson bracket. This fact will be used in the modified Melnikov method presented in chapter 7.3.

1.2.3 Canonical transformations

In this section we will introduce the definition of a canonical transformation and the Jacobi Theorem.

Definition 1.3 ([1])

We say that

$$\begin{aligned} F &: \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \\ F &: (t, q, p) \to (t, x, y) \end{aligned} \tag{1.8}$$

is a canonical (or symplectic) transformation when it satisfies the following three conditions

(C1) F is a diffeomorphism
(C2) F preserves time
(C3) There exists a function K_F such that $F^*\omega_2 = \omega_{K_F}$, where

$$\omega_{K_F} = \omega_1 + dK_F \wedge dt,$$

$\omega_2 = \sum_{i=1}^n dx_i \wedge dy_i$ and $\omega_1 = \sum_{i=1}^n dq_i \wedge dp_i$.

When the Hamiltonian is autonomous, in the above definition the function K_F is chosen to be equal to zero and $\omega_{K_F} = \omega_1$. This by (C3) means that the function F preserves the symplectic form

$$F^*\omega_2 = \omega_1. \tag{1.9}$$

The following Theorem states that after a canonical change of coordinates the vector field is still Hamiltonian.

Theorem 1.4 (Jacobi Theorem [1])

Let $H_1 : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ be a time dependent Hamiltonian. If F is a canonical transformation then the vector field X_{H_1} generated by the Hamiltonian H_1 is equal to

$$X_{H_1} = (F)^* X_{H_2} \tag{1.10}$$

where X_{H_2} is a vector field generated by a Hamiltonian H_2 which is given by

$$H_2(q, p, t) = H_1 \circ F(t, q, p) + K_F(q, p). \tag{1.11}$$

A time t shift along a solution of the Hamiltonian system (1.3) is a symplectic transformation. In order to formulate this fact rigorously let us introduce a notation $\phi(t, t_0, (q_0, p_0))$ for a general solution of the system (1.3) i.e.

$$\begin{aligned} &\frac{d}{dt}\phi(t, t_0, (q_0, p_0)) = J\nabla_{(q,p)} H(t, \phi(t, t_0, (q_0, p_0))) \\ &\phi(t_0, t_0, (q_0, p_0)) = (q_0, p_0). \end{aligned} \tag{1.12}$$

Using this notation we can formulate the following Theorem.

Theorem 1.5 ([22, Theorem 2])

Let $\phi(t, t_0, (q, p))$ be the general solution of the Hamiltonian system (1.3). Then for a fixed t and t_0 the map $F : (q, p) \to \phi(t, t_0, (q, p))$ is canonical (symplectic).

The Definition 1.3 of a canonical (symplectic) transformation is made in the general case of time dependent functions. During our discussions we will sometimes use very simple linear time independent canonical transformations. In such a case the conditions (C1), (C2), and (C3) reduce [22] to the following simple condition.

Remark 1.6

A linear transformation $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is canonical (symplectic) if

$$\Phi^T J \Phi = J, \tag{1.13}$$

where J is the matrix given by the equation (1.2).

1.2.4 Invariant manifolds

We say that a manifold $\Lambda \subset \mathbb{R}^n$ is invariant under the flow $\phi(t,x)$ when for all x_0 in Λ and all $t \in \mathbb{R}$ we have

$$\phi(t, x_0) \in \Lambda. \tag{1.14}$$

Definition 1.7

The stable and unstable manifolds for the flow $\phi(t,x)$ of the invariant manifold Λ are defined as

$$\begin{aligned} W^s_\Lambda &= \{x \in \mathbb{R}^n | \mathrm{dist}(\phi(t,x), \Lambda) \overset{t\to+\infty}{\to} 0\} \\ W^u_\Lambda &= \{x \in \mathbb{R}^n | \mathrm{dist}(\phi(t,x), \Lambda) \overset{t\to-\infty}{\to} 0\}. \end{aligned} \tag{1.15}$$

An analogous definition can be made for invertible maps. A manifold $\Lambda \subset \mathbb{R}^n$ is invariant for an invertible map $P : \mathbb{R}^n \to \mathbb{R}^n$ if for all $n \in \mathbb{Z}$ we have

$$P^n(\Lambda) \subset \Lambda. \tag{1.16}$$

Definition 1.8

The stable and unstable manifolds for an invertible map P of the invariant manifold Λ are defined as

$$\begin{aligned} W^s_\Lambda &= \{x \in \mathbb{R}^n | \mathrm{dist}(P^n(x), \Lambda) \overset{n\to+\infty}{\to} 0\} \\ W^u_\Lambda &= \{x \in \mathbb{R}^n | \mathrm{dist}(P^n(x), \Lambda) \overset{n\to-\infty}{\to} 0\}. \end{aligned} \tag{1.17}$$

When it will be important to distinguish for which particular map the manifolds are defined we will use the notation $W^s_\Lambda(P)$, $W^u_\Lambda(P)$ to indicate that the manifolds are for the map P. Let us finish the section with the definition of a transversal intersection.

Definition 1.9

Let $i, j \in \{u, s\}$. We say that the manifolds W^i_Λ and W^j_Λ intersect transversally at a point x_0 if the tangent spaces $T_{x_0}W^i_\Lambda$ and $T_{x_0}W^j_\Lambda$ span the state space $\mathbb{R}^n$

$$\mathrm{span}(T_{x_0}W^i_\Lambda, T_{x_0}W^j_\Lambda) = \mathbb{R}^n. \tag{1.18}$$

1.2.5 Gronwall Lemma

Lemma 1.10 ([8, p.37])

If the function u, v, and $c \geq 0$ on $[0,t]$, c is differentiable, and

$$v(t) \leq c(t) + \int_0^t u(s)v(s)ds, \tag{1.19}$$

then

$$v(t) \leq c(0) \exp \int_0^t u(s) ds + \int_0^t c'(s) \left[\exp \int_s^t u(\tau) d\tau \right] ds. \tag{1.20}$$

2
The Planar Restricted Circular Three Body Problem.

In this Chapter we will summarize a number of properties of the flow of the planar restricted circular three body problem (PRC3BP) described in [21]. The chapter does not contain any new results and is only a collection of the already known properties of the PRC3BP. The facts outlined in this chapter will be the starting point for the proof of the existence of Arnold diffusion in the elliptic three body problem given in the following Chapters.

Let us start with a brief introduction to the planar restricted circular three body problem. The equations of the PRC3BP describe the movement of a particle with an infinitely small mass (a comet or a spaceship) under the gravitational pull of two larger masses (two planets or a star and a single planet). We assume that the movement of all three bodies is contained in a plane. We also assume that the particle with the small mass has negligible impact on the movement of the two larger masses and also that the two masses move along circular orbits of periods 2π and constant angular velocity around the origin; hence the name circular. One of the two larger bodies has a small mass μ with comparison to the mass of the second body which is $1-\mu$. The radius of the orbit with the larger mass is equal to μ and the radius of the orbit of the smaller mass is $1-\mu$. If we set our coordinates so that they rotate together with the two bodies and that the center of mass is at the origin, it turns out [1] that the equation of motion of the third massless body is given by an autonomous Hamiltonian of the form

$$H(x,y,p_x,p_y) = \frac{(p_x+y)^2+(p_y-x)^2}{2} - \Omega(x,y), \tag{2.1}$$

where

$$\Omega(x,y) = \frac{x^2+y^2}{2} + \frac{1-\mu}{\sqrt{(x-\mu)^2+y^2}} + \frac{\mu}{\sqrt{(x+1-\mu)^2+y^2}}. \tag{2.2}$$

(The above equation and the change of coordinates from the stationary coordinates to the coordinates which rotate together with the two masses will be given in detail in Chapter 4). The motion of the massless particle is given by the following differential equations

$$\begin{array}{ll} \dot{x} = \frac{\partial H}{\partial p_x} & \dot{p}_x = -\frac{\partial H}{\partial x} \\ \dot{y} = \frac{\partial H}{\partial p_y} & \dot{p}_y = -\frac{\partial H}{\partial y}. \end{array} \tag{2.3}$$

Let us note that from (2.1) and (2.3) we have

$$\dot{x} = p_x + y \tag{2.4}$$

$$\dot{y} = p_y - x, \tag{2.5}$$

and therefore it is easy to pass from the coordinates x, y, p_x, p_y to the coordinates $x, y, \dot{x}, \dot{y}$. All the properties of the PRC3BP given below will be described in the $x, y, \dot{x}, \dot{y}$ coordinates as it is originally done in [21]. In the coordinates $x, y, \dot{x}, \dot{y}$ our system (2.3) has a well know Jacobi integral F given by

$$F(x, y, \dot{x}, \dot{y}) = -2H(x, y, \dot{x} - y, \dot{y} + x) = 2\Omega(x, y) - \left(\dot{x}^2 + \dot{y}^2\right), \tag{2.6}$$

and the differential equations (2.3) are equivalent to the equations

$$\begin{array}{l} \ddot{x} - 2\dot{y} = \Omega_x(x, y) \\ \ddot{y} + 2\dot{x} = \Omega_y(x, y). \end{array} \tag{2.7}$$

The Hamiltonian H and the Jacobi integral is constant along the solutions $(x(t), y(t), p_x(t), p_y(t))$ of (2.3)

$$H(x(t), y(t), p_x(t), p_y(t)) = -\frac{C}{2}, \tag{2.8}$$

$$F(x(t), y(t), p_x(t) + y(t), p_y(t) - x(t)) = C.$$

We will call the value C the energy of the solution $(x(t), y(t), p_x(t), p_y(t))$ (or if we wish to describe the system using the $\dot{x}, \dot{y}$ coordinates rather than p_x, p_y, then C will be the energy of the solution $(x(t), y(t), \dot{x}(t), \dot{y}(t))$).

2.1 Hill's region and the libration points.

Since the solutions of the PRC3BP (2.1) have constant energies, the movement of the flow (2.3) is restricted to the hypersurfaces

$$M(\mu, C) = \{(x, y, \dot{x}, \dot{y}) \in \mathbb{R}^4 | F(x, y, \dot{x}, \dot{y}) = C\}. \tag{2.9}$$

In the x, y coordinates this means that the movement is restricted to the so called Hill's region defined by

$$R(\mu, C) = \{(x, y) \in \mathbb{R}^2 | \Omega(x, y) \geq C/2\}. \tag{2.10}$$

The region $R(\mu, C)$ gives us a restriction of the area where the solutions with an energy C have to be confined. For any x, y for which we have $\Omega(x, y) = C/2$, by

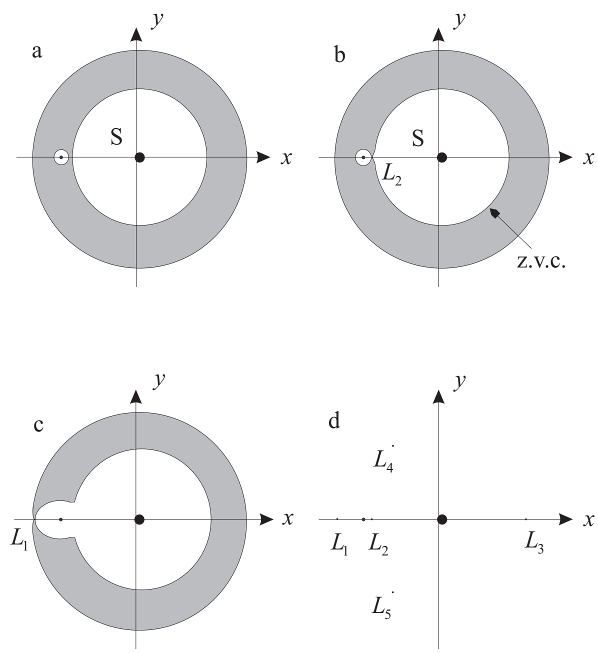

Figure 2.1 Forbidden region and the Libration points L_i.

equation (2.6) we know that $\dot{x} = \dot{y} = 0$. This means that the boundary of the region $R(\mu, C)$ is a zero velocity curve.

The shape of the Hills region may differ with C. Let us take a look at some possible shapes that the set $R(\mu, C)$ may take. Starting with C large, the Hill's region is composed of three separated areas: one surrounding the large mass $1-\mu$, which we will denote as the S region, one surrounding the smaller mass μ, and one in the outer region of the two masses (See Figure 2.1.a). As the energy C decreases, the two areas which surround the masses increase. For a certain energy which we will denote as C_2^μ (or C_2 when we have no doubt as to the μ for which we compute it) the two areas will join at a single equilibrium point L_2^μ (See Figure 2.1.b). This point and the dynamics of the solutions with energy $C \lesssim C_2^\mu$ shall be the focus of our attention. In such a case the set $R(\mu, C)$ has two components out of which one is bounded. We will denote this bounded component as $R_b(\mu, C)$. For an energy $C < C_2^\mu$ the two inner regions are joined but still separated from the outer region. As C decreases even more, the inner region increases, until it reaches the outer region at a single point, the so called libration point L_1^μ for $C = C_1^\mu$ (See Figure 2.1.c). As C decreases further three more equilibrium points appear in a similar fashion. One of them, L_3^μ is on the x axis and two others L_4^μ and L_5^μ are symmetrical, one above and one below the x axis (See Figure 2.1.d).

2.2 Symmetries of the PRC3BP

The planar restricted tree body problem (2.1) has the following symmetries. In the $x, y, \dot{x}, \dot{y}$ coordinates we have a symmetry

$$S(x, y, \dot{x}, \dot{y}, t) = (x, -y, -\dot{x}, \dot{y}, -t). \tag{2.11}$$

This means that if we find an orbit $q(t) = (q_x, q_y, q_{\dot{x}}, q_{\dot{y}})(t)$ which is a solution of (2.7), then it's S-symmetric image $S(q(t)) = (q_x, -q_y, -q_{\dot{x}}, q_{\dot{y}})(-t)$ is also a solution of (2.7). Let us note that the Libration point $L_2^\mu = (x_{L_2^\mu}, 0, 0, 0)$ is invariant under the S-symmetry. This in particular means that if we find an unstable manifold of L_2^μ, then the stable manifold can be obtained by the symmetry S.

The S-symmetry can be rewritten in the x, y, p_x, p_y coordinates as the following symmetry R

$$R(x, y, p_x, p_y, t) = (x, -y, -p_x, p_y, -t). \tag{2.12}$$

We will say that an orbit $q(t)$ is S-symmetric (or R-symmetric depending on the fact whether the orbit $q(t)$ is given in the $x, y, \dot{x}, \dot{y}$ or in the x, y, p_x, p_y coordinates) if

$$S(q(t), t) = (q(-t), -t), \tag{2.13}$$

($R(q(t), t) = (q(-t), -t)$). Many of the orbits of the PRC3BP considered by us in the future will be S and R-symmetric.

2.3 Intersections of invariant manifolds of the Lapunov orbits in the PRC3BP.

Let us now list some properties of the PRC3BP-flow discussed in [21]. As mentioned above, we will be interested in the dynamics of the flow with an energy close to the energy C_2^μ of the libration point L_2^μ. We know that L_2^μ is an equilibrium point for the flow $J\nabla H$ generated by (2.1)

$$J\nabla H(L_2^\mu) = 0. \tag{2.14}$$

The following Theorem gives us a formula for an orbit on the unstable manifold $W^u_{L_2}$. We state the Theorem in an identical form in which it is written in [21]. The formulation of the theorem is a bit vague and we will therefore follow the Theorem by a series of Remarks which will make the statement more rigorous.

Theorem 2.1 ([21, Theorem A])

For μ sufficiently small the branch of $W^u_{L_2}$ contained in the S region (see Figures 2.1 and 2.2) has a projection on $R_b(\mu, C)$ given by

$$d(t) = \mu^{1/3}(\frac{2}{3}N(\infty) - 3^{1/6} + M(\infty)\cos t + o(1)), \tag{2.15}$$

$$\alpha(t) = -\pi + \mu^{1/3}(N(\infty)t + 2M(\infty)\sin t + o(1)), \tag{2.16}$$

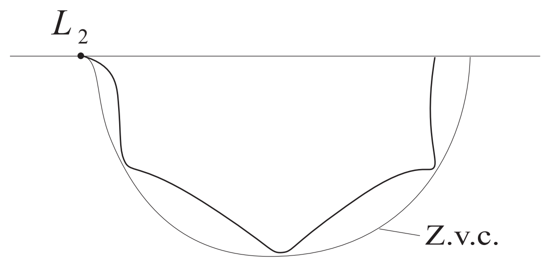

Figure 2.2 [21, Fig. 1.3] Projection of the right branch of $W^u_{L_2}$ onto the (x, y) plane.

where d is the distance to the z.v.c., α the angular coordinate, $N(\infty)$ and $M(\infty)$ are constants and the expressions remain true out of a given neighborhood of L_2. The parameter t means the physical time from a suitable origin. The terms $o(1)$ tend to zero when μ does and they are uniform in t for $t = O(\mu^{-1/3})$.

In particular the first intersection with the x axis is orthogonal to that axis, giving a S-symmetric homoclinic orbit for a sequence of values μ which has the following asymptotical expression:

$$\mu_k = \frac{1}{N(\infty)^3 k^3}(1 + o(1)). \tag{2.17}$$

Remark 2.2

To make the above statement more rigorous let us specify that the neighborhood out of which the formulas (2.15), (2.16) hold is dependent from μ and determined by the first intersection of $W^u_{L_2}$ with the section $\{y = -\bar{\bar{k}}\mu^{1/3}\}$ where $\bar{\bar{k}}$ is some large number independent from μ [21, page 121, par. 3] (The choice of the $\bar{\bar{k}}$ will be explained in more detail in Section 2.4). We therefore assume that at the time $t = 0$ the branch $W^u_{L_2}$ starts at some point in $\{y = -\bar{\bar{k}}\mu^{1/3}\}$. At this starting point the angle $\alpha(0)$ is close to $-\pi$ and as time increases the angle $\alpha(t)$ is enlarged. This means that the movement along $W^u_{L_2}$ in the S region flows first in the downward direction and circles around the origin until the angle α reaches the value 0 at which the intersection of the $W^u_{L_2}$ with the section $\{y = 0\}$ occurs. As the angle α changes from $-\pi$ to zero the radius of the movement changes according to the formula (2.15) for $d(t)$ and produces the waves from Figure 2.2. The intersection of $W^u_{L_2}$ with $\{y = 0\}$ is orthogonal to the section if both $\alpha(t) = 0$ and $d'(t) = 0$ at the same time. From this condition the equation (2.17) for μ_k is obtained.

Let us also specify that the formulas (2.15), (2.16) hold from the time $t = 0$ until the first intersection with the section $\{y = 0\}$, which occurs for a time $0 < t < D\mu^{-1/3}$, where $D \gtrsim \frac{\pi}{N(\infty)}$ is a constant. The term $o(1)$ and it's derivative over t tends uniformly to zero as μ tends to zero on the interval $[0, D\mu^{-1/3}]$.

In our further discussion we will be mostly interested in the PRC3BP with $\mu = \mu_k$. Let us note that in such a case the time T_k at which $W^u_{L_2}$ intersects

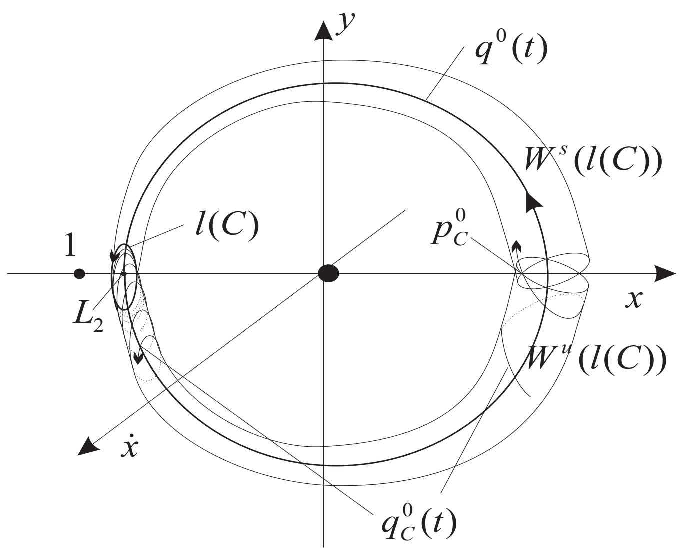

Figure 2.3 Intersections of $W^u(l(C))$ and $W^s(l(C))$.

$\{y = 0\}$ is asymptotically equal to $k\pi$ i.e.

$$|T_k - k\pi| \overset{k\to\infty}{\to} 0. \tag{2.18}$$

It is a well known fact that by the Lapunov Theorem (For details on the Lapunov Theorem see Chapter 3) for the PRC3BP there exists a family of Lapunov orbits $l_\mu(C)$ around L_2^μ. The parameter C is the energy of the orbit $l_\mu(C)$ and the orbits are parameterized by $C \in [C_2^\mu - \delta_\mu, C_2^\mu]$ where δ_μ is sufficiently small [21].

Theorem 2.3 ([21, Theorem B])

For μ and $\Delta C = C_2 - C$ sufficiently small, the branch of $W^u(l_\mu(C))$ contained in the S region intersects the plane $y = 0$ for $x > 0$ in a curve diffeomorphic to a circle (see Figure 2.3).

What is more for points in the (μ, C) plane such that there exists a μ_k of Theorem 2.1 for which

$$\Delta C > L\mu_k^{4/3}(\mu - \mu_k)^2 \tag{2.19}$$

holds (where L is a constant), there exist S-symmetric transversal homoclinic orbits. In particular, for $\mu = \mu_k$ there exist symmetrical transversal homoclinic orbits $q^0_{\mu_k C}$ for the periodic orbit $l_{\mu_k}(C)$ for every $C \in (\delta_{\mu_k}, C_2^{\mu_k})$.

Let us make a comment on how the transversality of the intersections of $W^s(l_\mu(C))$ and $W^u(l_\mu(C))$ of the Theorem 2.3 is understood. The invariant manifolds $W^s(l_\mu(C))$ and $W^u(l_\mu(C))$ are two dimensional "tubes" emanating from

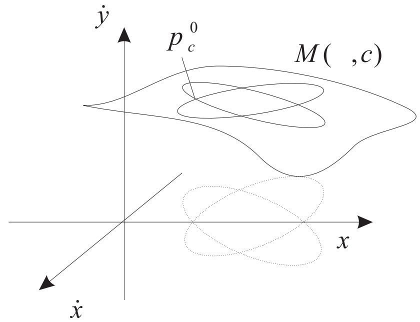

Figure 2.4 Intersection of $W^u(l(C))$, $W^s(l(C))$ with $\Sigma_{\{y=0\}}$ in the $x, \dot{x}, \dot{y}$ coordinates.

$l_\mu(C)$ (see Figure 2.3). The intersections of these tubes with the section $\Sigma_{\{y=0\}}$ projected onto the $x, \dot{x}$ coordinates are homeomorphic to circles. These projections intersect transversally in two points of the form $p^0_{C\mu} = (x^0_{C\mu}, 0, 0, \dot{y}^0_{C\mu})$. The fact that the projections onto the $x, \dot{x}$ intersect guarantees that the tubes will intersect in the three dimensional section $\Sigma_{\{y=0\}}$ follows from the fact that the solutions are immersed in the constant energy surface $M(\mu, C)$ and the coordinate $\dot{y}$ can be computed from the energy level C and $x, y, \dot{x}$ as

$$\dot{y} = \sqrt{2\Omega(x,y) - \dot{x}^2 - C}, \tag{2.20}$$

(see Figure 2.4). The homoclinic orbits $q^0_{\mu_k C}$ pass through these intersection points (see Figure 2.3).

From now on let us assume that we have chosen a certain $\mu = \mu_k$. The fact that we fix μ_k determines our equation (2.1), which will allow us to drop the index μ and μ_k from the following discussions. Thus we will write q^0, p^0 and L_2 instead of writing $q^0_{\mu_k}, p^0_{\mu_k}$ and $L_2^{\mu_k}$ and so on. We should keep in mind that all the following properties are dependent on the choice of our equation and therefore we will have different periodic orbits, invariant manifolds, intersection points etc. for the different μ_k. Since two of these notations will be important and often used in the following discussions we shall now write them out as a separate definitions so that they stand out and are clearly visible.

Definition 2.4

Let $q^0(t)$ be the homoclinic orbit given by Theorem 2.1 starting from $\Sigma_{\{y=0\}}$with $x > 0$ at the time $t = 0$ (see Figure 2.3)

$$\begin{aligned} q^0 &: \mathbb{R} \to \mathbb{R}^4 \\ q^0(0) &\in \Sigma_{\{y=0\}}. \end{aligned} \tag{2.21}$$

Definition 2.5

Let $q_C^0(t,t_0)$ denote the trajectory starting from one of the intersection points $p_C^0 = (x_C^0, 0, 0, \dot{y}_C^0)$ at a time t_0 (see Figure 2.3)

$$
\begin{aligned}
&q_C^0(\cdot, t_0) : \mathbb{R} \to \mathbb{R}^4 \\
&q_C^0(t_0, t_0) = (x_C^0, 0, 0, \dot{y}_C^0).
\end{aligned}
\tag{2.22}
$$

Let us finish this section by noting that the transversality of the intersection of $W^s(l_\mu(C))$ and $W^u(l_\mu(C))$ obtained in the Theorem 2.3 does not extend to the full four dimensional space. This can be summarized by the following remark.

Remark 2.6

Let us note that due to the fact that the invariant manifolds $W^s(l(C))$ and $W^u(l(C))$ are submerged in the three dimensional invariant energy manifold $M(\mu, C)$

$$
W^s(l(C)), W^u(l(C)) \subset M(\mu, C) = \{F(x, y, \dot{x}, \dot{y}) = C\}, \tag{2.23}
$$

their intersection cannot be transversal.

This does not mean though that the transversality of the intersection is only restricted to the $x, \dot{x}$ coordinates. For $\mu = \mu_k$ we are guaranteed to have a slightly better situation.

Remark 2.7

For C sufficiently close to C_2 the projection of the intersection of $W^s(l(C))$ and $W^u(l(C))$ at the point $p_C^0 = (x_C^0, 0, 0, \dot{y}_C^0)$ onto the $x, y, \dot{x}$ coordinates is transversal.

Proof

We already know that the projection onto $x, y, \dot{x}$ of the intersection restricted to $\Sigma_{\{y=0\}}$ is transversal in the $x, \dot{x}$ coordinates. We also know that the symmetric homoclinic orbit q^0 passes through the point $(x_C^0, 0, 0, \dot{y}_C^0)$ and therefore cuts through $\Sigma_{\{y=0\}}$ transversally in the direction of the y coordinate. This ensures the transversality of the projection of the intersection of $W^s(l(C))$ and $W^u(l(C))$ onto $x, y, \dot{x}$ in the y direction. □

2.4 The homoclinic orbit $q^0(t)$ when close to the libration point L_2.

In this section we will discuss how the homoclinic orbit $q^0(t)$ behaves in a close neighborhood of the Libration point L_2. We will highlight the following two facts.

The first is that in a small neighborhood of the point L_2 the flow of the PRC3BP can be approximated by the flow of the Hill's problem (this is summarized in Remark 2.8). The second fact is that from the point L_2 down to the section $\{y = -\mu^{1/4}\}$ the orbit $q^0(t)$ can be approximated by an appropriate orbit of the Hill's problem (this is summarized in Remark 2.9). The first fact will be used later on in Chapter 3 for the proof of the twist property of the PRC3BP close to the Libration point L_2. The second fact will be used in the computations of the Melnikov function of Chapter 8.

Let us clarify that this Section is a collection of rather technical results of [21] obtained in the course of the proof of Theorem 2.1. The aim of this Section is not to provide detailed proofs of the results, or even a sketch of the proof of Theorem 2.1, but simply to pick out the properties which will be needed in Chapters 3 and 8. Therefore at this stage the collection of results might seem random and unmethodical.

Let us start by writing out the Hill's problem and some of it's properties. The Hill's problem is given by a Hamiltonian

$$H^H(x_H, y_H, p_{xH}, p_{yH}) = \frac{(p_{xH} + y_H)^2 + (p_{yH} - x_H)^2}{2} - \Omega^H(x_H, y_H) \qquad (2.24)$$

where $\Omega^H(x_H, y_H) = \frac{1}{2}\left(3x_H^2 + 2(x_H^2 + y_H^2)^{-1/2}\right)$. In the above we have inserted the index H to distinguish between the variables x, y, of the original problem (2.7), and the variables x_H and y_H of the Hill's problem. The system (2.24) can be written in the $(x_H, y_H, \dot{x}_H, \dot{y}_H)$ coordinates as

$$\begin{aligned} \ddot{x}_H - 2\dot{y}_H &= 3x_H - x_H(x_H^2 + y_H^2)^{-3/2} = \Omega^H_{x_H}(x_H, y_H) \\ \ddot{y}_H + 2\dot{x}_H &= -y_H(x_H^2 + y_H^2)^{-3/2} = \Omega^H_{y_H}(x_H, y_H), \end{aligned} \qquad (2.25)$$

and has a Jacobi first integral given by

$$C^H = 2\Omega^H(x_H, y_H) - \dot{x}_H^2 - \dot{y}_H^2 = -2H^H(x_H, y_H, p_{xH}, p_{yH}). \qquad (2.26)$$

The Hamiltonian (2.24) can be derived from the equations of the PRC3BP (2.7) using the following change of coordinates [21, Ch. 4.]

$$\begin{aligned} x_H &= \mu^{-1/3}(x + 1 - \mu) \qquad & p_{xH} &= \mu^{-1/3} p_x \\ y_H &= \mu^{-1/3} y & p_{yH} &= \mu^{-1/3}(p_y + 1 - \mu), \end{aligned} \qquad (2.27)$$

and scaling μ down to zero. The change of coordinates (2.27) gives us (see [21, eq (4.1)] for details)

$$\begin{aligned} \dot{x}_H &= p_{xH} + y_H \\ \dot{y}_H &= p_{yH} - x_H \\ \dot{p}_{xH} &= 2x_H + p_{yH} - x_H(x_H^2 + y_H^2)^{-3/2} + \mu^{1/3}(6x_H^2 + 3y_H^2/2) + O(\mu^{2/3}) \\ \dot{p}_{yH} &= -p_{xH} - y_H - y_H(x_H^2 + y_H^2)^{-3/2} - \mu^{1/3}(3x_H y_H) + O(\mu^{2/3}), \end{aligned} \qquad (2.28)$$

Clearly for $\mu = 0$ the equations (2.28) are generated by the Hamiltonian (2.24). Let us also observe that from the formulas (2.28) we can see that in a small compact

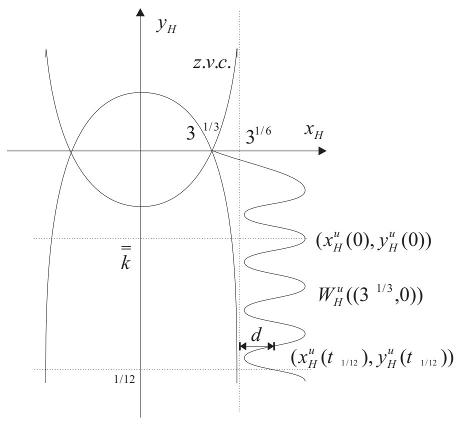

Figure 2.5 The orbit $q^H(t)$ in the x_H, y_H coordinates.

neighborhood of the Libration point L_2, in the Hill's coordinates x_H, y_H, p_{xH}, p_{yH}, the vector field f^{μ}_{3Body} of the PRC3BP is a $\mu^{1/3}$ perturbation of the vector field f_{Hill} of the Hill's problem.

$$f^{\mu}_{\text{3Body}} = f_{\text{Hill}} + O(\mu^{1/3}). \tag{2.29}$$

The Hill's problem (2.24) has two equilibrium points $L_1^H = (-3^{-1/3}, 0, 0, -3^{-1/3})$, $L_2^H = (3^{-1/3}, 0, 0, 3^{-1/3})$ [21]. The Jacobi constant $C^H = -2H^H$ of the two equilibrium points $L_1^H = (-3^{-1/3}, 0, 0, -3^{-1/3})$, $L_2^H = (3^{-1/3}, 0, 0, 3^{-1/3})$ is equal to

$$C_{L_2^H} = 3^{4/3}. \tag{2.30}$$

The zero velocity curve (see Figure 2.5) for this energy is $\{2\Omega^H - \dot{x}_H^2 - \dot{y}_H^2 = C_{L_2^H}\}$ and is given by the formula [21, page 116]

$$y_H = \pm\sqrt{\frac{4}{9}\left(3^{1/3} - x_H^2\right)^{-2} - x_H^2}, \quad |x_H| < 3^{1/6}. \tag{2.31}$$

For the values $x_H = \pm 3^{1/6}$ the zero velocity curve has two vertical asymptotes.

In the neighborhood of the equilibrium point L_2^H the local expression for the unstable manifold $W_H^u(L_2^H)$ of the Hill's problem is given by [21, page 116]

$$\begin{aligned} x_H &= 3^{-1/3} + s + \left(14\lambda^2 - 459\right) 3^{7/3} s^2 \left(1296 + 906\lambda^2\right)^{-1} + O(s^3) \\ y_H &= \left(\lambda^2 - 9\right) s \left(2\lambda\right)^{-1} + \left(81 - 13\lambda^2\right) 3^{7/3} s^2 \left[2\lambda\left(24\lambda^2 + 405\right)\right]^{-1} + O(s^3), \end{aligned} \tag{2.32}$$

where $s = \exp(\lambda t)$ and $\lambda = \sqrt{1 + 2\sqrt{7}} \approx 2.508\,3$.

Let us observe that far from L_2^H the terms $x_H(x_H^2 + y_H^2)^{-3/2}$ and $y_H(x_H^2 + y_H^2)^{-3/2}$ in the equations (2.25) of the Hill's problem

$$\begin{aligned} \ddot{x}_H - 2\dot{y}_H &= 3x_H - x_H(x_H^2 + y_H^2)^{-3/2} \\ \ddot{y}_H + 2\dot{x}_H &= -y_H(x_H^2 + y_H^2)^{-3/2} \end{aligned} \tag{2.33}$$

will be small. If we neglect these terms then we are left with a linear equation

$$\begin{aligned}\ddot{x}_H - 2\dot{y}_H &= 3x_H \\ \ddot{y}_H + 2\dot{x}_H &= 0,\end{aligned} \tag{2.34}$$

which has a solution given by

$$\begin{aligned}x_H &= \frac{2}{3}N + M\cos(t - t_0) \\ y_H &= B - Nt - 2M\sin(t - t_0).\end{aligned} \tag{2.35}$$

The above fact provides intuition for the occurrence of the waves of the unstable manifold $W^u_H(L^H_2)$ from the Figure 2.5. Indeed, it is shown [21, page 120] that the asymptotic behavior of an orbit on the unstable manifold $W^u_H(L^H_2)$ of the point $L^H_2 = (3^{-1/3}, 0, 0, 3^{-1/3})$ is

$$(x_H(t), y_H(t)) = (\frac{2}{3}N(\infty) + M(\infty)\cos(t - t_0), -N(\infty)t - 2M(\infty)\sin(t - t_0)), \tag{2.36}$$

where $M(\infty) \approx 2.1330587$ and $N(\infty) \approx 5.1604325$.

Let us now state our remarks which will be used later on in Chapters 3 and 8. The first states that the Libration point L^μ_2 of the PRC3BP can be approximated by the equilibrium point L^H_2 of the Hill's problem for sufficiently small μ.

Remark 2.8

In the Hill's coordinates x_H, y_H, p_{xH}, p_{yH}

1. in the neighborhood of the point L_2 the vector field given by (2.28) is analytic in x_H, y_H, p_{xH}, p_{yH} and $\mu^{1/3}$.
2. the libration point L^μ_2 tends to L^H_2 as μ tends to zero

$$\lim_{\mu\to 0} L^\mu_2 = L^H_2. \tag{2.37}$$

Proof

The first point comes directly from the equations (2.28). To prove the second observation let us consider a function

$$F(\mathbf{x}, \mu) = f^\mu_{3\text{Body}}(\mathbf{x}) \tag{2.38}$$

where $\mathbf{x} = (x_H, y_H, p_{xH}, p_{yH})$ and $f^\mu_{3\text{Body}}(\mathbf{x})$ is the vector field of the PRC3BP. Since

$$\frac{\partial F}{\partial \mathbf{x}}(L^H_2, 0) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 8 & 0 & 0 & 1 \\ 0 & -4 & -1 & 0 \end{pmatrix} \tag{2.39}$$

is an isomorphism we can apply the implicit function theorem to obtain a unique continuous function $\mathbf{x}(\mu)$ in the neighborhood of $\mu = 0$ such that $F(\mathbf{x}(\mu), \mu) = 0$. Since there is only a single Libration point between the masses μ and $1 - \mu$ the equilibrium point $\mathbf{x}(\mu)$ must be our Libration point L^μ_2. □

For the formulation of the second remark we will need the following notation. Let us denote by $q^H(t)$ the orbit of the Hill's problem which is contained in the unstable manifold $W^u_H(L^H_2)$ and starts at the section $\{y_H = \overline{\overline{k}}\}$ (see Remark 2.2 for $\overline{\overline{k}}$)

$$q^H(t) \subset W^u_H(L^H_2) \qquad (2.40)$$
$$q^H(0) \in \{y_H = \overline{\overline{k}}\}.$$

(See Figure 2.5). The following remark states that in an appropriate neighborhood of L^μ_2 the orbit $q^H(t)$ approximates the homoclinic orbit $q^0(t)$ for sufficiently small μ_k.

Remark 2.9

In the Hill's coordinates x_H, y_H, p_{xH}, p_{yH}

1. down to the section $\{y_H = -\mu^{-1/12}\}$ the distance between the orbits $q^H(t)$ and $q^0(t)$ is $O(\mu^{1/12})$.

$$q^H(t) - q^0(t - T_k) = O(\mu^{1/12}), \qquad (2.41)$$

   where $-T_k$ is the time at which the orbit $q^0(t)$ intersects $\{y_H = \overline{\overline{k}}\} = \{y = \mu^{1/3}\overline{\overline{k}}\}$ i.e. $q^0(-T_k) \in \{y = \mu^{1/3}\overline{\overline{k}}\}$ (see Remark 2.2).

2. down to the section $\{y_H = -\mu^{-1/12}\}$ the vales on the x_H coordinate of the orbit $q^0(t)$ homoclinic to L^μ_2 in the PRC3BP are bounded (see Definition 2.4 for the definition of $q^0(t)$).

3. down to the section $\{y_H = -\mu^{-1/12}\}$ the distance of $q^0(t)$ from the origin in the x_H, y_H coordinates is greater than $3^{-1/3}/2$ i.e.

$$\sqrt{x_H^2 + y_H^2} \geq 3^{-1/3}/2. \qquad (2.42)$$

4. For the times $t \in (-\infty, 0]$ the values $\dot{x}$, and $\dot{y}$ of the orbit $q^0(t)$ are uniformly $O(\mu^{1/3})$.

Proof

The first point comes from [21, p.122, par.1] by an argument that from the equations (2.28) we know that down to the section $\{y_H = -\mu^{-1/12}\}$, for bounded x_H, the solutions of the PRC3BP are $O(\mu^{\frac{1}{12}})$ approximated by the solutions of the Hill's problem.

For large t the orbit $q^H(t)$ of the Hill's problem homoclinic to L^H_2 is approximated by the equation (2.36). For small t the orbit $q^H(t)$ approaches the equilibrium point at a hyperbolic rate (2.32). This means that on the x_H coordinate this orbit is bounded for all $t \in \mathbb{R}$ (see also Figure 2.5). From the discussion from the beginning of the proof we know that down to the section $\{y_H = -\mu^{-1/12}\}$ the orbit $q^H(t)$ approximates $q^0(t)$. This means that down to the section $\{y_H = -\mu^{-1/12}\}$ the orbit $q^0(t)$ is also bounded on the x_H coordinate.

For the third point let us observe (see Figure 2.5) that on the orbit $q^H(t)$ we have

$$\sqrt{x_H^2 + y_H^2} \geq 3^{-1/3}. \tag{2.43}$$

This since down to the section $\{y_H = -\mu^{-1/12}\}$ the orbit $q^H(t)$ approximates $q^0(t)$ gives us (2.42).

The last point comes from the fact that for $t \in (-\infty, 0]$ the distance $d(t)$ from the zero velocity curve (2.15) is $O(\mu^{1/3})$. This gives a uniform bound $O(\mu^{1/3})$ on the velocity of the orbit $q^0(t)$ [21]. $\square$

Using the above Remark we can approximate our orbit $q^0(t)$ using $q^H(t)$ in the x_H, y_H, p_{xH}, p_{yH} coordinates down to the section $\{y_H = -\mu^{-1/12}\}$, and the accuracy of such an approximation is $O(\mu^{\frac{1}{12}})$. Coming back to our original coordinates x, y, p_x, p_y, since $y_H = \mu^{-1/3} y$, we can approximate $q^0(t)$ using $q^H(t)$ down to the section $\{y = -\mu^{-1/12}\mu^{1/3} = \mu^{1/4}\}$ and the accuracy of such an approximation is $O(\mu^{1/3}\mu^{1/12}) = O\left(\mu^{5/12}\right)$.

Let us finish this section with a remark concerning the choice of the section $\{y = -\bar{\bar{k}}\mu^{1/3}\}$, which is the starting point of the formulas from Theorem 2.1 (See Remark 2.2). During the course of the proof of Theorem A in [21] the large constant $\bar{\bar{k}}$ is chosen in such a way that for $y_H = -\bar{\bar{k}}$ the starting point $q^H(0)$ belongs to $W_H^u(L_2^H) \cap \{y_H = -\bar{\bar{k}}\}$ and is chosen to be at the top (or the bottom) of a wave of $W_H^u(L_2^H)$ [21, page 121, par. 3] (See Figure 2.5). Let us also note that the time $t_{\mu^{-1/12}}$ needed for $W_H^u(L_2^H)$ to reach the section $\{y_H = -\mu^{-1/12}\}$ is $O(\mu^{-1/12})$ [21, page 121, par. 3]. These facts will be used in one of the technical proofs in Chapter 8.

3 Dynamics of the flow close to the libration point L_2.

In this chapter we will describe the dynamics close to the libration point L_2. We will prove that there exists a family of the so called Lapunov periodic orbits emanating from the L_2. We will also show a method of how to prove that the Poincaré time 2π map restricted to the family of Lapunov orbits is a twist map. We will apply the method to an appropriate equilibrium point in the Hill's problem. Based on the fact that the Poincaré map for the Hill's problem is a twist map and also on the fact that the PRC3BP can be seen as a perturbation of the Hill's problem, we will show that for sufficiently small μ the Poincaré map is a twist map in the restricted three body problem. We will also discuss how the twist condition and the twist coordinates are related to the energy of the Lapunov orbits.

The main tool of this chapter is the Lapunov-Moser theorem (Theorem 3.2) which allows us to transform the solutions close to an equilibrium point into a Birkhoff type normal form by an appropriate change of coordinates. We will use the expansion into the power series of this form. It will turn out that if an appropriate coefficient in the expansion is nonzero then the twist property of a time 2π Poincaré map follows.

The main result of this chapter is the Theorem 3.15 which states that for a sufficiently small parameter μ in the PRC3BP the time 2π Poincaré map is a twist map at the Libration point L_2. This is a new result. It has a nontrivial consequence later on in Chapter 6, where it will be used to prove that the libration point L_2 persists under perturbation. Similar techniques were used in [22] to prove the persistence of the Libration point L_4 in the PRC3BP and therefore we would like to admit that the basic concept behind the proof is not a new one.

3.1 The linearization of the flow at the libration point L_2.

Let us first discuss how we can find the libration point L_2 and the energy C_2 associated with it. We are looking for an equilibrium point with a y coordinate equal to zero which lies between $-(1-\mu)$ and μ on the x axis (see Figure 2.1). Let us expand the equations of motion in the PRC3BP (2.3) to obtain the following equation

$$\begin{aligned}
\dot{x} &= p_x + y \\
\dot{y} &= p_y - x \\
\dot{p}_x &= p_y - \frac{(1-\mu)(x-\mu)}{((x-\mu)^2+y^2)^{3/2}} - \frac{\mu(1-\mu+x)}{((x+1-\mu)^2+y^2)^{3/2}} \\
\dot{p}_y &= -p_x - \frac{(1-\mu)y}{((x-\mu)^2+y^2)^{3/2}} - \frac{\mu y}{((x+1-\mu)^2+y^2)^{3/2}}.
\end{aligned} \tag{3.1}$$

Since the y coordinate of our point L_2 is equal to zero, from the first equation it is clear that $p_x = 0$. From the second and third equation we have

$$L_2 = (-k, 0, 0, -k), \tag{3.2}$$

where $k > 0$ is a solution of

$$-k - \frac{(1-\mu)(-k-\mu)}{((-k-\mu)^2)^{3/2}} - \frac{\mu(1-\mu-k)}{((1-\mu-k)^2)^{3/2}} = 0. \tag{3.3}$$

The Libration point L_2 lies on the x axis between the smaller body μ which lies at the point $(-1+\mu, 0)$ and the larger mass $1-\mu$ lying at the point $(\mu, 0)$. This means that we have

$$-\mu < k < 1-\mu. \tag{3.4}$$

The equations (3.3) and (3.4) are the key for finding the Libration point L_2. (From these equations it is also apparent that we do not have a simple analytical formula for the value of k.)

Let us now consider the linearization of the flow at the point L_2. In order to compute the linear terms of the equation (3.1) at $(-k, 0, 0, -k)$ let us first compute

$$\begin{aligned}
-\frac{\partial^2 H_C}{\partial x \partial x}(-k,0,0,-k) &= \frac{2(1-\mu)}{|k+\mu|^3} + \frac{2\mu}{|k-1+\mu|^3} \\
-\frac{\partial^2 H_C}{\partial y \partial y}(-k,0,0,-k) &= -\frac{1-\mu}{|k+\mu|^3} - \frac{\mu}{|k-1+\mu|^3} \\
-\frac{\partial^2 H_C}{\partial x \partial y}(-k,0,0,-k) &= -\frac{\partial^2 H_C}{\partial y \partial x}(-k,0,0,-k) = 0.
\end{aligned} \tag{3.5}$$

which gives the linear part of (3.1) at $(-k, 0, 0, -k)$ as

$$\begin{aligned}
\dot{x} &= p_x + y \\
\dot{y} &= p_y - x \\
\dot{p}_x &= p_y - x + (2\rho+1)x \\
\dot{p}_y &= -p_x - y - (\rho-1)y
\end{aligned} \tag{3.6}$$

where

$$\rho = \frac{1-\mu}{|k+\mu|^3} + \frac{\mu}{|k-1+\mu|^3}. \tag{3.7}$$

The above equation has been written in such a fashion, so that it is clear that it is generated by a Hamiltonian

$$H_l(x, y, p_x, p_y) = \frac{1}{2}\left((p_x + y)^2 + (p_y - x)^2 - ax^2 + by^2\right), \tag{3.8}$$

where $a = 2\rho + 1$ and $b = \rho - 1$. From the equation (3.6) we can compute that the eigenvalues α of the linear terms at L_2 are the roots of $\alpha^4 + (2-\rho)\alpha^2 + \left(1 + \rho - 2\rho^2\right)$.

Lemma 3.1

There are two real and two pure imaginary eigenvalues of the linear terms of (3.1) at L_2.

Proof

The eigenvalues are the roots of $\alpha^4 + (2-\rho)\alpha^2 + \left(1 + \rho - 2\rho^2\right)$. In order to prove that two of them are real and two are pure imaginary it is sufficient to show that the polynomial

$$x^2 + (2-\rho)x + \left(1 + \rho - 2\rho^2\right) = 0 \tag{3.9}$$

has two real solutions, of which one is positive and one negative. First of all let us show that $\rho > 1$. We know that k is a solution of the equation (3.3), which we can rearrange as

$$\begin{aligned} 0 &= -k - \frac{(1-\mu)(-k-\mu)}{((-k-\mu)^2)^{3/2}} - \frac{\mu(1-\mu-k)}{((1-\mu-k)^2)^{3/2}} \\ &= -k + \frac{(1-\mu)(k+\mu)}{|k+\mu|^3} + \frac{\mu(k-1+\mu)}{|k-1+\mu|^3} \\ &= -k + \rho\mu + k\rho - \frac{\mu}{|k-1+\mu|^3}, \end{aligned} \tag{3.10}$$

and therefore

$$\rho = \frac{1}{k+\mu}\left(\frac{\mu}{|k-1+\mu|^3} + k\right). \tag{3.11}$$

From (3.4) we know that $|k-1+\mu| < 1$ which gives us

$$\frac{1}{|k-1+\mu|^3} > 1, \tag{3.12}$$

which in turn by (3.11) guarantees that $\rho > 1$. Let us now turn to the roots of our quadratic equation (3.9). Let us observe that because $\Delta = \rho(9\rho - 8) > 0$ the equation (3.9) has two real solutions. In order to see if one of them is positive and one is negative it is sufficient to compute their product which is equal to $\left(1 + \rho - 2\rho^2\right)$. From the fact that $\rho > 1$ it is clear that this product is negative. □

3.2 From the Lapunov-Moser Theorem to twist maps in the neighborhood of the libration point L_2

In this section we will give general facts concerning four dimensional Hamiltonian systems with an equilibrium point and two real and two pure imaginary eigenvalues

$$\begin{aligned} \alpha_1 &= \lambda & \alpha_3 &= -\lambda \\ \alpha_2 &= i\kappa & \alpha_4 &= -i\kappa, \end{aligned} \tag{3.13}$$

where $\lambda, \kappa \in \mathbb{R}$. Based on the Lapunov-Moser Theorem [25] we will show that for such systems there exists a family of periodic orbits emanating from the equilibrium point (the so called Lapunov orbits). Based on the expansion into power series following from the Theorem we will show how to prove that the time 2π shift along the trajectory is a twist map on the set of these orbits in a small neighborhood of the equilibrium point.

Let us note that since in Lemma 3.1 we have proved that in the case of the PRC3BP the libration point L_2 has two real and two pure imaginary conjugate eigenvalues, all of the results given in this section will apply to the PRC3BP and the equilibrium point L_2.

First let us start with the formulation of the Lapunov-Moser Theorem.

Theorem 3.2 (the Lapunov-Moser Theorem [25])

Let

$$\begin{aligned} \dot{x}_\nu &= H_{y_\nu}(x, y) \\ \dot{y}_\nu &= -H_{x_\nu}(x, y) \end{aligned} \tag{3.14}$$

$\nu = 1, \ldots, n$, be an analytic Hamiltonian system with n degrees of freedom and an equilibrium solution $x = y = 0$. Let $\alpha_1, \ldots, \alpha_n, -\alpha_1, \ldots, -\alpha_n$ be the eigenvalues of the linearization of (3.14) at the equilibrium point $x = 0$. Assume that the eigenvalues

$$\alpha_1, \ldots, \alpha_n, -\alpha_1, \ldots, -\alpha_n \tag{3.15}$$

are $2n$ different complex numbers and that α_1, α_2 are independent over the reals, i.e. for any $t \in \mathbb{R}$

$$t\alpha_1 + \alpha_2 \neq 0. \tag{3.16}$$

Let us also assume that for any integer numbers n_1 and n_2

$$\alpha_\nu \neq n_1\alpha_1 + n_2\alpha_2 \quad \text{for } \nu \geq 3. \tag{3.17}$$

Then there exists a four parameter family of solutions of (3.14) of the form

$$\begin{aligned} x_\nu &= \phi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) \\ y_\nu &= \psi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) \end{aligned} \tag{3.18}$$

where

$$\xi_k(t) = \xi_k^0 e^{ta_k(\xi_1^0\eta_1^0, \xi_2^0\eta_2^0)}, \quad \eta_k(t) = \eta_k^0 e^{-ta_k(\xi_1^0\eta_1^0, \xi_2^0\eta_2^0)} \quad \text{for } k = 1, 2, \tag{3.19}$$

and

$$a_1(\xi_1^0\eta_1^0, \xi_2^0\eta_2^0) = \alpha_1 + ..., \qquad a_2(\xi_1^0\eta_1^0, \xi_2^0\eta_2^0) = \alpha_2 + ... \tag{3.20}$$

are convergent power series. The series ϕ_ν, ψ_ν converge in the neighborhood of the origin and the rank of the matrix

$$\begin{pmatrix} \phi_{\nu\xi_k} & \phi_{\nu\eta_k} \\ \psi_{\nu\xi_k} & \psi_{\nu\eta_k} \end{pmatrix}_{\substack{\nu=1,2 \\ k=1,2,\dots,n}} \tag{3.21}$$

is four. The solutions (3.18) depend on four complex parameters ξ_k^0, η_k^0.

Let us note that since we are interested in the application of the above Theorem to a four dimensional Hamiltonian system, in our case n is simply equal to two and the equations (3.18), (3.19) describe all the solutions near the neighborhood of the equilibrium point.

During the proof of the Theorem in [25] it is shown that

Lemma 3.3

When the system (3.14) is generated by a real Hamiltonian then if α_1 is real and α_2 is pure imaginary the real solutions must be of the form

$$\begin{aligned} x_\nu(t) &= \phi_\nu(\xi_1(t), \xi_2(t), \eta_1(t), \eta_2(t)) \\ y_\nu(t) &= \psi_\nu(\xi_1(t), \xi_2(t), \eta_1(t), \eta_2(t)) \end{aligned} \qquad \nu = 1, 2 \tag{3.22}$$

where

$$\begin{aligned} \xi_k(t) &= \xi_k^0 e^{ta_k(\xi_1^0\eta_1^0, \xi_2^0\eta_2^0)} \\ \eta_k(t) &= \eta_k^0 e^{-ta_k(\xi_1^0\eta_1^0, \xi_2^0\eta_2^0)} \end{aligned} \qquad \text{for } k = 1, 2, \tag{3.23}$$

$$\xi_1^0, \eta_1^0, \xi_2^0, \eta_2^0 \in \mathbb{C}$$

and the solution $(\xi_1(t), \xi_2(t), \eta_1(t), \eta_2(t))$ is invariant under the involution

$$J(\xi_1, \xi_2, \eta_1, \eta_2) = (\bar{\xi}_1, i\bar{\eta}_2, \bar{\eta}_1, i\bar{\xi}_2). \tag{3.24}$$

Remark 3.4

From the proof of the convergence of the series (3.18), (3.20) during the proof of Theorem 3.2 in [25], it follows that if we consider a family of analytic Hamiltonians

$$H_\lambda : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}, \tag{3.25}$$

which depends analytically on a parameter $\lambda \in \mathbb{R}$, then the radius of convergence of the series (3.18), (3.20) can be chosen uniformly for close values of λ.

Remark 3.5

The original version of the above Theorem 3.2 in [25] contains a small technical error. The involution used originally by the author was

$$J(\xi_1, \xi_2, \eta_1, \eta_2) = (\bar{\xi}_1, \bar{\eta}_2, \bar{\eta}_1, \bar{\xi}_2). \tag{3.26}$$

This stands in conflict with a requirement that an appropriate transformation in the proof is required to be canonical. We will come back to this issue in more detail in Remark 3.10. For now let us note that this has been corrected in the 1971 edition of the book by Siegel and Moser [19, page 102] where the corrected involution (3.27) is used.

In order to explain the Lemma 3.3 intuitively let us note that if we have a solution of the form (3.19) written in the ξ_v, η_v coordinates then this solution is carried by (3.18) into the x_v, y_v coordinates. Some of these solutions in the x_v, y_v will not turn out to be real solutions. Only the ones which satisfy the involution

$$J(\xi_1, \xi_2, \eta_1, \eta_2) = (\xi_1, \xi_2, \eta_1, \eta_2). \tag{3.27}$$

where J is given by (3.24), will be real. Let us demonstrate this on a simple example.

Example 3.6

Let us consider a harmonic oscillator

$$\begin{aligned} x' &= y \\ y' &= -x. \end{aligned} \tag{3.28}$$

We have two eigenvalues $\alpha_1 = i$ and $\alpha_2 = -i$ and their corresponding eigenvectors $(-i, 1)$ and $(1, -i)$. The solution of (3.28) is therefore given by

$$(x(t), y(t)) = \xi^0(-i, 1)e^{it} + \eta^0(1, -i)e^{-it}, \qquad \xi^0, \eta^0 \in \mathbb{C}. \tag{3.29}$$

If we would like to pick out the real solution we would need to have $\overline{(x(t), y(t))} = (x(t), y(t))$, which means that since

$$\overline{(x(t), y(t))} = \overline{\xi^0}(i, 1)e^{-it} + \overline{\eta^0}(1, i)e^{it} \tag{3.30}$$

we would need to have $(\xi^0, \eta^0) = (i\overline{\eta^0}, i\overline{\xi^0})$. This motivates the definition of our function J in (3.24) because the points invariant under J are the real points in the x, y coordinates.

Lemma 3.7

If α_1 is real and α_2 is pure imaginary then for all real solutions the series a_1 from the Theorem 3.2 is real and the series a_2 is pure imaginary. What is more if we choose a periodic solution

$$\begin{aligned} x_\nu(t) &= \phi_\nu(0, \xi_2(t), 0, \eta_2(t)) \\ y_\nu(t) &= \psi_\nu(0, \xi_2(t), 0, \eta_2(t)) \end{aligned} \qquad \nu = 1, 2 \tag{3.31}$$

where

$$\begin{aligned}\xi_k(t) &= \xi_k^0 e^{ta_k(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)} \\ \eta_k(t) &= \eta_k^0 e^{-ta_k(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)}\end{aligned} \qquad \text{for } k=1,2, \tag{3.32}$$

then if the solution is real then there exist two real numbers r and ϕ such that

$$\begin{aligned}\xi_2(t) &= re^{ta_2(0,ir^2)+i\phi} \\ \eta_2(t) &= ire^{-ta_2(0,ir^2)-i\phi}.\end{aligned} \tag{3.33}$$

Proof

From Lemma 3.3 we know that the real solutions satisfy the involution (3.27). We therefore have

$$\xi_1^0 e^{ta_1(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)} = \overline{\xi_1^0 e^{ta_1(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)}} \tag{3.34}$$

$$\xi_2^0 e^{ta_2(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)} = i\left(\overline{\eta_2^0 e^{-ta_2(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)}}\right) \tag{3.35}$$

$$\eta_1^0 e^{-ta_1(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)} = \overline{\eta_1^0 e^{-ta_1(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)}} \tag{3.36}$$

$$\eta_2^0 e^{-ta_2(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)} = i\left(\overline{\xi_2^0 e^{ta_2(\xi_1^0\eta_1^0,\xi_2^0\eta_2^0)}}\right) \tag{3.37}$$

if we choose $t=0$ then from the above we can see that ξ_1^0 and η_1^0 must be real and that $\xi_2^0 = i\overline{\eta_2^0}$. Using the fact that $\xi_1^0,\eta_1^0 \in \mathbb{R}$ with (3.34) and (3.36) we can see that a_1 must be real. Using (3.35) and (3.37) and the fact that $\xi_2^0 = i\overline{\eta_2^0}$ we can see that a_2 must be pure imaginary.

All periodic solutions have the initial conditions $\xi_1^0 = \eta_1^0 = 0$. If we choose an initial condition $\xi_2^0 = re^{i\phi}$ then for the solution to be real we must have $\xi_2^0 = i\overline{\eta_2^0}$. In such case the equation (3.19) gives us the periodic solutions as

$$\begin{aligned}\xi_2(t) &= \xi_2^0 e^{ta_2(0,\xi_2^0\eta_2^0)} = re^{ta_2(0,ir^2)+i\phi} \\ \eta_2(t) &= \eta_2^0 e^{-ta_2(0,\xi_2^0\eta_2^0)} = ire^{-ta_2(0,ir^2)-i\phi}.\end{aligned} \tag{3.38}$$

□

The above Lemma shows that all periodic solutions which are real in the x_ν, y_ν coordinates and lie close to the equilibrium point, are given by the equation

$$l_r(t) = (0, re^{ta_2(0,ir^2)+i\phi}, 0, ire^{-ta_2(0,ir^2)-i\phi}), \tag{3.39}$$

when seen in the ξ_ν, η_ν coordinates. Let us denote the set which contains these orbits by

$$B_R = \{(0, re^{i\theta}, 0, ire^{-i\theta}) | \theta \in [0,2\pi), 0 \le r \le R\} \tag{3.40}$$

where R is sufficiently small for the series $a_2(0, ir^2)$ to be convergent for $r \le R$.

Let $P : \mathbb{R}^4 \to \mathbb{R}^4$ be the time 2π shift along the trajectory i.e.

$$P(q(t)) = q(t+2\pi) \tag{3.41}$$

where $q(t)$ is a solution of (3.14). We will show a method which will allow us to prove whether P restricted to our set B_R, for R sufficiently small, is an analytic twist map.

Lemma 3.8

If in the series a_2 from Theorem 3.2 i.e.

$$a_2(\xi_1\eta_1, \xi_2\eta_2) = \alpha_2 + a_{2,1}\xi_1\eta_1 + a_{2,2}\xi_2\eta_2 + ... \tag{3.42}$$

we have $a_{2,2} \neq 0$, then for a sufficiently small R, the time 2π shift along the trajectory P restricted to the set B_R is an analytic twist map i.e.

$$P(r,\theta) = (r, \theta + f(r)) \tag{3.43}$$
$$\frac{df}{dr} \neq 0.$$

Proof

Since in the ξ, η coordinates on B_R the map P takes form

$$P\begin{pmatrix} 0 \\ re^{ta_2(0,ir^2)+i\phi} \\ 0 \\ ire^{-ta_2(0,ir^2)-i\phi} \end{pmatrix} = \begin{pmatrix} 0 \\ re^{(t+2\pi)a_2(0,ir^2)+i\phi} \\ 0 \\ ire^{-(t+2\pi)a_2(0,ir^2)-i\phi} \end{pmatrix} \tag{3.44}$$

we can see that

$$P(r,\theta) = (r, \theta - i2\pi a_2(0, ir^2)). \tag{3.45}$$

Since

$$a_2(0, ir^2) = \alpha_2 + a_{2,2}ir^2 + O(r^4) \tag{3.46}$$

it is evident that if $a_{2,2} \neq 0$, then for sufficiently small r

$$-i2\pi\frac{d}{dr}\left(a_2(0, ir^2)\right) \neq 0. \tag{3.47}$$

□

3.3 Computation of the twist coefficient $a_{2,2}$ from the Hamiltonian

In the previous section by Lemma 3.8 we have shown that in order to prove the twist condition it is sufficient to construct the series $a_2 = \alpha_2 + a_{2,1}\xi_1\eta_1 + a_{2,2}\xi_2\eta_2 + ...$ from Theorem 3.2, and show that $a_{2,2} \neq 0$. In this section we will show how the construction can be performed. We will follow the construction presented by Moser [25] in the proof of the Lapunov-Moser Theorem 3.2.

3.3.1 Summary of the transformations needed for the construction.

Since the construction is performed in two stpdf, let us first draw a quick diagram which will help in navigating throughout the procedure

$$\begin{array}{ccccc} \mathbb{C}^4 & \stackrel{\Psi}{\to} & \mathbb{R}^4 & \stackrel{\Phi}{\to} & \mathbb{R}^4 \\ (\xi_1,\xi_2,\eta_1,\eta_2) & \stackrel{\Psi}{\mapsto} & (x_1,x_2,y_1,y_2) & \stackrel{\Phi}{\mapsto} & (x,y,p_y,p_y). \end{array} \quad (3.48)$$

The transformation Φ will allow us to change from the system (3.1) in the x, y, p_y, p_y coordinates into a system with a simplified form

$$\begin{aligned} \dot{x}_\nu &= \alpha_\nu x_\nu + f_\nu(x,y) \\ \dot{y}_\nu &= -\alpha_\nu y_\nu + g_\nu(x,y) \end{aligned} \qquad \nu = 1,2. \quad (3.49)$$

where α_1 and α_2 are the eigenvalues of the equilibrium point and f and g are power series starting from quadratic terms. This will be done by transforming the linear terms of (3.1) into a diagonal form in Section 3.3.3. The transformation Ψ,

$$\Psi(\xi,\eta) = (\phi_1(\xi,\eta), \phi_2(\xi,\eta), \psi_1(\xi,\eta), \psi_2(\xi,\eta)) \quad (3.50)$$

will be constructed by comparison of coefficients coming from the differential equations in ξ, η and x_ν, y_ν coordinates.

We will start with the construction of Ψ.

3.3.2 Construction of the function Ψ and the term $a_{2,2}$ by comparison of coefficients.

In this section we will present a method of [25] which will alow us to compute the power series Ψ and the series a_2 by a method of comparison of coefficients in a special case when the Hamiltonian H generates a differential equation of the form

$$\begin{aligned} \dot{x}_\nu &= \alpha_\nu x_\nu + f_\nu(x,y) \\ \dot{y}_\nu &= -\alpha_\nu y_\nu + g_\nu(x,y) \end{aligned} \qquad \nu = 1,2, \quad (3.51)$$

where f and g are power series starting from quadratic terms. We will construct power series $\phi_\nu, \psi_\nu, a_\nu$ where

$$\begin{aligned} \phi_\nu(\xi_1,\xi_2,\eta_1,\eta_2) &= \textstyle\sum_{k=1}^2 \delta_{\nu k}\xi_k + h.o.t. \\ \psi_\nu(\xi_1,\xi_2,\eta_1,\eta_2) &= \textstyle\sum_{k=1}^2 \delta_{\nu k}\eta_k + h.o.t. \end{aligned} \quad (3.52)$$

such that

$$\begin{aligned} x_\nu &= \phi_\nu(\xi_1,\xi_2,\eta_1,\eta_2) \\ y_\nu &= \psi_\nu(\xi_1,\xi_2,\eta_1,\eta_2), \end{aligned} \quad (3.53)$$

satisfy (3.51) if

$$\begin{aligned} \dot{\xi}_k &= a_k(\xi_1\eta_1,\xi_2\eta_2)\xi_k \\ \dot{\eta}_k &= -a_k(\xi_1\eta_1,\xi_2\eta_2)\eta_k. \end{aligned} \quad (3.54)$$

Let us note that the series $\phi_\nu, \psi_\nu, a_\nu$ are not yet uniquely determined by (3.52), (3.53), (3.54). They are unique only if a certain normalization to the above equations is added [25]. In order to define this normalization we shall need some additional notations.

Let a series F in ξ_k, η_k be of the form

$$F = \sum_{n_1,n_2,m_1,m_2} c_{n_1,m_1,n_2,m_2} \xi_1^{n_1} \eta_1^{m_1} \xi_2^{n_2} \eta_2^{m_2}. \tag{3.55}$$

We denote by $[F]$ a series

$$[F] := \sum_{n_1,n_2} c_{n_1,n_1,n_2,n_2} \left(\xi_1\eta_1\right)^{n_1} \left(\xi_2\eta_2\right)^{n_2}. \tag{3.56}$$

It turns out [25, Sec. 2.] that the series ϕ_v, ψ_v, a_k are uniquely determined if the series $\left[\frac{\phi_k}{\xi_k}\right]$ and $\left[\frac{\psi_k}{\eta_k}\right]$ are normalized in some way. For the purpose of our construction we will pick the simplest possible normalization which is

$$\begin{aligned} \left[\frac{\phi_k}{\xi_k}\right] &= 1 \\ \left[\frac{\psi_k}{\eta_k}\right] &= 1 \end{aligned} \qquad \text{for } k = 1, 2. \tag{3.57}$$

We will now show how to construct the series ϕ_v, ψ_v, a_k under the constraint (3.57). In order to find the series ϕ_v, ψ_v, a_k let us first rewrite the equation (3.51) as

$$\begin{aligned} \dot{x}_\nu &= \textstyle\sum_{k=1}^2 \left(\frac{\partial \phi_\nu}{\partial \xi_k} a_k \xi_k - \frac{\partial \phi_\nu}{\partial \eta_k} a_k \eta_k\right) = \alpha_\nu \phi_\nu + f_\nu(\phi, \psi) \\ \dot{y}_\nu &= \textstyle\sum_{k=1}^2 \left(\frac{\partial \psi_\nu}{\partial \xi_k} a_k \xi_k - \frac{\partial \psi_\nu}{\partial \eta_k} a_k \eta_k\right) = -\alpha_\nu \psi_\nu + g_\nu(\phi, \psi). \end{aligned} \qquad \nu = 1, 2. \tag{3.58}$$

We will construct our series by comparing the coefficients in the above equations. Let us denote by $\phi_{\nu,N}, \psi_{\nu,N}, a_{\nu,N}$ the coefficients in the series $\phi_\nu, \psi_\nu, a_\nu$ which come from the homogenous polynomials of the order N. We can rewrite the part of the above equations which contains all the terms of the order N as

$$\begin{aligned} &\textstyle\sum_{k=1}^2 \alpha_k \left(\xi_k \frac{\partial}{\partial \xi_k} - \eta_k \frac{\partial}{\partial \eta_k}\right) \phi_{\nu,N} + \ldots + \delta_{\nu k} \xi_k a_{k,N-1} = \alpha_\nu \phi_{\nu,N} + \ldots \\ &\textstyle\sum_{k=1}^2 \alpha_k \left(\xi_k \frac{\partial}{\partial \xi_k} - \eta_k \frac{\partial}{\partial \eta_k}\right) \psi_{\nu,N} + \ldots - \delta_{\nu k} \eta_k a_{k,N-1} = -\alpha_\nu \psi_{\nu,N} + \ldots \end{aligned} \tag{3.59}$$

where the dots indicate all the terms which can be computed from $\phi_{\nu,l}, \psi_{\nu,l}, a_{\nu,l-1}$ with $l = 1, \ldots, N-1$.

The nature of equations (3.59) suggest that the series can be constructed by induction starting with the lowest terms. It turns out though that not all of the coefficients can be computed from (3.59). This is because some of the terms in (3.59) cancel each other out. The value of coefficients corresponding to such terms is chosen from the normalization condition (3.57). To make the above statement rigorous let us consider a homogenous polynomial $c\xi_1^{n_1}\eta_1^{m_1}\xi_2^{n_2}\eta_2^{m_2}$ of order N from $\phi_{\nu,N}$. Such term will cancel out in (3.59) if

$$\sum_{k=1}^2 \alpha_k \left(\xi_k \frac{\partial}{\partial \xi_k} - \eta_k \frac{\partial}{\partial \eta_k}\right) c\xi_1^{n_1}\eta_1^{m_1}\xi_2^{n_2}\eta_2^{m_2} - \alpha_\nu c\xi_1^{n_1}\eta_1^{m_1}\xi_2^{n_2}\eta_2^{m_2} = 0. \tag{3.60}$$

This can happen only if we have

$$\sum_{k=1}^{2} \alpha_k \left(n_k - m_k\right) - \alpha_\nu = 0. \tag{3.61}$$

One of the assumptions of the Theorem 3.2 is that for any $t \in \mathbb{R}$ we have $t\alpha_1 + \alpha_2 \neq 0$, which means that (3.61) is true only for the terms of the form $c\xi_v \left(\xi_1\eta_1\right)^{n_1} \left(\xi_2\eta_2\right)^{n_2}$. First of all this means that with respect to the equations (3.59) the coefficient c in the term $c\xi_v \left(\xi_1\eta_1\right)^{n_1} \left(\xi_2\eta_2\right)^{n_2}$ from $\phi_{v,N}$ can be chosen to be arbitrary. On the other hand we have chosen the normalization (3.57). Such a normalization determines that in these terms we set $c = 0$. Since we choose the terms of the form $c\xi_v \left(\xi_1\eta_1\right)^{n_1} \left(\xi_2\eta_2\right)^{n_2}$ from $\phi_{v,N}$ to be equal to zero the terms of $\delta_{\nu k}\xi_k a_{k,N-1}$ are uniquely determined through the equations (3.59) by $\phi_{\nu,l}, \psi_{\nu,l}, a_{\nu,l-1}$ with $l = 1, \ldots, N-1$. This will allow us to write an explicit formula for the coefficient $a_{2,2}$ later on. A similar discussion can be made for $\psi_{v,N}$. From the above we can see that once we choose our normalization (3.57) we can compute our series by induction starting with the low order terms.

Our aim is to compute the coefficient $a_{2,2}$ which in the above equations is connected with the term $\xi_2\xi_2\eta_2$ of the order $N = 3$. The simplest method to achieve this is to perform the comparison of coefficients in the equations (3.58) with $\xi_1 = \eta_1 = 0$ and with $\nu = 2$. We can compute our term $a_{2,2}$ from the first of the two equations (3.58) which in this case takes form

$$\alpha_2 \left(\xi_2 \frac{\partial}{\partial \xi_2} - \eta_2 \frac{\partial}{\partial \eta_2}\right) \phi_2^{2,1} \xi_2\xi_2\eta_2 + \ldots + \xi_2 a_{2,2} \xi_2\eta_2 = \alpha_2 \phi_2^{2,1} \xi_2\xi_2\eta_2 + \ldots \tag{3.62}$$

where $\phi_2^{2,1}$ is the coefficient in ϕ_2, which stands before the term $\xi_2\xi_2\eta_2$, and the dots indicate the terms which can be computed from $\phi_{\nu,l}, \psi_{\nu,l}, a_{\nu,l-1}$ with $l = 1, 2$. As we have mentioned before for such terms our equation should simplify even more. Indeed, if we notice that

$$\alpha_2 \left(\xi_2 \frac{\partial}{\partial \xi_2} - \eta_2 \frac{\partial}{\partial \eta_2}\right) \phi_2^{2,1} \xi_2\xi_2\eta_2 - \alpha_2 \phi_2^{2,1} \xi_2\xi_2\eta_2 = 0 \tag{3.63}$$

then we can see that the coefficient $a_{2,2}$ is dependent only from the terms $\phi_{\nu,l}$, $\psi_{\nu,l}$, $a_{\nu,l-1}$ with $l = 1, 2$. These terms can be easily computed from (3.59) for $N = 2$.

Lemma 3.9

If

$$\begin{array}{l} f_\nu(x_1, x_2, y_1, y_2) = \sum_{i,j,k,l \geq 1} f_{ijkl}^{\nu} x_1^i x_2^j y_1^k y_2^l \\ g_\nu(x_1, x_2, y_1, y_2) = \sum_{i,j,k,l \geq 1} g_{ijkl}^{\nu} x_1^i x_2^j y_1^k y_2^l \end{array} \qquad \nu = 1, 2. \tag{3.64}$$

and we choose our normalization to be

$$\begin{array}{l} \left[\frac{\phi_k}{\xi_k}\right] = 1 \\ \left[\frac{\psi_k}{\eta_k}\right] = 1 \end{array} \qquad \text{for } k = 1, 2, \tag{3.65}$$

then

$$a_{2,2} = \frac{1}{\alpha_2}(-f^2_{1,1,0,0}f^1_{0,1,0,1} - f^2_{0,1,1,0}g^1_{0,1,0,1} + f^2_{0,0,1,1}g^1_{0,2,0,0} - f^2_{0,2,0,0}f^2_{0,1,0,1} \\ + 2g^2_{0,2,0,0}f^2_{0,0,0,2} + f^2_{1,0,0,1}f^1_{0,2,0,0} - g^2_{0,1,0,1}f^2_{0,1,0,1}) + f^2_{0,2,0,1} \tag{3.66}$$

Proof

The above can be checked from the formula (3.58) by direct computation. This has also been checked by comparison of coefficients in Maple. □

3.3.3 The construction of Φ and the transformation of the equation into the desired form.

In order to show that the time 2π shift along the trajectory is a twist map on the set B_R composed of periodic orbits in the neighborhood of the libration point L_2, we need to check if the coefficient $a_{2,2} \neq 0$, where

$$a_2 = \alpha_2 + a_{2,1}\xi_1\eta_1 + a_{2,2}\xi_2\eta_2 \dots \tag{3.67}$$

is the expansion given by the Lapunov Theorem 3.2. From the previous section we know that if the equation is of the form

$$\begin{aligned} \dot{x}_\nu &= \alpha_\nu x_\nu + f^\nu(x,y) \\ \dot{y}_\nu &= -\alpha_\nu y_\nu + g^\nu(x,y) \end{aligned} \qquad \nu = 1,2. \tag{3.68}$$

where f^ν and g^ν are power series starting from quadratic terms, then the Lemma 3.9 gives us an explicit formula for $a_{2,2}$. In this section we will show how to transform the equation (3.1) into the form (3.68).

Moser [25] presented a method of how this should be done. We should choose a transformation Φ by changing the coordinates in such a way, that the linear part of the equations (3.1) in the new coordinates becomes generated by a diagonal matrix. What is more, the transformation Φ should be canonical i.e.

$$\Phi J \Phi^T = J \tag{3.69}$$

where

$$J = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix} \quad \text{and} \quad Id = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3.70}$$

On top of that Φ should satisfy the following reality condition

$$J_z\Phi = \Phi J_w, \tag{3.71}$$

where

$$\begin{aligned} J_z(x,y,p_x,p_y) &= (\bar{x},\bar{y},\bar{p}_x,\bar{p}_y) \\ J_w(x_1,x_2,y_1,y_2) &= (\bar{x}_1, i\bar{y}_2, \bar{y}_1, i\bar{x}_2) \end{aligned} \tag{3.72}$$

Remark 3.10

In the original proof [25] Moser required that Φ should both be canonical (3.69) and satisfy a reality condition

$$J_z \Phi = \Phi \tilde{J}_w, \tag{3.73}$$

with

$$\begin{aligned} J_z(x, y, p_x, p_y) &= (\bar{x}, \bar{y}, \bar{p}_x, \bar{p}_y) \\ \tilde{J}_w(x_1, x_2, y_1, y_2) &= (\bar{x}_1, \bar{y}_2, \bar{y}_1, \bar{x}_2). \end{aligned} \tag{3.74}$$

Such two conditions cannot both be satisfied at the same time (the correct reality condition is the condition (3.71) which has been corrected in [19]). Let us observe this on the example of the harmonic oscillator from Example 3.6.

Example 3.11 (Example 3.6 continued)

Let us again consider the harmonic oscillator

$$\begin{aligned} x' &= y \\ y' &= -x. \end{aligned} \tag{3.75}$$

We will show that it is impossible to transform the matrix of linear terms of (3.75) through a linear transformation Φ into the Jordan form

$$\Phi^{-1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \Phi = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \tag{3.76}$$

so that Φ is both canonical (3.69) and the reality condition (3.73) is satisfied at the same time.

Since Φ transforms (3.75) into the Jordan form its columns are composed of eigenvectors. From Example 3.6 we already know that $(-i, 1)$ and $(1, -i)$ are eigenvectors with corresponding eigenvalues i and $-i$. This means that Φ is of the form

$$\Phi = \begin{pmatrix} -ia & b \\ a & -ib \end{pmatrix}, \tag{3.77}$$

with some $a, b \in \mathbb{C}$. The canonical condition (3.69) gives us

$$\Phi J \Phi^T = \begin{pmatrix} 0 & -2ba \\ 2ba & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.78}$$

and therefore

$$2ba = -1. \tag{3.79}$$

On the other hand the reality condition (3.73) gives

$$\begin{aligned} J_z \Phi \begin{pmatrix} x \\ y \end{pmatrix} &= \Phi \tilde{J}_w \begin{pmatrix} x \\ y \end{pmatrix} \\ \begin{pmatrix} i\bar{a}\bar{x} + \bar{b}\bar{y} \\ \bar{a}\bar{x} + i\bar{b}\bar{y} \end{pmatrix} &= \begin{pmatrix} b\bar{x} - ia\bar{y} \\ -ib\bar{x} + a\bar{y} \end{pmatrix} \end{aligned} \tag{3.80}$$

which means that

$$b = i\bar{a}. \tag{3.81}$$

The conditions (3.79) and (3.81) are clearly conflicting.

Once a transformation Φ which satisfies both of the conditions (3.69) and (3.71) is found we are ready to compute the power series f^ν and g^ν. Let V be the vector field generated by the Hamiltonian H. If we expand the Hamiltonian H into a power series, then we can compute the expansion of the vector field V from the equation

$$V = J\nabla H. \tag{3.82}$$

Let x denote the variable (x, y, p_x, p_y) and z denote (x_1, x_2, y_1, y_2) (See Section 3.3.1). Having the the expansion of V we can compute the power series f^ν and g^ν from

$$z' = (\Phi^{-1}x)' = \Phi^{-1}(V(x)) = \Phi^{-1}(V(\Phi(z))). \tag{3.83}$$

Let us keep in mind that the computation of the power series f^ν and g^ν is needed in order to check whether $a_{2,2} \neq 0$. From Lemma 3.9 we know that $a_{2,2}$ can be computed from

$$\begin{aligned} a_{2,2} = \frac{1}{\alpha_2}(&-f^2_{1,1,0,0}f^1_{0,1,0,1} - f^2_{0,1,1,0}g^1_{0,1,0,1} + f^2_{0,0,1,1}g^1_{0,2,0,0} - f^2_{0,2,0,0}f^2_{0,1,0,1} \\ &+ 2g^2_{0,2,0,0}f^2_{0,0,0,2} + f^2_{1,0,0,1}f^1_{0,2,0,0} - g^2_{0,1,0,1}f^2_{0,1,0,1}) + f^2_{0,2,0,1}, \end{aligned} \tag{3.84}$$

and therefore it is sufficient to compute the terms of f^ν and g^ν of order smaller than or equal to three. This means that we need to expand H up to the terms of order four, compute the expansion of V from (3.82) and compute the terms of f^ν and g^ν from (3.83).

3.4 Twist in the Hill's problem and in the PRC3BP for sufficiently small μ.

As an example of application of the above described procedure, in this section we will show that the time 2π shift along the trajectory is a twist map around the equilibrium points of the Hill's problem. We will use this result later on in the section to show that for sufficiently small μ we also have a twist in the neighborhood of the libration point L_2 of the three body problem. This will follow from the fact that for sufficiently small μ the PRC3BP can be seen as a analytical perturbation of the Hill's problem.

3.4.1 Twist in the Hill's problem

Let us recall from the Section 2.4 that the Hill's problem is generated by the Hamiltonian

$$H = \frac{(p_x + y)^2 + (p_y - x)^2}{2} - \Omega^H(x, y), \tag{3.85}$$

where $\Omega^H(x, y) = \frac{1}{2}\left(3x^2 + 2(x^2 + y^2)^{-1/2}\right)$. The differential equation generated by the Hamiltonian (3.85)

$$x' = J\nabla H(x), \tag{3.86}$$

has two equilibrium points of the form $L_i^H = (-1)^i(3^{-1/3}, 0, 0, 3^{-1/3})$, $i = 1, 2$. We will apply the procedure and compute $a_{2,2}$ for the equilibrium point L_2^H. The linear terms of (3.86) in L_2^H are given by the matrix

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 8 & 0 & 0 & 1 \\ 0 & -4 & -1 & 0 \end{pmatrix}. \tag{3.87}$$

The eigenvalues at the point are $\alpha_1 = \sqrt{1+2\sqrt{7}}$ and $\alpha_2 = \sqrt{1-2\sqrt{7}}$, first is real and the second is pure imaginary. We will choose the function Φ composed of the eigenvectors of the eigenvalues $\pm\alpha_1$ and $\pm\alpha_2$

$$\Phi = \begin{pmatrix} \lambda_1 & \beta & \lambda_2 & i\beta \\ -9\lambda_1 \frac{1}{\alpha_1(\sqrt{7}+4)} & 9\beta\frac{1}{\alpha_2(\sqrt{7}-4)} & 9\lambda_2\frac{1}{\alpha_1(\sqrt{7}+4)} & -9i\beta\frac{1}{\alpha_2(\sqrt{7}-4)} \\ 9\lambda_1\frac{\sqrt{7}+3}{\alpha_1(\sqrt{7}+4)} & 9\beta\frac{\sqrt{7}-3}{\alpha_2(\sqrt{7}-4)} & -9\lambda_2\frac{\sqrt{7}+3}{\alpha_1(\sqrt{7}+4)} & -9i\beta\frac{\sqrt{7}-3}{\alpha_2(\sqrt{7}-4)} \\ -\lambda_1\frac{2}{3+\sqrt{7}} & \beta\frac{2}{\sqrt{7}-3} & -\lambda_2\frac{2}{3+\sqrt{7}} & i\beta\frac{2}{\sqrt{7}-3} \end{pmatrix}, \tag{3.88}$$

with $\lambda_1, \lambda_2, \beta \in \mathbb{R}$. The transformation Φ satisfies the reality condition (3.71)

$$J_z\Phi = \Phi J_w. \tag{3.89}$$

For our transformation we will choose the coefficients $\lambda_1, \lambda_2, \beta$ as

$$\lambda_1 = -\lambda_2 = \sqrt{\frac{1}{252}\alpha_1\left(\sqrt{7}+4\right)\sqrt{7}} \tag{3.90}$$
$$\beta = \sqrt{\frac{1}{252}\left|\alpha_2\right|\left(4-\sqrt{7}\right)\sqrt{7}}.$$

Such a choice of coefficients guarantees that Φ is canonical i.e. the condition (3.69)

$$\Phi J \Phi^T = J \tag{3.91}$$

is satisfied. The fact that the choice of coefficients (3.90) guarantees the conditions (3.89) and (3.91) can be shown through direct computation.

Since Φ is composed of the eigenvectors of the matrix A we also have

$$\Phi^{-1}A\Phi = \mathrm{diag}(\alpha_1, \alpha_2, -\alpha_1, -\alpha_2). \tag{3.92}$$

Computing the power series f^ν and g^ν from (3.83) and the term $a_{2,2}$ using the formula (3.66) we obtain

$$a_{2,2}^{Hill} = \frac{\frac{2187}{16}(1-2\sqrt{7})3^{2/3}(5767\sqrt{7}-15274)}{(1+2\sqrt{7})^2(\sqrt{7}-3)^2(4\sqrt{7}-7)(\sqrt{7}-14)^2} \approx 8.483, \tag{3.93}$$

which by Lemma 3.8 proves the following Lemma.

Lemma 3.12

There exists a number $R_{\text{Hill}} > 0$ such that the time 2π shift along the trajectory P^{Hill} on the set of Lapunov orbits $B_{R_{\text{Hill}}} = \{(r_H, \theta) | 0 \leq r_H < R_{\text{Hill}}\}$ around the equilibrium point L_2^H

$$P^{\text{Hill}} : B_{R_{\text{Hill}}} \to B_{R_{\text{Hill}}} \tag{3.94}$$

is a twist map; i.e.

$$P^{\text{Hill}}(r_H, \theta) = (r_H, \theta + f(r_H)), \tag{3.95}$$

$$\frac{\partial f}{\partial r_H}(r_H) > 0 \quad \text{for } 0 < r_H < R_{\text{Hill}}.$$

Remark 3.13

From Remark 2.8 we know that in the Hill's coordinates $x_H, y_H, p_{xH,} p_{yH}$ (defined by (2.27)), close to the point $L_2^H = \left(3^{-1/3}, 0, 0, 3^{-1/3}\right)$ the vector field $f_{3\text{Body}}^{\mu}$ of the PRC3BP is analytic with respect to $x_H, y_H, p_{xH,} p_{yH}$ and $\mu^{1/3}$ and that for $\mu = 0$ we have

$$f_{3\text{Body}}^{0} = f_{\text{Hill}}. \tag{3.96}$$

This means that from the Remark 3.4 we know that the radius of convergence R_{Hill} will also be valid for small parameters $\mu > 0$.

3.4.2 Twist in the PRC3BP for small μ

Let us now apply the above result to show that we also have a twist around L_2 in the PRC3BP. The result follows from the fact that for sufficiently small μ the Hill's problem is an approximation of the PRC3BP.

Let us start with a lemma concerning the relation between the twist coefficient $a_{2,2}^{\mu}$ of the PRC3BP with a small mass μ with the twist coefficient $a_{2,2}^{Hill}$ of the Hill's problem.

Lemma 3.14

Let

$$a_2^{\mu}(0, r^2) = \alpha_2^{\mu} + a_{2,2}^{\mu} r^2 + O(r^4) \tag{3.97}$$

be the expansion from Theorem 3.2 at the libration point L_2^{μ} of the PRC3BP, then

$$\lim_{\mu \to 0} \mu^{2/3} a_{2,2}^{\mu} = a_{2,2}^{\text{Hill}}. \tag{3.98}$$

Proof

By Remark 2.8 we know that in the Hill's coordinates $x_H, y_H, p_{xH,} p_{yH}$ the vector field $f_{3\text{Body}}^{\mu}$ of the PRC3BP is is analytic with respect to $x_H, y_H, p_{xH,} p_{yH}$ and

$\mu^{1/3}$. What is more we know that the libration point L_2^μ of the PRC3BP tends to L_2^H as μ tends to zero

$$\lim_{\mu\to 0} L_2^\mu = L_2^H. \tag{3.99}$$

This means that the derivative Df^μ_{3Body} of the vector field of PRC3BP at L_2^μ can be approximated by the derivative Df_{Hill} at the point L_2^H

$$\lim_{\mu\to 0} \left\| Df^\mu_{\mathrm{3Body}}(L_2^\mu) - Df_{\mathrm{Hill}}(L_2^H) \right\| = 0. \tag{3.100}$$

Since the operator Φ from our construction brings the derivative of the vector field to the Jordan form, the operator $\Phi^\mu_{\mathrm{3Body}}$ for the PRC3BP can be chosen close (depending analytically on $\mu^{1/3}$) to the operator Φ_{Hill} of the Hill problem

$$\lim_{\mu\to 0} \left\| \Phi^\mu_{\mathrm{3Body}} - \Phi_{\mathrm{Hill}} \right\| = 0. \tag{3.101}$$

From Remark 2.8, again from the smoothness of coefficients of Taylor expansion with respect to $\mu^{1/3}$ of the vector field f^μ_{3Body} in the neighborhood of L_2^H, we know that the terms up to the third order of the expansion of the vector field f^μ_{3Body} around the equilibrium point L_2^μ are continuously dependent on μ. Since the term $a_{2,2}$ from our construction depends only on the operator Φ and the terms of the order three or less of the expansion of the vector field, we know that in the Hill's coordinates x_H, y_H, p_{xH}, p_{yH} the coefficient $a^{\mu,H}_{2,2}$ constructed for he PRC3BP will tend to the coefficient $a^{\mathrm{Hill}}_{2,2}$ of the Hill's problem

$$\lim_{\mu\to 0} a^{\mu,H}_{2,2} = a^{\mathrm{Hill}}_{2,2}. \tag{3.102}$$

Going back through the scaling (2.27) from the Hill's coordinates x_H, y_H, p_{xH}, p_{yH} to our original coordinates x, y, p_x, p_y we will have $r = \mu^{1/3} r_H$ and

$$a_2^{\mu,H}(r_H) = a_2^\mu(r). \tag{3.103}$$

Let us note that in the Hill's coordinates we have

$$a_2^{\mu,H}(0, ir_H^2) = \alpha_2^\mu + a^{\mu,H}_{2,2} i r_H^2 + O(r_H^4). \tag{3.104}$$

On the other hand using our original coordinates we can write

$$a_2^\mu(0, ir^2) = \alpha_2^\mu + a^\mu_{2,2} i r^2 + O(r^4). \tag{3.105}$$

We can compute

$$a^\mu_{2,2} r^2 = a^\mu_{2,2}(\mu^{1/3} r_H)^2 = \mu^{2/3} a^\mu_{2,2}(r_H)^2 \tag{3.106}$$

and therefore if we rewrite (3.105) using the Hill's coordinates r_H and compare with the series (3.104) then we shall obtain the following equality

$$\mu^{2/3} a^\mu_{2,2} = a^{\mu,H}_{2,2}. \tag{3.107}$$

The above together with (3.102) gives us our result. □

From the above Lemma and the fact that by Remark 3.4 we know that the radius of convergence of the series form Lapunov-Moser Theorem 3.2 is uniform for small μ, we can conclude that for sufficiently small μ we will have the twist property at L_2^μ in the PRC3BP.

Theorem 3.15

For any $R < R_{\text{Hill}}$ (where R_{Hill} is given by Lemma 3.12) there exists a $\mu(R) > 0$, such that for all $0 \leq \mu < \mu(R)$ the time 2π shift along the trajectory P_μ of the PRC3BP, in the Hill's coordinates, on the set of Lapunov orbits $B_R = \{(r_H, \theta)|0 \leq r_H < B_R\}$ around L_2^μ

$$P_\mu : B_R \to B_R, \tag{3.108}$$

is a twist map; i.e.

$$P_\mu(r_H, \theta) = (r_H, \theta + f_\mu(r_H)) \tag{3.109}$$

$$\frac{df_\mu}{dr_H}(r_H) \neq 0 \quad \text{for } 0 < r_H \leq R.$$

3.5 Numerical verification of the twist condition in the PRC3BP for some of the values μ_k.

In this section we will illustrate numerically the results obtained in Section 3.4. By applying the procedure described in Sections 3.3.2 and 3.3.3 numerically for the libration point $L_2^{\mu_k}$, where the value μ_k is the mass for which there exists the homoclinic orbit $q^0_{\mu_k}$ form Theorem 2.1, we obtain the following results

k	μ_k	$a_{2,2}$	$\mu_k^{2/3} a_{2,2}$
2	0.4253863522E-2	380	9.977
3	0.6752539971E-3	1199	9.230
4	0.2192936884E-3	2469	8.982
10	0.92907436E-5	19571	8.649
11	0.68212830E-5	24001	8.633
12	0.51549632E-5	28883	8.619
21	0.8807195E-6	93141	8.558
22	0.7619792E-6	102538	8.554
23	0.6636634E-6	112388	8.551
50	0.582146E-7	566758	8.513
60	0.336890E-7	815650	8.508
70	0.212152E-7	1109723	8.504
200	0.9096E-9	9044098	8.490

In the above table the values μ_k for $2 \leq k \leq 12$ are taken form the numerical results in [21]. For larger values of k we have used the asymptotic formula for μ_k from Theorem 2.1

$$\mu_k = \frac{1}{N(\infty)^3 k^3}(1 + o(1)), \tag{3.110}$$

with the number $N(\infty) = 5.1604325$ obtained by [21]. From the above table we can see that in the PRC3BP, according to our expectations $|a^{\mu}_{2,2}|$ tends to infinity and $\mu^{2/3} a^{\mu}_{2,2}$ tends to $a^{Hill}_{2,2} \approx 8.483$ as μ tends to zero. The fact that we have used the decreasing series μ_k for our example instead of any other decreasing series does not play an important role. The above property can be observed for an arbitrary decreasing series of μ.

3.6 The relation between the radius of the periodic orbit and its energy

Having described the transition functions Φ and Ψ (see Section 3.3.1) between the original coordinates x, y, p_x, p_y of our Hamiltonian system (3.1) and the coordinates $\xi_1, \xi_2, \eta_1, \eta_2$ of the Lapunov-Moser Theorem 3.2, we can now turn to the relation between the radius r of the orbit $l_r(t)$ with its energy. We know that the Hamiltonian H (2.1) of the PRC3BP is constant on all solutions and therefore it is also constant on $l_r(t)$. We can therefore define a function $h(r)$ as the energy level of these solutions

$$h(r) = -2H\left(\Phi(\Psi(l_r)) + L_2\right). \tag{3.111}$$

Lemma 3.16

For sufficiently small r the function

$$h(r) = -2H(\Phi\left(\Psi(l_r)\right) + L_2), \tag{3.112}$$

which gives the energy level of the Lapunov orbit l_r is equal to

$$h(r) = C_2 + h_2 r^2 + o(r^2), \tag{3.113}$$

where $h_2 = -D^2 H(L_2)\left(\Phi(0, 1, 0, i)\right)$ and C_2 is the energy of the Libration point L_2. What is more the distance between the periodic orbit

$$l(C) := \Phi\left(\Psi(l_r)\right) + L_2 \tag{3.114}$$

associated with the energy $C = h(r) < C_2$ and the libration point L_2 is

$$\text{dist}(l(C), L_2) = O(\Delta C), \tag{3.115}$$

where $\Delta C = \sqrt{C_2 - C}$.

Proof

In the ξ, η coordinates the formula for the periodic orbit l_r is given by the Lemma 3.7

$$l_r(t) = (0, re^{ta_2(0,ir^2)+i\phi}, 0, ire^{-ta_2(0,ir^2)-i\phi}). \tag{3.116}$$

We will see what the energy level of this orbit is in the x, y, p_x, p_y coordinates. The orbits with the same r and a different ϕ differ only by a time shift and are essentially the same and so their energies are identical. For simplicity of notations we will therefore assume that $\phi = 0$. The energy (3.111) of an orbit is constant in time and it is therefore sufficient to compute the energy for the time $t = 0$ i.e.

$$h(r) = -2H(\Phi(\Psi(l_r(0))) + L_2).$$

Let us first note that the construction of $\Psi = (\phi_2, \phi_2, \psi_1, \psi_2)$ in Section 3.3.2 produced power series of the form (3.52)

$$\begin{aligned} \phi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) &= \textstyle\sum_{k=1}^{2} \delta_{\nu k}\xi_k + h.o.t. \\ \psi_\nu(\xi_1, \xi_2, \eta_1, \eta_2) &= \textstyle\sum_{k=1}^{2} \delta_{\nu k}\eta_k + h.o.t. \end{aligned} \tag{3.117}$$

hence

$$\Psi(l_r(0)) = (\phi_2, \phi_2, \psi_1, \psi_2)(l_r(0)) = (0, r, 0, ir) + O(r^2). \tag{3.118}$$

The transformation Φ (see Section 3.3.3) is linear and therefore

$$\Phi(\Psi(l_r(0))) = r\Phi(0, 1, 0, i) + O(r^2). \tag{3.119}$$

We can now compute $h(r)$ as

$$\begin{aligned} h(r) &= -2H(\Phi(\Psi(l_r(0))) + L_2) \\ &= -2H(L_2) - 2DH(L_2)(\Phi(\Psi(l_r(0)))) \\ &\quad - D^2H(L_2)(\Phi(\Psi(l_r(0)))) + o(\Phi(\Psi(l_r(0)))) \end{aligned} \tag{3.120}$$

We know that L_2 is an equilibrium point of the vector field $J\nabla H$ and therefore

$$DH(L_2)(\Phi(\Psi(l_r(0)))) = \nabla H(L_2) \cdot \Phi(\Psi(l_r(0))) = 0. \tag{3.121}$$

Thus

$$\begin{aligned} h(r) &= -2H(L_2) - 2D^2H(L_2)(\Phi(\Psi(l_r(0)))) + o(\Phi(\Psi(l_r(0)))^2) \\ &= C_2 - r^2D^2H(L_2)(\Phi(0, 1, 0, i)) + o(r^2). \end{aligned} \tag{3.122}$$

The claim that $h(r)$ is one to one comes directly from the above equation. Finally from (3.116), (3.117) and the fact that Φ is linear we can compute

$$\begin{aligned} \mathrm{dist}(l(C), L_2) &= \mathrm{dist}(\Phi(\Psi(l_r)), 0) \\ &= O(r) \\ &= O(\Delta C). \end{aligned} \tag{3.123}$$

□

The formula (3.113) means in particular that for sufficiently small $r > 0$ the function $h(r)$ is one to one. For a given energy c we can denote by $l(c)$ the Lapunov orbit in the x, y, p_x, p_y coordinates

$$l(c)(t) = \Phi(\Psi(l_{h^{-1}(c)}(t))) + L_2. \tag{3.124}$$

A question remains whether $h(r)$ is greater or smaller than C_2. This can be observed using the following argument. For all energy levels $C > C_2$ the libration point L_2 is contained in the interior of the forbidden region and therefore there cannot exist a family of periodic orbits emanating from it (see Figure 2.1). This means that $h(r)$ must be smaller than C_2 which means that family of periodic orbits $l(c)$ is parameterized by $c \leq C_2$. Let C denote the smallest energy for which the series from the Theorem 3.2 are convergent i.e.

$$C = h(R), \tag{3.125}$$

where R is the radius of the set B_R (3.40). We will define a notation B_C for the set of all Lapunov orbits with energies between C and C_2 i.e.

$$B_C = \{l(c)|C \leq c \leq C_2\}. \tag{3.126}$$

We can rewrite the twist property on the set B_C by the following

Lemma 3.17

For sufficiently small μ there exists a $C(\mu) < C_2^\mu$ sufficiently close to C_2^μ such that the time 2π shift along the trajectory P^μ of the PRC3BP on the set of Lapunov orbits $B_{C(\mu)} = \{l(c)|C(\mu) \leq c \leq C_2^\mu\}$ is an analytic twist map i.e.

$$P(c, \phi) = (c, \phi + f(c)) \tag{3.127}$$

and

$$\frac{df}{dc} \neq 0 \quad \text{for all } c \in [C(\mu), C_2^\mu]. \tag{3.128}$$

Proof

The above is a direct consequence of the Lemma 3.8, Theorem 3.15 and Lemma 3.16.

Let us just point out that in the Theorem 3.15 we did not have the twist property for $r = 0$, but in the (c, ϕ) coordinates we have the twist also for $c = C_2^\mu = h(0)$. To explain this let us note that from the proof of Lemma 3.8 the twist follows from the equation (3.47)

$$-i2\pi \frac{d}{dr}\left(a_2(0, ir^2)\right) \neq 0, \tag{3.129}$$

where

$$a_2(0, ir^2) = \alpha_2 + a_{2,2}ir^2 + O(r^4). \tag{3.130}$$

Clearly (3.129) does not hold for $r = 0$. In the (c, ϕ) coordinates though, if we let $c = h(r)$ and using (3.113) compute

$$\frac{da_2(0, i(h^{-1}(c))^2)}{dc}(C_2^\mu) = \left(\frac{1}{h'(r)}\frac{da_2(0, ir^2))}{dr}\right)(0) = \frac{a_{2,2}}{h_2} \neq 0, \tag{3.131}$$

then we obtain the twist property at C_2^μ. □

4 The equations for the PRC3BP and PRE3BP in rotating coordinates.

In the planar restricted elliptic three body problem (PRE3BP) we investigate the motion of two large masses μ and $(1-\mu)$ and a third small massless particle whose motion is restricted to a two dimensional plane. We assume that the third mass does not influence the motion of the first two masses. This means that the two large masses rotate around the origin on elliptic Kepler orbits which come from the solution of the two body problem (hence the name elliptical). The equations for the PRE3BP describe the movement of the massless particle under the gravitational pull of the two large masses.

In this chapter we will show that in the rotating frame this motion can be viewed as a perturbation of the motion of the particle in the PRC3BP i.e. the Hamiltonian of the elliptic problem can be written as

$$H^e(x,y,p_x,p_y,t) = H(x,y,p_x,p_y) + eG(x,y,p_x,p_y,t) + O(e^2) \tag{4.1}$$

where H is the Hamiltonian of the circular problem (2.1) and e is the eccentricity of the elliptic Kepler orbit of the two large masses. This is the main result of the Chapter and is formulated in Lemma 4.3. The form (4.1) will be used throughout our future discussion. In particular it will be used when proving the persistence results of the Lapunov orbits in Chapter 6. It will also be used when deriving and applying the Melnikov method in Chapters 7 and 8.

Let us note that the above is not a new approach. An almost identical form to (4.1), has been used by Xia [35] for his Arnold diffusion result.

In the PRC3BP (2.1) the equations of motion are time independent. This is due to the fact that since both of the masses μ and $(1-\mu)$ rotate around the origin on circular orbits we can introduce a rotating coordinate system in which the bodies are motionless [1]. In the elliptical case this will not be possible. When we introduce the rotating frame, for small eccentricities e the two large masses will be almost motionless but they will oscillate in a 2π periodic fashion. This oscillation will be described by the term $eG + O(e^2)$ in equation (4.1).

In order to obtain the form (4.1) of the Hamiltonian we will first have to show how the rotating coordinates can be introduced. We will start the section with showing how this is done in the circular case and then move on to applying the procedure to the elliptic case.

4.1 The equations for the PRC3BP in rotating coordinates.

The equation of motion of a particle with an infinitely small mass in the circular three body problem is given [1] by the Hamiltonian

$$H(q,p,t) = \frac{p_1^2 + p_2^2}{2} - \frac{1-\mu}{r_1} - \frac{\mu}{r_2}, \tag{4.2}$$

where

$$\begin{aligned} r_1^2 &= (q_1 - \mu\cos t)^2 + (q_2 - \mu\sin t)^2, \\ r_2^2 &= (q_1 + (1-\mu)\cos t)^2 + (q_2 + (1-\mu)\sin t)^2. \end{aligned} \tag{4.3}$$

In this section we will introduce an appropriate change of coordinates F, so that in the new coordinates the equation becomes autonomous.

Before we introduce any coordinate changes, let us first note that in the PRC3BP the large mass $(1-\mu)$ rotates around the origin on a circle with a radius μ in the anticlockwise direction. This motion is given by the equation

$$q_{1-\mu}(t) = \mu\,(\cos t, \sin t)\,. \tag{4.4}$$

Similarly the motion of the small mass μ is given by

$$q_{\mu}(t) = -(1-\mu)\,(\cos t, \sin t)\,. \tag{4.5}$$

We can see that the terms r_1 and r_2 in (4.2) can be written as

$$\begin{aligned} r_1^2 &= \left\| q - q_{1-\mu}(t) \right\|^2, \\ r_2^2 &= \left\| q - q_{\mu}(t) \right\|^2. \end{aligned} \tag{4.6}$$

The idea behind the change of coordinates is to move our coordinate system along with the rotation of the masses $(1-\mu)$ and μ. If we do this then the masses $(1-\mu)$ and μ instead of rotating in the anticlockwise direction around the origin, will stand still in the points $(\mu, 0)$ and $(-(1-\mu), 0)$ respectively. Such a change of coordinates should allow us to make the terms r_1 and r_2 independent form t.

Let us now formally define our coordinate change F. Let

$$F : R \times R^2 \to R \times R^2 : (t, q_1, q_2, p_1, p_2) \to (t, x, y, p_x, p_y) \tag{4.7}$$

where

$$\begin{aligned} x &= q_1 \cos t + q_2 \sin t, \quad p_x = p_1 \cos t + p_2 \sin t, \\ y &= -q_1 \sin t + q_2 \cos t, \quad p_y = -p_1 \sin t + p_2 \cos t. \end{aligned} \tag{4.8}$$

A straightforward computation gives us the inverse of this transformation

$$\begin{aligned} q_1 &= x \cos t - y \sin t, \quad p_1 = p_x \cos t - p_y \sin t, \\ q_2 &= x \sin t + y \cos t, \quad p_2 = p_x \sin t + p_y \cos t. \end{aligned} \tag{4.9}$$

Before we rewrite our Hamiltonian H in the new coordinates let us note that the transformation F^{-1} is canonical.

Lemma 4.1

F^{-1} is a canonical transformation. That is, F^{-1} satisfies the following three conditions

(C1) F^{-1} is a diffeomorphism
(C2) F^{-1} preserves time
(C3) There exists a function $K_{F^{-1}}$ such that $(F^{-1})^* \omega_2 = \omega_{K_{F^{-1}}}$, where

$$\omega_{K_{F^{-1}}} = \omega_1 + dK_{F^{-1}} \wedge dt, \tag{4.10}$$

$\omega_2 = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ and $\omega_1 = dx \wedge dp_x + dy \wedge dp_y$.

Proof

The first two conditions are evident. The third condition is obtained by direct computation

$$\left(F^{-1}\right)^* \omega_2 = \left(F^{-1}\right)^* (dq_1 \wedge dp_1 + dq_2 \wedge dp_2) = \omega_1 + d(yp_x - xp_y) \wedge dt, \tag{4.11}$$

which gives us

$$K_{F^{-1}} = yp_x - xp_y. \tag{4.12}$$

□

Since F^{-1} is canonical, then by Jacobi Theorem 1.4 the vector field X_H generated by H can be computed as

$$X_H = (F^{-1})^* X_{H_{rot}} \tag{4.13}$$

where H_{rot} is the Hamiltonian of the problem in the rotating coordinates given by

$$H_{rot}(t, x, y, p_x, p_y) = H \circ F^{-1}(t, x, y, p_x, p_y) + yp_x - xp_y, \tag{4.14}$$

and $X_{H_{rot}}$ is the vector field is generated by the Hamiltonian H_{rot}. In the case of our Hamiltonian (4.2) of the PRC3BP this gives us

$$\begin{aligned} H_{rot} = &\frac{(p_x \cos t - p_y \sin t)^2 + (p_x \sin t + p_y \cos t)^2}{2} \\ &- \frac{1-\mu}{r_1} - \frac{\mu}{r_2} + yp_x - xp_y, \end{aligned} \tag{4.15}$$

where r_1 and r_2 in the x, y, p_x, p_y coordinates are given by

$$
\begin{aligned}
r_1^2 &= ((x\cos t - y\sin t) - \mu\cos t)^2 \\
&\quad + ((x\sin t + y\cos t) - \mu\sin t)^2, \\
r_2^2 &= ((x\cos t - y\sin t) + (1-\mu)\cos t)^2 \\
&\quad + ((x\sin t + y\cos t) + (1-\mu)\sin t)^2.
\end{aligned} \tag{4.16}
$$

After simplifying the above equations we obtain our time independent Hamiltonian (2.1)

$$H_{rot} = \frac{(p_y - x)^2 + (p_x + y)^2}{2} - \Omega(x,y), \tag{4.17}$$

where

$$\Omega(x,y) = \frac{x^2+y^2}{2} + \frac{1-\mu}{\sqrt{(x-\mu)^2 + y^2}} + \frac{\mu}{\sqrt{(x+1-\mu)^2+y^2}}. \tag{4.18}$$

4.2 The equations for the PRE3BP in rotating coordinates.

In this section we will apply the same change of coordinates as above to the equations of motion in the elliptic problem (PRE3BP). The new coordinates will rotate with constant velocity in such a way so that after the time $T = 2\pi$ the two bodies μ and $1-\mu$ will end up in the same positions from which they have started at at the time $t = 0$.

In the PRE3BP we do not assume that the two larger bodies have circular orbits. We allow them to have elliptic orbits with an eccentricity $0 \le e < 1$. We will start this section with deriving the equations of motion of the two larger bodies. Next we will use the equations in our Hamiltonian which describes the movement of the third massless body and apply our change of coordinates to obtain the Hamiltonian of the PRE3BP in rotating coordinates.

4.2.1 The movement of the two larger masses in the PRE3BP

We have assumed that the two larger bodies have masses μ and $1-\mu$. Since the third massless body does not influence the movements of the larger bodies, their motion is given by the following equations of the two body problem [22, Section I.C.1]

$$
\begin{aligned}
\ddot{q}_\mu &= \frac{(1-\mu)(q_{1-\mu} - q_\mu)}{\left\|q_\mu - q_{1-\mu}\right\|^3} \\
\ddot{q}_{1-\mu} &= \frac{\mu(q_\mu - q_{1-\mu})}{\left\|q_\mu - q_{1-\mu}\right\|^3}.
\end{aligned} \tag{4.19}
$$

By the fact that the momentum of the system has to be constant we know that the center of mass

$$q_0 = \frac{\mu q_\mu + (1-\mu)q_{1-\mu}}{\mu(1-\mu)} \tag{4.20}$$

moves with a constant speed on a straight line [2, Section 2.10.5]. Since in the PRC3BP we have assumed that the center of mass is at the origin here we will also make the same assumption which means that we set $q_0 = 0$.

If we define a vector q as the difference between q_μ and $q_{1-\mu}$

$$q = q_\mu - q_{1-\mu}, \tag{4.21}$$

then we can see that using (4.19) we can write the following equation for q

$$\begin{aligned}
\ddot{q} &= \ddot{q}_\mu - \ddot{q}_{1-\mu} \\
&= \frac{(1-\mu)(q_{1-\mu} - q_\mu)}{\|q_\mu - q_{1-\mu}\|^3} - \frac{\mu(q_\mu - q_{1-\mu})}{\|q_\mu - q_{1-\mu}\|^3} \\
&= -\frac{(1-\mu)q}{\|q\|^3} - \frac{\mu q}{\|q\|^3} \\
&= -\frac{q}{\|q\|^3}.
\end{aligned} \tag{4.22}$$

The equation

$$\ddot{q} = -\frac{q}{\|q\|^3}. \tag{4.23}$$

is an equation of the Kepler problem and it's solution in polar coordinates $q = (r, \psi)$ is given by [22, Section IV.C.7]

$$r(t) = \frac{c^2}{1 + e\cos\psi(t)}, \tag{4.24}$$

where

$$\frac{d\psi}{dt} = \frac{c}{r(t)^2}, \tag{4.25}$$

and c is a constant (c is the angular momentum of the Kepler problem). It turns out that using the above solution we can find the equations for q_μ and $q_{1-\mu}$. In order to do this let us use the polar coordinates $q_\mu = (r_\mu, \psi_\mu)$, $q_{1-\mu} = (r_{1-\mu}, \psi_{1-\mu})$. We have assumed that the center of mass is at the origin which means that

$$\begin{aligned}
\psi_\mu &= \psi \\
\psi_{1-\mu} &= \psi \\
\mu r_\mu + (1-\mu) r_{1-\mu} &= 0.
\end{aligned} \tag{4.26}$$

Using this fact we can find the equations for r_μ and $r_{1-\mu}$

$$\begin{aligned}
r_{1-\mu} &= \mu(r_\mu - r_{1-\mu}) = \mu r \\
r_\mu &= -\frac{1}{\mu}(1-\mu) r_{1-\mu} = -(1-\mu) r.
\end{aligned} \tag{4.27}$$

We would like to set up our elliptic orbits in such a way that their period is equal to 2π. Based on this requirement we must choose an appropriate value of c. In order to do this let us first note that the area of the ellipse given by (4.24) is equal to

$$\begin{aligned}\int_0^{2\pi} \frac{1}{2} r^2 d\psi &= \int_0^{2\pi} \frac{1}{2}\left(\frac{c^2}{1+e\cos\psi}\right)^2 d\psi \\ &= \frac{\pi c^4}{(1-e^2)^{3/2}}. \end{aligned} \tag{4.28}$$

On the other hand if we require that the period of our elliptic orbit is 2π then using (4.25) we can compute this area as

$$\int_0^{2\pi} \frac{1}{2} r(t)^2 \dot{\psi}(t) dt = \int_0^{2\pi} \frac{1}{2} c dt = \pi c \tag{4.29}$$

which compared to (4.28) gives us the value of c

$$c = \sqrt{1-e^2}. \tag{4.30}$$

For such a choice of c the period of the Kepler orbit of q will be equal to 2π which by (4.26) and (4.27) implies that the period of the orbits q_μ and $q_{1-\mu}$ is also 2π. We have therefore shown that the movement of the two larger bodies is given by

$$\begin{aligned} q_\mu(t) &= -(1-\mu)(x_{12}(t), y_{12}(t)) \\ q_{1-\mu}(t) &= \mu(x_{12}(t), y_{12}(t)), \end{aligned} \tag{4.31}$$

where

$$\begin{aligned} x_{12}(t) &= r(t)\cos\psi(t) \\ y_{12}(t) &= r(t)\sin\psi(t), \end{aligned} \tag{4.32}$$

and

$$\begin{aligned} r(t) &= \frac{1-e^2}{1+e\cos\psi(t)} \\ \frac{d}{dt}\psi(t) &= \frac{\sqrt{1-e^2}}{r^2(t)}. \end{aligned} \tag{4.33}$$

From the fact that the Kepler orbit is 2π periodic we have that the angle $\psi(t) - t$ is 2π periodic. This means that in particular

$$\psi(2k\pi) - 2k\pi = 0 \quad \text{for } k \in \mathbb{Z}. \tag{4.34}$$

Using (4.31) we obtain the Hamiltonian of the PRE3BP

$$\begin{aligned} H(q_1, q_2, p_1, p_2) = &\frac{p_1^2 + p_2^2}{2} + \frac{1-\mu}{\sqrt{(q_1 - \mu x_{12})^2 + (q_2 - \mu y_{12})^2}} \\ &+ \frac{\mu}{\sqrt{(q_1 + (1-\mu)x_{12})^2 + (q_2 + (1-\mu)y_{12})^2}}. \end{aligned} \tag{4.35}$$

Lemma 4.2

For small eccentricity e the motion $x_{12}(t)$ and $y_{12}(t)$ can be expressed as

$$x_{12}(t) = (1 - e\cos\psi)\cos\psi + O(e^2) \tag{4.36}$$
$$y_{12}(t) = (1 - e\cos\psi)\sin\psi + O(e^2) \tag{4.37}$$
$$\psi(t) = t + 2e\sin t + O(e^2) \tag{4.38}$$

Proof

Let us define

$$\phi(t) = t + 2e\sin t. \tag{4.39}$$

The first step in proving the Lemma is to show that ψ can be rewritten as

$$\psi(t) = \phi(t) + O(e^2). \tag{4.40}$$

From (4.33) and (4.34) we know that

$$\psi'(t) = \frac{1 + 2e\cos\psi(t) + e^2\cos^2\psi(t)}{(1-e^2)^{3/2}}, \tag{4.41}$$
$$\psi(2k\pi) - 2k\pi = 0 \quad \text{for } k \in \mathbb{Z},$$

what is more form (4.39) it is evident that

$$\phi(2k\pi) - 2k\pi = 0 \quad \text{for } k \in \mathbb{Z}. \tag{4.42}$$

Since both $\phi(t) - t$ and $\psi(t) - t$ are 2π periodic it is sufficient to prove (4.38) for $t \in [0, 2\pi]$. Let us note that since

$$(1-e^2)^{-3/2} = 1 + O(e^2) \tag{4.43}$$

from (4.41) we have

$$\psi'(t) = 1 + 2e\cos\psi(t) + O(e^2). \tag{4.44}$$

We can use the above to compute

$$\begin{aligned}
|\phi(t) - \psi(t)| &= \left|\int_0^t (\phi'(s) - \psi'(s))ds\right| \\
&= \left|\int_0^t \left(1 + 2e\cos s - \left(1 + 2e\cos\psi(t) + O(e^2)\right)\right) ds\right| \\
&\le 2e\int_0^t |\cos s - \cos\psi(s)|\, ds + tO(e^2) \\
&\le 2e\int_0^t |s - \psi(s)|\, ds + tO(e^2) \qquad (4.45) \\
&= 2e\int_0^t |s + 2e\sin s - 2e\sin s - \psi(s)|\, ds + tO(e^2) \\
&\le 2e\int_0^t |\phi(s) - \psi(s)|\, ds + 2e\int_0^t |2e\sin s|\, ds + tO(e^2) \\
&\le \int_0^t |\phi(s) - \psi(s)|\, ds + te^2 M
\end{aligned}$$

for some constant $M > 0$. Using the Gronwall Lemma (Lemma 1.10) with

$$\begin{aligned} c(t) &= te^2 M \\ u(t) &= 1, \end{aligned} \tag{4.46}$$

we obtain an estimate

$$\begin{aligned} |\phi(t) - \psi(t)| &\leq \int_0^t e^2 M \exp(t-s) ds \\ &= e^2 M \left(\exp(t) - 1\right) \\ &\leq e^2 M \left(\exp\left(2\pi\right) - 1\right), \end{aligned} \tag{4.47}$$

which proves (4.38). In order to show (4.36) let us compute

$$\begin{aligned} x_{12}(t) - (1 - e\cos\psi(t))\cos\psi(t) &= r(t)\cos\psi(t) - (1 - e\cos\psi(t))\cos\psi(t) \\ &= \frac{\left(1-e^2\right)\cos\psi(t)}{1+e\cos\psi(t)} - (1 - e\cos\psi(t))\cos\psi(t) \\ &= e^2 \frac{\cos^3\psi(t) - \cos\psi(t)}{1+e\cos\psi(t)} \\ &= O(e^2). \end{aligned} \tag{4.48}$$

The proof of (4.37) is analogous. □

4.2.2 The Hamiltonian of the PRE3BP in rotating coordinates

For the circular problem ($e = 0$) from the previous section we know that after changing the coordinates into a rotating coordinate system, the equations become autonomous. Since we would like to treat the elliptic problem as a perturbation of the circular problem, we are now going to rewrite (4.35) in the rotating coordinate system. This requires the change of coordinates F, where F is given by (4.7). Such a change of coordinates will result in the Hamiltonian (4.35) taking the desired form (4.1). This is given by the following Lemma.

Lemma 4.3

For any $M, \delta > 0$ for all x, y such that $|x|, |y| \leq M$, $|(x,y) - (\mu, 0)| \geq \delta$ and $|(x,y) - (\mu - 1, 0)| \geq \delta$ the Hamiltonian of the PRE3BP in the rotating coordinates takes form

$$H^e = H + eG + O(e^2) \tag{4.49}$$

where H is the Hamiltonian of the circular problem given by (4.17) and

$$G = G(x,y) = \frac{1-\mu}{(r_1)^3} f(x,y,\mu,t) + \frac{\mu}{(r_2)^3} f(x,y,\mu-1,t). \tag{4.50}$$

where

$$\begin{aligned} r_1^2 &= (x-\mu)^2 + y^2 \\ r_2^2 &= (x+1-\mu)^2 + y^2 \\ f(x,y,\alpha,t) &= -y\alpha[3\sin t - \sin^3 t] + x\alpha[\cos t + \cos^3 t] - \alpha^2\cos(t). \end{aligned} \tag{4.51}$$

What is more for the above mentioned values of x and y the vector field generated by the Hamiltonian H^e of the PRE3BP takes form

$$f^e = f + eg + O(e^2) \tag{4.52}$$

where

$$\begin{aligned} f &= J\nabla H \\ g &= J\nabla G. \end{aligned} \tag{4.53}$$

Proof

From the Lemma 4.1 we know that the transformation F is canonical and therefore by the Jacobi Theorem 1.4 the the vector field X_H generated by the Hamiltonian H (4.35), satisfies the equation

$$X_H = (F^{-1})^* X_{H^e}, \tag{4.54}$$

where

$$H^e(t,x,y,p_x,p_y) = H \circ F^{-1}(t,x,y,p_x,p_y) + yp_x - xp_y. \tag{4.55}$$

This gives us

$$\begin{aligned} H^e &= H \circ F + K_F \\ &= \frac{p_1^2+p_2^2}{2} - \frac{1-\mu}{\sqrt{(q_1-\mu x_{12})^2 + (q_2-\mu y_{12})^2}} \\ &\quad - \frac{\mu}{\sqrt{(q_1+(1-\mu)x_{12})^2 + (q_2+(1-\mu)y_{12})^2}} + yp_x - xp_y. \end{aligned} \tag{4.56}$$

Let us first compute the term $(q_1-\mu x_{12})^2 + (q_2-\mu y_{12})^2$. From the formulas (4.36) and (4.37) for x_{12} and y_{12} and from (4.9) we can compute

$$\begin{aligned} (q_1-\mu x_{12})^2 + (q_2-\mu y_{12})^2 = \mu^2 &+ x^2 + y^2 \\ &- 2x\mu[\cos t\cos\psi + \sin t\sin\psi] \\ &+ 2y\mu[\sin t\cos\psi - \cos t\sin\psi] \\ &+ 2e\mu y[\cos\psi\cos t\sin\psi - 2\cos^2\psi\sin t] \\ &+ 2e\mu x[\cos\psi\sin t\sin\psi + 2\cos^2\psi\cos t] \\ &- e2\mu^2\cos\psi + xO(e^2) + yO(e^2) + O(e^2), \end{aligned} \tag{4.57}$$

where all three terms $O(e^2)$ are independent from x and y. We can approximate $\sin\psi$ and $\cos\psi$ by using the formula (4.38) and expanding into the Taylor series around t. This gives us

$$\begin{aligned}\sin\psi &= \sin(t + 2e\sin t + O(e^2)) \\ &= \sin(t) + \cos t\left(2e\sin t + O(e^2)\right) + O(e^2) \\ &= \sin(t) + 2e\sin t\cos t + O(e^2),\end{aligned} \tag{4.58}$$

and similarly

$$\begin{aligned}\cos\psi &= \cos(t + 2e\sin t + O(e^2)) \\ &= \cos t - \sin t\left(2e\sin t + O(e^2)\right) + O(e^2) \\ &= \cos t - 2e\sin^2 t + O(e^2).\end{aligned} \tag{4.59}$$

We can use the above to compute the terms from (4.57)

$$\begin{aligned}\cos t\cos\psi + \sin t\sin\psi &= 1 + O(e^2) \\ \sin t\cos\psi - \cos t\sin\psi &= -2e\sin t + O(e^2) \\ \cos\psi\cos t\sin\psi - 2\cos^2\psi\sin t &= -\cos^2 t\sin t + O(e) \\ \cos\psi\sin t\sin\psi + 2\cos^2\psi\cos t &= \cos t + \cos^3 t + O(e).\end{aligned} \tag{4.60}$$

By substituting the above into the equation (4.57) we obtain

$$\begin{aligned}(q_1 - \mu x_{12})^2 + (q_2 - \mu y_{12})^2 &= (x-\mu)^2 + y^2 \\ &\quad - 2ey\mu[2\sin t + \cos^2 t\sin t] \\ &\quad + 2ex\mu[\cos t + \cos^3 t] \\ &\quad - e2\mu^2\cos(t) \\ &\quad + xO(e^2) + yO(e^2) + O(e^2) \\ &= (x-\mu)^2 + y^2 + ef(x,y,\mu,t) \\ &\quad + xO(e^2) + yO(e^2) + O(e^2)\end{aligned} \tag{4.61}$$

where $f(x,y,\mu,t)$ is given by (4.51) and all the terms $O(e^2)$ are independent from x and y. Analogically we can compute the term $(q_1+(1-\mu)x_{12})^2+(q_2+(1-\mu)y_{12})^2$ from (4.56) as

$$\begin{aligned}(q_1 + (1-\mu)x_{12})^2 + (q_2 + (1-\mu)y_{12})^2 &= (x+1-\mu)^2 + y^2 + ef(x,y,\mu-1,t) \\ &\quad + xO(e^2) + yO(e^2) + O(e^2).\end{aligned} \tag{4.62}$$

We have almost computed (4.56). What is left now, is to change the coordinates in the first term of (4.56), which simply gives us

$$\frac{p_1^2 + p_2^2}{2} = \frac{(p_x\cos t - p_y\sin t)^2 + (p_x\sin t + p_y\cos t)^2}{2} = \frac{p_x^2 + p_y^2}{2}. \tag{4.63}$$

After substituting all of the terms into (4.56) we obtain the Hamiltonian in the rotating coordinate system

$$
\begin{aligned}
H^e &= H \circ F + K_F \qquad (4.64)\\
&= \frac{(p_x+y)^2+(p_y-x)^2}{2} - \frac{x^2+y^2}{2}\\
&\quad - \frac{1-\mu}{\sqrt{(x-\mu)^2+y^2+ef(x,y,\mu,t)+xO(e^2)+yO(e^2)+O(e^2)}}\\
&\quad - \frac{\mu}{\sqrt{(x+1-\mu)^2+y^2+ef(x,y,\mu-1,t)+xO(e^2)+yO(e^2)+O(e^2)}}.
\end{aligned}
$$

where all $O(e^2)$ are independent of x, y, p_x, p_y. By expanding the above into series around r_1^2 and r_2^2, since we have assumed that $|x|, |y| \leq M$, $|(x,y)-(\mu,0)| \geq \delta$ and $|(x,y)-(\mu-1,0)| \geq \delta$, we obtain both (4.49) and (4.52). □

From the above proof we can obtain the following technical Remark. This Remark will become important later on in Chapter 6 where we will require that the problems PRC3BP and PRE3BP are "close" to one another.

Remark 4.4

If we fix a certain $l > 0$ then from the equation (4.64) by computing partial derivatives we can see that for $|x|, |y| \leq M$, $|(x,y)-(\mu,0)| \geq \delta$ and $|(x,y)-(\mu-1,0)| \geq \delta$ we will have

$$\|J\nabla H^e\|_{C^l} \leq M_1(e_0,l) \qquad (4.65)$$

for some bound $M_1(e_0,l) > 0$ and all $e \leq e_0$, where the norm $\|\cdot\|_{C^l}$ is defined as

$$\|f\|_{C^l} = \sup\{|\frac{\partial^{k_1+k_2+k_3+k_4}}{\partial x^{k_1}\partial y^{k_2}\partial p_x^{k_3}\partial p_y^{k_4}} f(x)| : k_1+k_2+k_3+k_4 \leq l\}. \qquad (4.66)$$

What is more, by expanding the partial derivatives $\frac{\partial^{k_1+k_2+k_3+k_4}}{\partial x^{k_1}\partial y^{k_2}\partial p_x^{k_3}\partial p_y^{k_4}}$ of (4.64) around r_1^2 and r_2^2 we will have

$$\left\| \frac{1}{e}\left(J\nabla H^e - J\nabla H^0\right) \right\|_{C^l} \leq M_2(e_0,l) \qquad (4.67)$$

for all $e \leq e_0$ and all x, y such that $|x|, |y| \leq M$, $|(x,y)-(\mu,0)| \geq \delta$ and $|(x,y)-(\mu-1,0)| \geq \delta$, for some bound $M_2(e_0,l) > 0$.

Let us finish this chapter with a remark how the above results fit in with our framework described in Chapter 2.

Remark 4.5

In our future discussions we are going to be concerned with the behavior of the homoclinic orbit $q_C^0(t)$ (See Figure 2.3) under a perturbation $e \neq 0$. The orbit belongs to the bounded set $R_b(\mu, C)$ (See Figure 2.1) and is separated from both masses. This means that in the neighborhood of this orbit we shall have $|x|, |y| \leq M$, $|(x, y) - (\mu, 0)| \geq \delta$ and $|(x, y) - (\mu - 1, 0)| \geq \delta$ which means that we can use our formulas (4.49), (4.52) from Lemma 4.3 and the bounds (4.65) and (4.67) obtained in Remark 4.4.

5
Arnold diffusion and the intuition behind the method.

In this chapter we will give a brief overview of the method which will be used to prove the occurrence of Arnold diffusion in the PRE3BP. Since the overall procedure involves a number of intertangled notations which on their own could be hard to follow, this chapter is devoted to presenting a geometrical intuition behind the method. We will try to illustrate the procedure with pictures on which all the relevant notations would appear. This representation is difficult since in our model we are working in a four, and sometimes even five dimensional space (when counting time), and are restricted to two dimensional drawings. To overcome this problem we will sometimes illustrate the same concept making drawings from different angles and using different coordinates.

Let us note that some of the notations and methods which will appear might not be rigorously defined or explained. The aim of the chapter is to draw an overall picture and not to present a detailed argument. The procedure which is sketched in this chapter will be rigorously described in the Chapters 6 and 7.

We will start with drawing pictures and representing the dynamics of PRC3BP which have already been described in Chapters 2 and 3. Later on in the chapter we will show which of the results survive if we view the PRE3BP as a perturbation of the PRC3BP. Our main problem will be associated with the fact that contrary to the PRC3BP, the elliptic problem does not come from an autonomous Hamiltonian system and therefore we will no longer have the invariant foliation connected with the energy level. We will discuss how this problem can be solved using an Melnikov type argument. We will finish off the chapter with a short description of the Arnold diffusion which will emerge from our discussion.

5.1 The dynamics in the circular problem.

The behavior of the invariant manifolds close to the libration point L_2 has been described in Chapter 3. Let us draw a few pictures of this behavior which will help us in future visualization of the dynamics of PRC3BP. At the libration point L_2 we have the four coordinates $\xi_1, \eta_1 \in \mathbb{R}$, $r \in \mathbb{R}^+$ and $\theta \in \mathbb{T}^1 = (0, 2\pi]$. The ξ_1, η_1 coordinates are the hyperbolic directions of the flow and the r,θ coordinates are the twist coordinates responsible for the existence of the Lapunov orbits. At L_2 we can visualize the flow as the product of two planes, one in the ξ_1, η_1 and the other in r,θ coordinates (see Figure 5.1). For $\xi_1 = \eta_1 = 0$ each circle of the radius r represents a Lapunov orbit l_r with an energy level $c = h(r)$ (when convenient we will denote the orbit by $l(c)$). Each Lapunov orbit is a one dimensional hyperbolic invariant torus and can be graphically represented by a circle in a three dimensional space (see Figure 5.2). These kinds of invariant sets are often referred to in literature as whiskered tori. The unstable manifold $W^u(l(c))$ and the stable manifold $W^s(l(c))$ are both two dimensional. We can draw these manifolds as it is done in Figure 5.2, but also we can look at them as two dimensional tubes starting from $l(c)$ as was done in Figure 2.3.

Another convenient way for drawing the invariant tori is to draw the coordinates r and θ in an orthogonal frame c, θ where $c = h(r)$ (see Figure 5.3). Using this representation it easy to add a third coordinate. For example, when adding the coordinate ξ_1 or η_1 we get a picture from Figure 5.4 a) and b), and when the third axis represents both of the two coordinates then the layers on the c level represent the foliation $M(\mu, c)$, Figure 5.4 c). From the Figure 5.4 c) we can see that in particular there cannot exist a heteroclinic orbit from $l(c_1)$ to $l(c_2)$ when $c_1 \neq c_2$. The aim of this chapter is to illustrate that this could be possible for a perturbation of the PRC3BP. We will come to this later on in Section 5.3.

In Section 2.3 in the Definition 2.4 we have introduced the orbit q_c^0 homoclinic to the Libration point L_2. Let us show how this orbit can be drawn in our setting. The orbit q_c^0 is contained in both the stable and the unstable manifold of the orbit $l(c)$. We can illustrate this on two pictures given in Figure 5.5. From these pictures it is quite evident that this intersection cannot be transversal. To get a better picture of this intersection in the full four dimensional space it might be a good idea to take a look at both Figure 2.3 and Figure 5.5 at the same time. Even though the Figure 2.3 seems to capture more detail, the interpretation of Figure 5.5 will help us to understand the dynamics when we will no longer have the invariant foliation $M(\mu, c)$.

5.2 The dynamics in the elliptic problem.

The set of the Lapunov orbits $B_C = \{l(c) | C \leq c \leq C_2\}$ is normally hyperbolic (for details on normal hyperbolicity please refer to Chapter 6). This fact is clearly illustrated by Figure 5.4. From normal hyperbolicity theory we know that a normally hyperbolic set persists under small perturbations. This means that if we

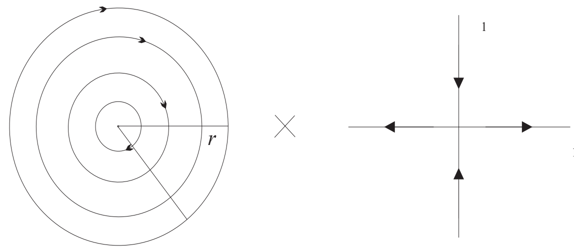

Figure 5.1 The flow in the neighborhood of L_2.

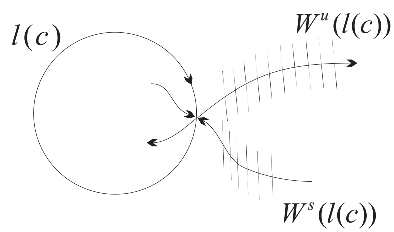

Figure 5.2 An invariant whiskered torus $l(c)$.

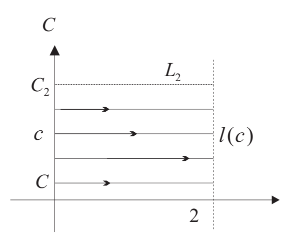

Figure 5.3 Lapunov orbits in the c, θ coordinates.

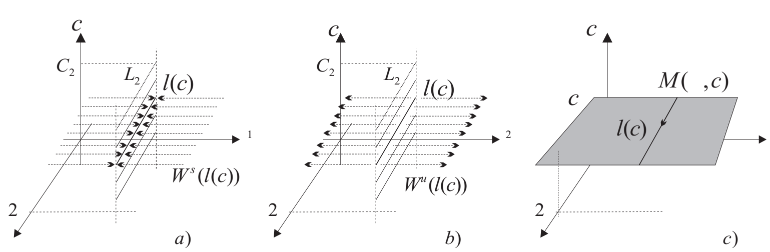

Figure 5.4 The Lapunov orbits $l(c)$ in a three dimensional representation.

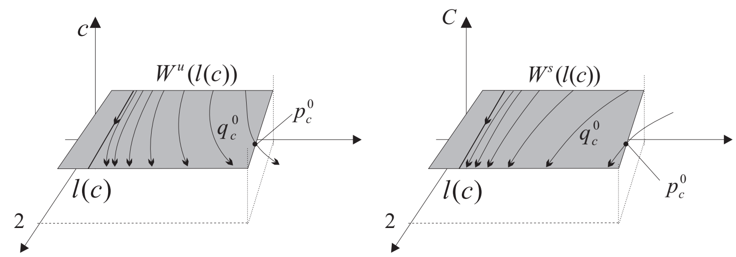

Figure 5.5 The homoclinic orbit q_c^0 and one of the two intersection points of the manifolds $W^u(l(c))$ and $W^s(l(c))$.

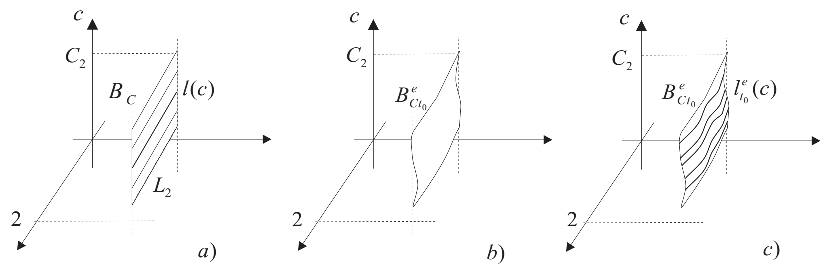

Figure 5.6 The the set of the Lapunov orbits B_C a), Persistence of B_C b), Persistence of the Lapunov orbits $l(c)$ c).

regard the planar restricted elliptic three body problem as a perturbation of the circular problem, then the set will persist for sufficiently small eccentricities e. The persistence should be understood in the following sense. The elliptic three body problem is given by a time 2π periodic differential equation given by the Hamiltonian H^e given by the formula (4.49). Since the equation is 2π periodic, we cen define a time 2π Poincaré map

$$P^e_{t_0} : \Sigma_{t_0} \to \Sigma_{t_0+2\pi},$$

where $\Sigma_{t_0} = \{(x, y, \dot{x}, \dot{y}, t)|t = t_0\}$. The set of Lapunov orbits B_C is an invariant set for the map $P^0_{t_0}$ of the circular problem for any t_0. This invariant set is perturbed to a nearby invariant set $B^e_{Ct_0}$ for the map $P^e_{t_0}$ (see Figure 5.6 a) and b)). From Section 3.3 we know that the map $P^0_{t_0}$ is a twist map on the set B_C. By Kolmogorov, Arnold, Moser (KAM) Theorem most of the invariant rings $l(c)$ survive under the perturbation and are perturbed to nearby invariant sets $l^e_{t_0}(c)$ for the map $P^e_{t_0}$ (see Figure 5.6 c)). This will be rigorously discussed and proved in Chapter 6, Sections 6.1 and 6.2.

Our main focus will be on the dynamics associated with the quasi periodic orbits $l^e_{t_0}(c)$. Based on the results of Simo we know how the intersections of the invariant manifolds $W^u(l(c))$ and $W^s(l(c))$ behave in the circular three body problem (see Figures 2.3 and 2.4). Let $W^u(l^e_{t_0}(c), P^e_{t_0})$ and $W^s(l^e_{t_0}(c), P^e_{t_0})$ denote respectively the unstable manifold and the stable manifold of the quasi periodic orbit $l^e_{t_0}(c)$ with respect to the map $P^e_{t_0}$. After perturbing the circular three body

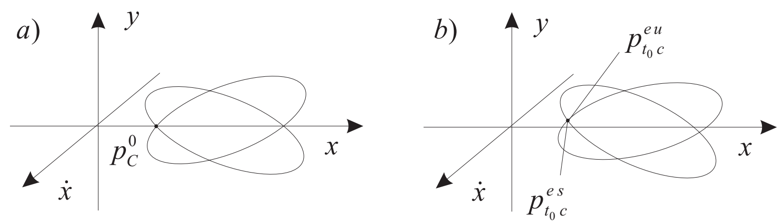

Figure 5.7 The intersection of $W^u(l(c)) \cap \Sigma_{\{y=0\}}$ and $W^s(l(c)) \cap \Sigma_{\{y=0\}}$ a). The $W^u(l^e_{t_0}(c), P^e_{t_0}) \cap \Sigma_{\{y=0\}}$ and $W^s(l^e_{t_0}(c), P^e_{t_0}) \cap \Sigma_{\{y=0\}}$ in the $x, y, \dot{x}$ coordinates b).

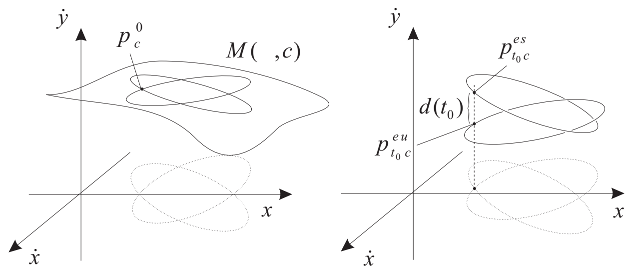

Figure 5.8 The intersection of $W^u(l(c))$ and $W^s(l(c))$ in the $x, y, \dot{y}$ coordinates a). The $W^u(l^e_{t_0}(c), P^e_{t_0}) \cap \Sigma_{\{y=0\}}$ and $W^s(l^e_{t_0}(c), P^e_{t_0}) \cap \Sigma_{\{y=0\}}$ in the $x, y, \dot{y}$ coordinates b).

problem into an elliptical one with sufficiently small $e > 0$ the projection of $W^u(l^e_{t_0}(c), P^e_{t_0}) \cap \Sigma_{\{y=0\}}$ and $W^s(l^e_{t_0}(c), P^e_{t_0}) \cap \Sigma_{\{y=0\}}$ should produce the same qualitative picture as for the circular case (see Figure 5.7). In the case of the circular problem the intersection in the $x, y, \dot{x}$ coordinates guaranteed the intersection in the full four dimensional space, but for the elliptic problem we do not have the invariant foliation $M(\mu, c)$, which means that it is possible that the intersection in the $x, y, \dot{x}$ coordinates does not imply an actual intersection (see Figure 5.8). This is a serious problem which we will have to overcome in the future. On the other hand the fact that for the elliptic problem we are not restricted by the foliation will allow the existence of heteroclinic orbits between $l^e_{t_0}(c_1)$ and $l^e_{t_0}(c_2)$ for $c_1 \neq c_2$, which would not be possible in the circular case. The intersection of $W^u(l^e_{t_0}(c), P^e_{t_0}) \cap \Sigma_{\{y=0\}}$ and $W^s(l^e_{t_0}(c), P^e_{t_0}) \cap \Sigma_{\{y=0\}}$ in the $x, y, \dot{x}$ coordinates will in general not guarantee an intersection but produce a pair of points

$$p^{e\,u}_{t_0\,c} = (x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^{e\,u}_{t_0\,c})$$

and

$$p^{e\,s}_{t_0\,c} = (x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^{e\,s}_{t_0\,c})$$

(see Figure 5.8). If we will be able to show that $p^{e\,u}_{t_0\,c} = p^{e\,s}_{t_0\,c}$ then we will have an intersection of the manifolds $W^u(l^e_{t_0}(c), P^e_{t_0})$ and $W^s(l^e_{t_0}(c), P^e_{t_0})$.

To get a slightly different perspective let us illustrate the situation presented in Figure 5.8 using the c and θ coordinates. This is done in Figure 5.9 . Using this figure we can graphically represent two orbits q^e_{cs} and q^e_{cu} which start from the points

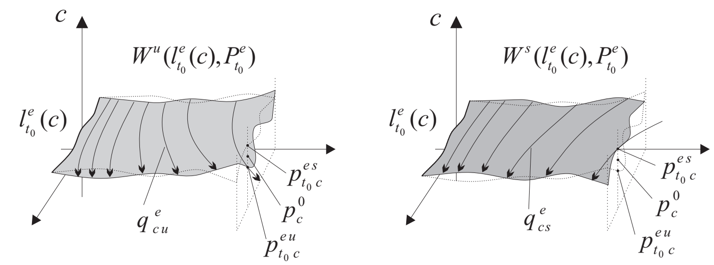

Figure 5.9 The orbits q^e_{cu} and q^e_{cs} starting from the points $p^{eu}_{t_0 c}$ and $p^{es}_{t_0 c}$.

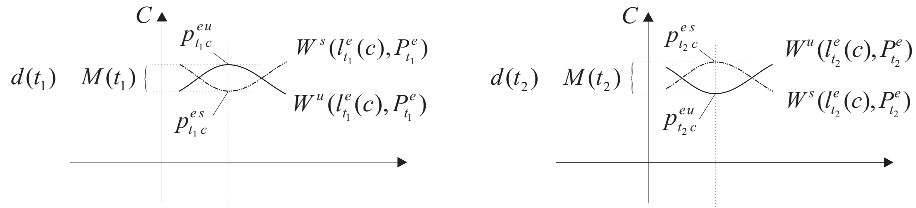

Figure 5.10 Change of the sign of the Melnikov function $M(t_1) < 0 = M(t_0) < M(t_2)$.

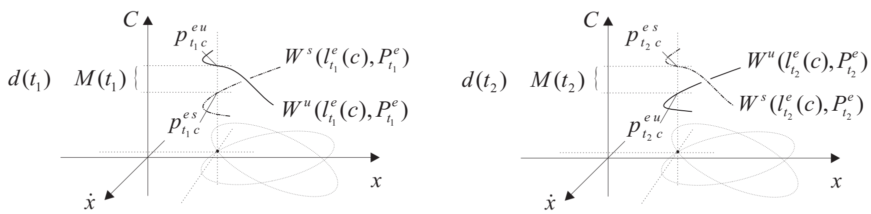

Figure 5.11 Change of the sign of the Melnikov function $M(t_1) < 0 = M(t_0) < M(t_2)$ in the x, $\dot{x}$, C coordinates.

$p^{e\,u}_{t_0\,c}$ and $p^{e\,s}_{t_0\,c}$ respectively, at the time $t = t_0$. These orbits will be important in the later discussion in Section 7.3. The two orbits q^e_{cs} and q^e_{cu} can be approximated by the homoclinic orbit q^0. What is more, it will turn out that the signed distance between the points $p^{e\,s}_{t_0\,c}$ and $p^{e\,u}_{t_0\,c}$ can be computed by an appropriate Melnikov integral $M(t_0)$ along the homoclinic orbit q^0. For sufficiently small μ and c sufficiently close to C_2, the integral $M(t_0)$ will depend only on the choice of the section Σ_{t_0}. If the sign of the Melnikov function $M(t_0)$ will change for a given t_0 then this will mean that for this t_0 we will have an intersection of the invariant manifolds $W^u(l^e_{t_0}(c), P^e_{t_0})$ and $W^s(l^e_{t_0}(c), P^e_{t_0})$ at some point $(x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^e_{t_0\,c})$ (see Figures 5.10 and 5.11, and also Figures 5.9 and 5.8 for comparison).

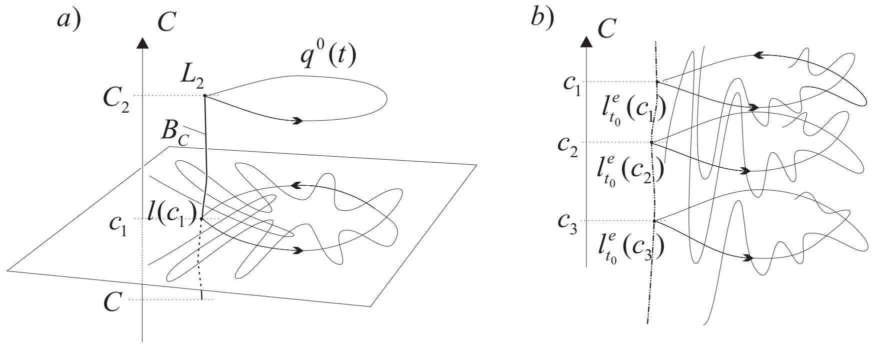

Figure 5.12 The dynamics of the circular problem which is restricted to each energy level a) compared with Arnold diffusion b)

5.3 Arnold diffusion

In the above section we have outlined a method which will be used to prove that the stable and the unstable manifolds $W^s(l^e_{t_0}(c), P^e_{t_0})$ and $W^u(l^e_{t_0}(c), P^e_{t_0})$ intersect each other. If the intersection of the manifolds is transversal then this leads to complicated dynamics which is often referred to in literature as Arnold diffusion.

The fact that $W^s(l^e_{t_0}(c), P^e_{t_0})$ and $W^u(l^e_{t_0}(c), P^e_{t_0})$ intersect each other transversally causes the stable and unstable manifolds of the surviving perturbations of neighboring Lapunov orbits to do the same [31]. To be more precise, if an energy c_1 is sufficiently close to c and if the Lapunov orbit $l(c_1)$ survives under perturbation and is perturbed to a nearby quasi periodic orbit $l^e_{t_0}(c_1)$, then $W^s(l^e_{t_0}(c_1), P^e_{t_0})$ and $W^u(l^e_{t_0}(c_1), P^e_{t_0})$ intersect transversally. As if that was not enough it also turns out that for such c_1 sufficiently close to c we will also have intersections of $W^u(l^e_{t_0}(c_1), P^e_{t_0})$ with $W^s(l^e_{t_0}(c), P^e_{t_0})$ and $W^s(l^e_{t_0}(c_1), P^e_{t_0})$ with $W^u(l^e_{t_0}(c), P^e_{t_0})$ [31]. The above argument can be repeated for c_2 sufficiently close to c_1 which by induction leads to a series of $l^e_{t_0}(c_1), \ldots, l^e_{t_0}(c_n)$ invariant tori interconnected with each other by stable and unstable manifolds. Such series are called transition chains. From the above argument for each energy level c_i we have chaotic dynamics coming from the fact that $W^s(l^e_{t_0}(c_i), P^e_{t_0})$ and $W^u(l^e_{t_0}(c_i), P^e_{t_0})$ intersect transversally. Together with the dynamics coming from each energy level c_i, we have a transition from the level c_i to an nearby level c_k for $k = 1, \ldots, n$ (see Figure 5.12). The above described dynamics is called Arnold diffusion for the Transition chain $l^e_{t_0}(c_1), \ldots, l^e_{t_0}(c_n)$. This setting guarantees rich symbolic dynamics of the system. Such method of obtaining the symbolic dynamics from transition tori has been investigated by Marsden and Holmes in [15].

6
Normal hyperbolicity, KAM Theorem and the persistence of Lapunov orbits

In this chapter we introduce the concept of a normally hyperbolic manifold and apply the normally hyperbolic manifold theory to show that the set of Lapunov orbits persists under perturbation. We also introduce the Kolmogorov-Arnold-Moser (KAM) theorem and apply the theorem to show that on the perturbed set of Lapunov orbits most of the orbits persist under perturbation and are perturbed into quasi periodic orbits of the time 2π Poincaré map of the PRE3BP. This is the main result of this chapter and will be formulated in the Theorem 6.16.

The procedure which we apply in this chapter is a mirror of the one used in [9]. Hence most of the definitions, theorems and the method of the proof which is used are taken form [9] where an identical result is shown but in the case of a geodesic flow of $\mathbb{T}^2$. The one difference is that we will use a different version of the KAM theorem than the one used in [9]. The results of [9] were obtained using the Herman's version of KAM theorem [12], whereas we will use a result of Broer [7]. The latter is more convenient for us because the formulation of the theorem puts more emphasis on the smooth dependence on the initial conditions on and on the perturbation parameter. This smooth dependence will play an important role for the Melnikov method of Chapter 7. Similar arguments based on normal hyperbolicity and KAM Theorem have been used in [15], [9], [23], [35] and [36] to obtain similar results in different settings. The only truly sensitive point is making sure that all the appropriate bounds and continuity conditions are met when applying the procedure to our particular case.

6.1 Normal hyperbolicity and the persistence of the set of Lapunov orbits

In this section we will present a number of results concerning the regularity, persistence, and smooth dependence on the initial conditions of normally hyperbolic invariant manifolds. We will also apply these results to show that the set of Lapunov orbits is invariant under the perturbation from the PRC3BP to the PRE3BP.

Let us start with a definition of a normally hyperbolic manifold.

Definition 6.1 ([9, A1])

Let M be a manifold in $\mathbb{R}^n$ and Φ_t a C^r, $r \geq 1$ flow on it. We say that a manifold $\Lambda \subset M$ invariant under Φ_t is α-β normally hyperbolic when there is a bundle decomposition

$$TM = T\Lambda \oplus E^s \oplus E^u, \tag{6.1}$$

invariant under the flow, and numbers $C > 0$, $0 < \beta < \alpha$, such that for $x \in \Lambda$

$$v \in E^s_x \Leftrightarrow |D\Phi_t(x)v| \leq Ce^{-\alpha t}|v| \quad \forall t > 0 \tag{6.2}$$

$$v \in E^u_x \Leftrightarrow |D\Phi_t(x)v| \leq Ce^{\alpha t}|v| \quad \forall t < 0 \tag{6.3}$$

$$v \in T_x\Lambda \Leftrightarrow |D\Phi_t(x)v| \leq Ce^{\beta|t|}|v| \quad \forall t \tag{6.4}$$

We will now show that the set of Lapunov orbits around L_2 is normally hyperbolic.

Lemma 6.2

For a sufficiently small mass μ and for $C < C^\mu_2$ sufficiently close to C^μ_2, the set $B_C = \{l(c)|C \leq c \leq C^\mu_2\}$ combined of the Lapunov orbits of the PRC3BP (see Section 3.6 for more details on the set B_C) is α-β normally hyperbolic for the flow generated by (3.14), where $\alpha > 0$ is close to the real eigenvalue α_1 at the Libration point L^μ_2 and $\beta > 0$ can be chosen arbitrarily close to zero.

Proof

Let $M \subset \mathbb{C}^4$ be the set given by the real solutions in the ξ, η coordinates (for the definition of these coordinates see Section 3.2 and Theorem 3.2 in particular). From Lemmas 3.3 and 3.7 we know that this set is a subset of the following set

$$M \subset \{(\xi_1, \xi_2, \eta_1, \eta_2)|\xi_1, \eta_1 \in \mathbb{R}, \xi_2 = i\overline{\eta_2} = re^{i\phi}, r \in \mathbb{R}^+, \phi \in [0, 2\pi)\}. \tag{6.5}$$

We will look at the set in the (ξ_1, η_1, r, ϕ) coordinates where ξ_1, η_1 are the coordinates of the hyperbolic expansion and the r, ϕ are the coordinates of the twist rotation around the libration point. In these coordinates we have

$$M \subset \{(\xi_1, \eta_1, r, \phi)|\xi_1, \eta_1 \in \mathbb{R}, r \in \mathbb{R}^+, \phi \in [0, 2\pi)\}. \tag{6.6}$$

The set B_C of the Lapunov orbits, in these coordinates is given by

$$B_C = \{(0,0,r,\phi)|C \leq h(r) \leq C_2^{\mu}, \phi \in [0,2\pi)\}, \tag{6.7}$$

(See (3.111) for the definition of $h(r)$). From (3.39) we know that the flow on B_C is given by

$$\Phi_t(0,0,r,\phi) = (0,0,r,\left[\phi + t|a_2(0,ir^2)|\right]_{\mathrm{mod}2\pi}). \tag{6.8}$$

From the above we can see that if we define

$$\begin{aligned} E^u &= \{(\xi_1,0,0,0)|\xi_1 \in \mathbb{R}\}, \\ E^s &= \{(0,\eta_1,0,0)|\eta_1 \in \mathbb{R}\}, \\ TB_C &= \{(0,0,r,\phi)|C \leq h(r) \leq C_2^{\mu}, \phi \in [0,2\pi)\}, \end{aligned} \tag{6.9}$$

then we will have

$$TM = E^u \oplus E^s \oplus TB_C. \tag{6.10}$$

For $x = (0,0,r_0,\phi_0) \in B_C$, since the direction ξ_1 and η_1 is related to the hyperbolic expansion and contraction with eigenvalues α_1 and $-\alpha_1$ at $x_0 = (0,0,0,0)$, there exists a $\alpha > 0$ close to α_1, $c > 1$ and sufficiently small $R > 0$ such that for all x with $r_0 < R$ we have

$$\begin{aligned} |D\Phi_t(x)v| &\leq ce^{\alpha t}\,|v| \quad \text{for } v \in E^u \text{ and } t < 0 \\ |D\Phi_t(x)v| &\leq ce^{-\alpha t}\,|v| \quad \text{for } v \in E^s \text{ and } t > 0. \end{aligned} \tag{6.11}$$

For $v = (0,0,v_r,v_\phi) \in TB_C$ we have

$$\begin{aligned} |D\Phi_t(x)v| &= \left| \begin{pmatrix} 1 & 0 \\ t\frac{\partial |a_2|}{\partial r}(0,r_0) & 1 \end{pmatrix} \begin{pmatrix} v_r \\ v_\phi \end{pmatrix} \right| \\ &= \left| \begin{matrix} v_r \\ v_\phi + t\frac{\partial |a_2|}{\partial r}(0,r_0)v_r \end{matrix} \right|. \end{aligned} \tag{6.12}$$

For sufficiently small $R > 0$, for which a_2 is convergent, for all x with $r_0 < R$ by choosing the coefficient c suitably large we will have

$$|D\Phi_t(x)v| \leq ce^{\beta|t|}|v| \quad \text{for } v \in T_xB_C, \tag{6.13}$$

for any given $\beta > 0$. To make sure that for all our x in B_C we will have $r_0 < R$ we can choose the energy C sufficiently close to C_2^{μ}. This will guarantee that the distance between x and L_2 is sufficiently small and all of the above inequalities will hold. The last thing that we have to check is whether we have $0 < \beta < \alpha$. This will hold if at the beginning of the proof we will choose a sufficiently small μ. By Remark 2.8 point 1, a sufficiently small μ will guarantee that α will be close to the real eigenvalue $\alpha_1^{Hill} = \sqrt{1+2\sqrt{7}} \approx 2.508\,3$ of the Hill's problem at L_2^{Hill}. Since for β we can choose any number greater than zero, this gives us $0 < \beta < \alpha$ for sufficiently small μ. □

We know that the set of Lapunov orbits of PRC3BP is normally hyperbolic. Let us now present a result about the existence of invariant stable and unstable manifolds for normally hyperbolic manifolds.

Theorem 6.3 ([9, A7])

Let Λ be a compact α-β normally hyperbolic manifold (possibly with a boundary) for the C^r flow Φ_t, satisfying the Definition 6.1. Then there exists a sufficiently small neighborhood U of Λ and a sufficiently small $\delta > 0$ such that

1. The manifold Λ is $C^{\min(r,r_1-\delta)}$, where $r_1 = \alpha/\beta$.
2. For any x in Λ, the set

$$\begin{aligned} W_x^s &= \{y \in U : \mathrm{dist}(\Phi_t(y), \Phi_t(x)) \leq Ce^{(-\alpha+\delta)t} \text{ for } t > 0\} \\ &= \{y \in U : \mathrm{dist}(\Phi_t(y), \Phi_t(x)) \leq Ce^{(-\beta-\delta)t} \text{ for } t > 0\} \end{aligned} \tag{6.14}$$

   is a C^r manifold and $T_x W_x^s = E_x^s$.
3. The bundles E_x^s are $C^{\min(r,r_0-\delta)}$ in x, where $r_0 = (\alpha - \beta)/\beta$, and

$$\begin{aligned} W_\Lambda^s &= \{y \in U : \mathrm{dist}(\Phi_t(y), \Lambda) \leq Ce^{(-\alpha+\delta)t} \text{ for } t > 0\} \\ &= \{y \in U : \mathrm{dist}(\Phi_t(y), \Lambda) \leq Ce^{(-\beta-\delta)t} \text{ for } t > 0\} \end{aligned} \tag{6.15}$$

   is a $C^{\min(r,r_0-\delta)}$ manifold. Moreover $T_x W_\Lambda^s = E_x^s$. Finally

$$W_\Lambda^s = \bigcup_{x \in \Lambda} W_x^s. \tag{6.16}$$

   Moreover, we can find a $\rho > 0$ sufficiently small and a $C^{\min(r,r_0-\delta)}$ diffeomorphism from the bundle of balls of radius ρ in E_Λ^s to $W_\Lambda^s \cap U$.

Remark 6.4

An analogous theorem can be stated for W_Λ^u by considering the flow Φ_{-t}.

We will treat the elliptic problem as a perturbation of the circular problem. In our case the normally hyperbolic manifold before the perturbation $B_C = \{l(c) | C \leq c \leq C_2^\mu\}$ is compact. It turns out that such normally hyperbolic manifolds persist under perturbation to become locally invariant.

Definition 6.5 ([33])

Let $\Lambda \subset R^n$ be a compact, connected C^r manifold with a boundary in $\mathbb{R}^n$. We say that Λ is locally invariant under a flow Φ_t if for each $p \in \Lambda$ there exists a time interval $I_p = \{t \in \mathbb{R} | t_1 < t < t_2$ where $t_1 \leq 0 < t_2$ or $t_1 < 0 \leq t_2\}$ such that $\Phi_t(p) \in \Lambda$ for all $t \in I_p$.

The fact that a normally hyperbolic manifold is perturbed into a locally invariant normally hyperbolic manifold is given by the following theorem.

Theorem 6.6 ([9, A.14])

Let $\Lambda \subset M$ (not necessarily compact) be α-β normally hyperbolic for the flow Φ_t generated by the vector field X, which is uniformly C^r in a neighborhood U of Λ such that $\mathrm{dist}(M \setminus U, \Lambda) > 0$. Let Ψ_t be the flow generated by another vector field Y which is C^r and sufficiently C^1 close to X. Then we can find a manifold Γ which is α'-β' hyperbolic for Y and $C^{\min(r,r_1-\delta)}$ close to Λ, where $r_1 = \alpha/\beta$.

The constants α', β' are arbitrarily close to α, β if Y is sufficiently C^1 close to X.

The manifold Γ is the only $C^{\min(r,r_1-\delta)}$ normally hyperbolic manifold C^0 close to Λ and locally invariant under the flow of Y.

The above Theorem is extended to give us a smooth dependence on the parameter by the following two remarks.

Remark 6.7 ([9, observation 1. page 390])

Assume that we have a family of flows $\Phi_{t,e}$, generated by vector fields X_e which are jointly C^r in all its variables (the base point x and the parameter e). Let Λ_e be the normally hyperbolic manifold Γ from Theorem 6.6 for the flow $\Phi_{t,e}$. Then there exists a $C^{\min(r,r_1-\delta)}$ mapping $F : \Lambda \times I \to M$, where $r_1 = \alpha/\beta$ and $I \subset \mathbb{R}$ is an interval containing zero, such that $F(\Lambda, e) = \Lambda_e$ and $F(\cdot, 0)$ is the identity.

Remark 6.8 ([9, observation 2. page 390])

For a family of flows $\Phi_{t,e}$ with the same assumptions as in Remark 6.7, there exists a $C^{\min(r,r_1-\delta)}$ ($r_1 = \alpha/\beta$) mapping $R^s : W^s_\Lambda \times I \to M$ such that $R^s(W^s_\Lambda, e) = W^s_{\Lambda,e}$, $R^s(\cdot, e)|_\Lambda = F(\cdot, e)$, $R^s(W^s_x, e) = W^s_{F(x,e),e}$.

An analogous mapping R^u also exists for W^u_Λ.

We will apply the above Theorem 6.6 and Remark 6.7 to obtain a persistence result for the normally hyperbolic set of Lapunov orbits around the libration point L_2. Since in our case the perturbation is not autonomous as is in the Theorem 6.6 we will have to consider the problem in an extended phase space by adding the extra time variable. Let us introduce the appropriate notations. Let $\phi^e_{t,s} : \mathbb{R}^4 \to \mathbb{R}^4$ be given by

$$\phi^e_{t,s}(x) = q(s+t), \tag{6.17}$$

where $q(\cdot)$ is the solution for the PRE3BP (4.35) with an initial condition $q(s) = x$. We will define our flow on the extended phase space $\Phi^e_t : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4 \times \mathbb{R}$ as

$$\Phi^e_t(x, s) = (\phi^e_{t,s}(x), s+t). \tag{6.18}$$

By Lemma 6.2 for $e = 0$ we have a α-β normally hyperbolic invariant manifold for Φ^0_t of the form

$$\Lambda = B_C \times \mathbb{R}, \tag{6.19}$$

where B_C is the set of the Lapunov orbits around L_2 in the PRC3BP. Now we are ready to state our perturbation result.

Lemma 6.9

For sufficiently small μ there exists a $e_0(\mu)$ such that for all $0 < e \leq e_0(\mu)$ the normally hyperbolic set $\Lambda = B_C \times \mathbb{R}$ of the PRC3BP in the extended phase space is perturbed to a $O(e)$ close C^∞ normally hyperbolic manifold

$$\Lambda_e = \{(\Lambda_{t,e}, t)\,|\,\Lambda_{t,e} \subset \mathbb{R}^4, t \in \mathbb{R}\} \tag{6.20}$$

which is locally invariant under the flow of PRE3BP given by (4.49). What is more the manifold $\Lambda_{t,e}$ is 2π periodic in t i.e

$$\Lambda_{t,e} = \Lambda_{t+2\pi,e}. \tag{6.21}$$

Proof

Applying the Theorem 6.6 and Remark 6.7 we obtain a family of normally hyperbolic manifolds Λ_e invariant under Φ_t^e and a function $F : \Lambda \times I \to M$ such that

$$F(\Lambda, e) = \Lambda_e. \tag{6.22}$$

By the Remark 6.7 the function F is $C^{\min(r, r_1-\delta)}$, where $r_1 = \alpha/\beta$. Since the vector field for the PRE3BP in the neighborhood of L_2 is C^∞ the function F will be $C^{r_1-\delta}$. From the proof of Lemma 6.2 we know that for sufficiently small μ we have $\alpha \approx \sqrt{1+2\sqrt{7}}$ and that $\beta > 0$ can be chosen to be arbitrarily close to zero. This means that for sufficiently small μ, the function F is C^k for any given $k > 0$. The invariant manifold Λ_e is given in the extended phase space and is therefore equal to

$$\Lambda_e = \{(\Lambda_{t,e}, t)\,|\,\Lambda_{t,e} \subset \mathbb{R}^4, t \in \mathbb{R}\}. \tag{6.23}$$

In the extended phase space the solutions are unique. What is more the PRE3BP is 2π periodic in time. This means that the manifolds $\Lambda_{t,e}$ will also be 2π periodic in t.

What is now left to show is that the set Λ_e is $O(e)$ close to Λ. Since $\Lambda_{t,e}$ is 2π periodic it is sufficient to show that $\{(\Lambda_{t,e}, t)\,|t \in [0, 2\pi]\}$ is $O(e)$ close to $B_C \times [0, 2\pi]$. Let us restrict the interval I from the domain of the function F to be bounded. This will give us

$$\sup\{|DF(x,t,e)| : (x,t,e) \in B_C \times [0,2\pi] \times I\} \leq M \tag{6.24}$$

for some bound $M > 0$. We can use the above to obtain our result by the following estimate

$$\begin{aligned} \mathrm{dist}(B_C \times [0,2\pi], \{(\Lambda_{t,e}, t)\,|t \in [0,2\pi]\}) &\leq \sup\{|F(x,t,e) - F(x,t,0)| : x \in B_C\} \\ &\leq eM. \end{aligned} \tag{6.25}$$

□

Let us now introduce the following notation.

$$B^e_{Ct_0} = \Lambda_{t_0,e}, \tag{6.26}$$

where $\Lambda_{t_0,e}$ is given by the Lemma 6.9. The set $B^e_{Ct_0}$ is the perturbation under $e > 0$ of the set of Lapunov orbits

$$B_C = \{l(c)|C \leq c \leq C_2^{\mu}\}. \tag{6.27}$$

We can perform an analogous construction and obtain a smaller set $B^e_{ct_0} \subset B^e_{Ct_0}$ for any $c \in (C, C_2)$ which will be the perturbation of the set

$$B_c = \{l(\tilde{c})|c \leq \tilde{c} \leq C_2^{\mu}\}. \tag{6.28}$$

By Lemma 6.9 we know that $B^e_{Ct_0}$ is $O(e)$ close to B_C and that $\Lambda_e = \{(B^e_{Ct}, t)|t \in \mathbb{R}\}$ is locally invariant for Φ^e_t the extended flow (6.18) of the PRE3BP. The flow is generated by the Hamiltonian (4.49)

$$H^e = H + eG + O(e^2), \tag{6.29}$$

where from the Lemma 4.3 we know that G is bounded in the neighborhood of L_2. This means that for any $x \in B^e_{Ct_0}$

$$\Phi^e_{2\pi}(x, t_0) = (\phi^e_{2\pi,t_0}(x), t_0 + 2\pi) = (\phi^0_{2\pi,t_0}(x) + O(e), t_0 + 2\pi). \tag{6.30}$$

Therefore if we choose any $c \in (C, C_2)$ then for a sufficiently small e the Poincaré time 2π map from the smaller set $B^e_{ct_0}$ onto the larger set $B^e_{Ct_0}$

$$\begin{aligned} P^e_{t_0} &: B^e_{ct_0} \to B^e_{Ct_0} \\ P^e_{t_0}(x) &= \phi^e_{2\pi,t_0}(x), \end{aligned} \tag{6.31}$$

will be properly defined.

Remark 6.10

Let us note that in the above, in the definition of $P^e_{t_0}$ in (6.31), we have been somewhat careful and have restricted the domain of our Poincaré function $P^e_{t_0}$ from the set $B^e_{Ct_0}$ to a smaller set $B^e_{ct_0}$ so that we are sure that this function is properly defined. Later on in the chapter (after applying the KAM Theorem) we will know that most of the Lapunov orbits will survive the perturbation as KAM-tori. This will allow us to simplify our setting and assume that we have

$$P^e_{t_0} : B^e_{Ct_0} \to B^e_{Ct_0} \tag{6.32}$$

with the C chosen to be the energy of one of those orbits.

The results of normally hyperbolic theory do not as yet state any results about the symplectic structure on the perturbed manifolds $B^e_{Ct_0}$ which will be needed for the application of the KAM Theorem in the following section. This will now be the subject of our discussion.

Let ω denote the standard symplectic form in $\mathbb{R}^4$ i.e.

$$\omega = dx \wedge dp_x + dy \wedge dp_y, \tag{6.33}$$

where $(x, y, p_x, p_y) \in \mathbb{R}^4$. Let $\omega^e_{t_0}$ denote the induced form on $\Lambda_{t_0,e} = B^e_{Ct_0}$. The fact that the symplectic structure on the perturbed manifolds is preserved is given by the following lemma.

Lemma 6.11 ([9, page 367])

There exist close to identity C^∞ coordinate maps $c^e_{t_0} : \Lambda_{t_0,e} \to \Lambda_{t_0,0} = B_C$ which transport the symplectic forms $\omega^e_{t_0}$ into the standard one. Moreover these maps can be chosen to be C^∞ jointly with the parameters.

6.2 KAM Theorem and its application to the set of Lapunov orbits

In this section we will introduce the Kolmogorov-Arnold-Moser Theorem (KAM) and apply it for the Poincaré time 2π map (6.31) $P^e_{t_0}$ of the PRE3BP. We will show that most of the Lapunov orbits $l(c)$ on the set B_C are perturbed to one dimensional invariant tori $l^e_{t_0}(c)$ for the Poincaré map $P^e_{t_0}$. What is more we will show that these invariant tori depend analytically on the parameter e.

Definition 6.12

A real number α is called a Diophantine number of exponent $\tau > 2$ if there exists a constant $\gamma > 0$ such that

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{\gamma}{q^\tau} \tag{6.34}$$

for all $p \in \mathbb{Z}$, $q \in \mathbb{N}$.

For a fixed $\tau > 0$ and $\gamma > 0$ with γ sufficiently small the set of all diophantine numbers is a Cantor set [7]. The measure the complement of this set is $O(\gamma)$ as γ decreases to zero [7].

Let us consider a C^∞ area preserving map $P_e : [0,1] \times \mathbb{T} \to [0,1] \times \mathbb{T}$ of the form

$$P_e(c, \phi) = (c, \phi + a(c)) + O(e), \tag{6.35}$$

where the map a satisfies the twist condition $\frac{da}{dc}(c) \neq 0$ for $c \in [0,1]$. For a given $\tau > 0$ and $\gamma > 0$ the Cantor set of Diophantine numbers is pulled back through the function a to a Cantor set on $[0,1]$. We will denote this set as $\mathfrak{C}$. For the above given map P_e we have the following theorem.

Theorem 6.13 (The KAM Theorem. [7, Theorem 2])

If the map P_e is $O(e)$ close to P_0 in the C^l norm (see (4.66) for the definition of $\|\cdot\|_{C^l}$) for $l \geq 6$

$$\|P_e - P_0\|_{C^l} = O(e),$$

and if we assume that $\gamma > 0$ is sufficiently small, then for $e > 0$ sufficiently small there exists a C^∞ transformation of the annulus $\Phi_e : [0,1] \times \mathbb{T} \to [0,1] \times \mathbb{T}$, conjugating the restriction $P_0|_{\mathfrak{C}\times\mathbb{T}}$ to a subsystem of P_e i.e. the following graph commutes

$$\begin{array}{ccc} \Phi_e(\mathfrak{C}\times\mathbb{T}) & \stackrel{P_e}{\to} & \Phi_e(\mathfrak{C}\times\mathbb{T}) \\ \uparrow \Phi_e & & \uparrow \Phi_e \\ \mathfrak{C}\times\mathbb{T} & \stackrel{P_0}{\to} & \mathfrak{C}\times\mathbb{T} \end{array} . \tag{6.36}$$

Moreover Φ_e is C^l in e.

Remark 6.14

Let us note that in the above Theorem we require that our map P_e is defined on $[0,1]\times\mathbb{T}$. This requirement is not necessary and the Theorem also works for area preserving maps $P_e:[0,a]\times\mathbb{T}\to[0,1]\times\mathbb{T}$ where $0<a<1$. Such was the case originally considered by Moser in [24].

From our perspective, what is of interest in the Theorem is the persistence of periodic orbits close to the Libration equilibrium point L_2 and therefore we will consider it's small neighborhood $[0,a]\times\mathbb{T}$.

We will apply the KAM Theorem to obtain invariant tori on the perturbed set $B^e_{Ct_0}$ of the Lapunov orbits. In order to do this we must first show that the time 2π Poincaré map for the ecliptic problem is area preserving. This is done in the following lemma.

Lemma 6.15

For sufficiently small μ there exists an $e_0(\mu)$ such that for all $0<e\leq e_0(\mu)$ the time 2π Poincaré map $P^e_{t_0}$ for the PRE3BP, restricted to the normally hyperbolic set $B^e_{ct_0}$

$$P^e_{t_0}:B^e_{ct_0}\to B^e_{Ct_0} \tag{6.37}$$

where $c\in(C,C^\mu_2)$ and C is sufficiently close to C^μ_2, is an area preserving map.

Proof

We know that the Poincaré map

$$P^e_{t_0}:\mathbb{R}^4\to\mathbb{R}^4 \tag{6.38}$$

is generated by a Hamiltonian system. This fact by Theorem 1.5 guarantees that it has to be a symplectic map for the standard symplectic form

$$\omega=dx\wedge dp_x+dy\wedge dp_y. \tag{6.39}$$

In order to show that the map $P^e_{t_0}$ restricted to $B^e_{Ct_0}$ is area preserving it is sufficient to show that the form $\omega|_{B^e_{Ct_0}}$ is not degenerate on the set $B^e_{Ct_0}$. If we can show this then from the fact that $P^e_{t_0}$ is symplectic we will also know that $P^e_{t_0}|_{B^e_{Ct_0}}$ is symplectic and therefore that it is area preserving on $B^e_{Ct_0}$. Because for sufficiently small e the set $B^e_{Ct_0}$ is arbitrarily close to B_C, in order to show that the form ω is not degenerate on the set $B^e_{Ct_0}$ it is sufficient to show that ω is not degenerate on the set B_C. The set B_C, sufficiently close to the Libration point L_2^μ, in turn can be approximated by the vector space V^μ given by the eigenvectors of the complex eigenvalues $\pm\alpha_2$ at L_2^μ which are responsible for the rotation on the set B_C. By Remark 2.8 the space V^μ can be approximated by the space V given by the pure complex eigenvalues $\pm\alpha_2^{Hill} = \pm\sqrt{1-2\sqrt{7}}$ of the Hill's problem which means that it is sufficient to check that ω is non degenerate on V.

From (3.88) we know that the eigenvectors connected with the purely complex eigenvalues $\pm\alpha_2^{Hill}$ at L_2^{Hill} are

$$\begin{aligned} w_1 &= (1, \frac{9}{\alpha_2\left(\sqrt{7}-4\right)}, \frac{9\left(\sqrt{7}-3\right)}{\alpha_2\left(\sqrt{7}-4\right)}, \frac{2}{\sqrt{7}-3})^T \quad \text{for } \alpha_2^{Hill}, \\ w_2 &= (1, \frac{-9}{\alpha_2\left(\sqrt{7}-4\right)}, \frac{-9\left(\sqrt{7}-3\right)}{\alpha_2\left(\sqrt{7}-4\right)}, \frac{2}{\sqrt{7}-3})^T \quad \text{for } -\alpha_2^{Hill}. \end{aligned} \tag{6.40}$$

This means that the tangent to B_C^{Hill} vector field V is equal to

$$\begin{aligned} V &= \operatorname{span}\{w_1^{re}, w_1^{im}\} \\ &= \operatorname{span}\{(\sqrt{7}-3, 0, 0, 2)^T, (0, 1, \sqrt{7}-3, 0)^T\} \end{aligned} \tag{6.41}$$

where $w_1 = w_1^{re} + i w_1^{im}$. In order to check whether ω is non degenerate on V let us consider the immersion

$$\pi = (\pi_1, \pi_2, \pi_3, \pi_4) : V \to \mathbb{R}^4 \tag{6.42}$$

and compute $\pi^*(\omega)$. A point v in V can be represented by

$$v = v_1(\sqrt{7}-3, 0, 0, 2)^T + v_2(0, 1, \sqrt{7}-3, 0)^T \tag{6.43}$$

where $v_1, v_2 \in \mathbb{R}$. We can therefore compute $\pi^*(\omega)(v_1, v_2)$ as

$$\begin{aligned} \pi^*(\omega)(v_1, v_2) &= (\frac{\partial \pi_1}{\partial v_1} dv_1 + \frac{\partial \pi_1}{\partial v_2} dv_2) \wedge (\frac{\partial \pi_3}{\partial v_1} dv_1 + \frac{\partial \pi_3}{\partial v_2} dv_2) \\ &\quad + (\frac{\partial \pi_2}{\partial v_1} dv_1 + \frac{\partial \pi_2}{\partial v_2} dv_2) \wedge (\frac{\partial \pi_4}{\partial v_1} dv_1 + \frac{\partial \pi_4}{\partial v_2} dv_2) \\ &= (\sqrt{7}-3) dv_1 \wedge (\sqrt{7}-3) dv_2 + dv_2 \wedge 2 dv_1 \\ &= \left((\sqrt{7}-3)^2 - 2\right) dv_1 \wedge dv_2 \\ &\neq 0, \end{aligned} \tag{6.44}$$

which means that ω is not degenerate on V. □

Now we are ready to state the main result of this chapter.

Theorem 6.16

For a sufficiently small $\mu > 0$ and for $C < C_2^\mu$ sufficiently close to C_2^μ there exists a $0 < e_0(\mu)$ and a C^∞ function

$$F_{t_0} : B_C \times [0, e_0(\mu)] \to \mathbb{R}^4 \tag{6.45}$$

such that

$$F_{t_0}(B_C, e) = B^e_{Ct_0}. \tag{6.46}$$

What is more for any $e \in [0, e_0(\mu)]$ there exists a Cantor set $\mathfrak{C} \subset [C, C_2^{\mu}]$, such that for any $c \in \mathfrak{C}$ the Lapunov orbit $l(c)$ is perturbed into $l^e_{t_0}(c) = F_{t_0}(l(c), e)$ which is an $O(e)$ close to $l(c)$ invariant torus for the Poincaré map $P^e_{t_0}$.

Remark 6.17

Intuitively the above Theorem simply states that most of the Lapunov orbits $l(c)$ survive the perturbation and that the perturbed orbits $l^e_{t_0}(c)$ depend smoothly on the parameter e.

Proof (proof of the Theorem 6.16)

In order to avoid a crowd of notations we will skip the index μ in what follows. Let us just keep in mind that the discussion is made for a fixed small μ and that with the change of μ all the below objects will change as well.

We know that the set $B^e_{Ct_0}$ was defined to be equal to the manifold $\Lambda_{t_0,e}$ which was constructed in the proof of Lemma 6.9. By Lemma 6.11 there exists an exact symplectic map $c^{t_0}_e : B^e_{Ct_0} \to B_C$ which is close to identity, which transforms the form on $B^e_{Ct_0}$ into a standard one. In Lemma 6.15 we have shown that for $c \in (C, C_2)$ the map

$$P^e_{t_0} : B^e_{ct_0} \to B^e_{Ct_0} \tag{6.47}$$

is area preserving. We also know that for $e = 0$ the map $P^0_{t_0} : B_c \to B_C$ is a twist map. To be more precise, from Lemma 3.17 we know that for sufficiently small μ in the (c, θ) coordinates we have

$$P(c, \theta) = (c, \theta + f(c)) \tag{6.48}$$
$$\frac{df}{dc}(c) \neq 0 \quad \text{for } C \leq c \leq C_2.$$

We will use the KAM Theorem 6.13 to show that most of the Lapunov orbits $l(c)$ survive under a sufficiently small perturbation to become invariant tori. In our case the Poincaré map $P^e_{t_0}$ does not yet fit the setting of the Theorem. The problem is that in our case the domain changes with the parameter e which is not the case in Theorem 6.13 where the domain is fixed. This can be obtained by using the functions $c^{t_0}_e : B^e_{Ct_0} \to B_C$ from Lemma 6.11. We can define

$$P_e = c^{t_0}_e \circ P^e_{t_0} \circ \left(c^{t_0}_e\right)^{-1} : B_c \to B_C \tag{6.49}$$

Let us note that for $e = 0$ the function $c^{t_0}_0$ is simply equal to identity and $P_0 = P^0_{t_0}$ is the time 2π Poincaré map of the PRC3BP. From the fact that $P^e_{t_0}$ is area preserving and the fact that $P^0_{t_0}$ is a twist map follow the same properties for our map P_e and P_0. What is now left is to show that $\|P_e - P_0\|_{C^l} = O(e)$. This follows from the equation (4.67) from the Remark 4.4, where we have a bound on the vector field $J\nabla H^e - J\nabla H^0$ given as

$$\left\| J\nabla H^e - J\nabla H^0 \right\|_{C^l} \leq e M_2(e_0, l) \tag{6.50}$$

for all $e \leq e_0$. This means that for any l, for the time 2π Poincaré map $P^e - P^0$ generated by this vector field we will have a bound $\|P_e - P_0\|_{C^l} \leq eM_3(e_0, l)$.

Now we will shift from the set B_C to the radius-angle coordinates $[0,1] \times \mathbb{T}^1$. We know that for $e = 0$ the point L_2 in B_C is invariant for the map $P_{e=0}$. By the fact that $DP_{e=0}(L_2)$ is the matrix of rotation around L_2, we can apply the implicit function theorem to obtain a function of stationary points $L(e)$ of the maps P_e, with $L(0) = L_2$. We change the coordinates from B_C to $[0,1] \times \mathbb{T}^1$ so that the radius is measured around the stationary points $L(e)$. From the fact that we have $\|P_e - P_0\|_{C^l} = O(e)$ in the coordinates B_C and the fact that the maps P_e are C^l in both the coordinates $x \in B_C$ and in the parameter e, we will have the same property

$$\|P_e - P_0\|_{C^l} = O(e), \tag{6.51}$$

in the $[0,1] \times \mathbb{T}^1$ coordinates. We can therefore apply the KAM Theorem 6.13 to the map P_e.

We are now ready to construct our function F_{t_0}. The function F_{t_0} is determined by the following diagram

$$\begin{array}{ccc} B^e_{Ct_0} & \overset{P^e_{t_0}}{\rightarrow} & B^e_{Ct_0} \\ \downarrow c^{t_0}_e & & \downarrow c^{t_0}_e \\ B_C & \overset{P_e}{\rightarrow} & B_C \\ \uparrow \Phi_e & & \uparrow \Phi_e \\ B_C & \overset{P_0}{\rightarrow} & B_C \end{array} \tag{6.52}$$

where $\Phi_e = \Phi_{e,t_0}$ is the map obtained from the KAM Theorem 6.13. Our function F_{t_0} is defined as

$$F_{t_0}(x, e) = \left(c^{t_0}_e\right)^{-1} \circ \Phi_{e,t_0}(x). \tag{6.53}$$

□

6.3 Smooth dependence of the invariant manifolds on the parameter e

In the previous section we have shown that the set of Lapunov orbits B_C is perturbed into a nearby set $B^e_{Ct_0}$. We have also shown that this perturbation can be expressed by a C^∞ function $F_{t_0} : B_C \times [0, e(\mu)] \to \mathbb{R}^4$ and that for all energies c from the Cantor set $\mathfrak{C} = \mathfrak{C}(e)$ the Lapunov orbits $l(c)$ are perturbed into

$$l^e_{t_0}(c) = F_{t_0}(l(c), e), \tag{6.54}$$

which are invariant tori for the Poincaré map $P^e_{t_0}$ i.e.

$$P^e_{t_0}(l^e_{t_0}(c)) = l^e_{t_0}(c). \tag{6.55}$$

In this section we will show that in a small neighborhood $U \subset \mathbb{R}^4$ of the set B_C the function F_{t_0} can be extended to a function $R^s_{t_0} : U \times [0, e(\mu)] \to \mathbb{R}^4$ which not only describes the perturbation of the set B_C but also gives us the perturbed stable manifold $W^s_{B_C}(P^0_{t_0})$ of the time 2π Poincaré map $P^0_{t_0}$. This is given by the following lemma.

Lemma 6.18

Let $0 < e(\mu)$ and $F_{t_0} : B_C \times [0, e(\mu)] \to B^e_{Ct_0}$ be the parameter and function given by the Theorem 6.16. For such $e(\mu)$ and F_{t_0} there exists a neighborhood U of the set of Lapunov orbits B_C and a C^∞ function $R^s_{t_0} : U \times [0, e(\mu)] \to \mathbb{R}^4$ such that

$$R^s_{t_0}(W^s_{B_C}(P^0_{t_0}), e) = W^s_{B^e_{Ct_0}}(P^e_{t_0}) \tag{6.56}$$

$$R^s_{t_0}(\cdot, e)|_{B_C} = F_{t_0}(\cdot, e) \tag{6.57}$$

$$R^s_{t_0}(W^s_x(P^0_{t_0}), e) = W^s_{F_{t_0}(x,e)}(P^e_{t_0}). \tag{6.58}$$

Proof

Let us note that the above Lemma is simply a reformulation of the Remark 6.8. The Remark 6.8 is a statement made for flows and the above Lemma is a mirror statement for a time 2π Poincaré map. The intuition behind the proof is the following. We will first apply the Remark 6.8 to the perturbed flow

$$\Phi^e_t : \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4 \times \mathbb{R}, \tag{6.59}$$

on the extended phase space (see equation (6.18) for the definition of Φ^e_t). Since on the additional time variable the flow Φ^e_t is simply the movement in time (which is independent from the choice of e), we will obtain our function $R^s_{t_0}$ and it's properties by an intersection of the results obtained through the Remark 6.8 for (6.59) with a Poincaré section $\Sigma_{t_0} = \{(x, t_0) | x \in \mathbb{R}^4\}$.

In the extended phase space we know that the set

$$\Lambda = B_C \times \mathbb{R} \tag{6.60}$$

is normally hyperbolic for the flow Φ^0_t and that by Lemma 6.9, for small $e > 0$ it perturbs to a set

$$\Lambda_e = \{(\Lambda_{t,e}, t)\, | \Lambda_{t,e} = B^e_{Ct}, t \in \mathbb{R}\}, \tag{6.61}$$

which is normally hyperbolic for the flow Φ^e_t. Let us define a function $F : \Lambda \times [0, e(\mu)] \to \mathbb{R}^4 \times \mathbb{R}$ as

$$F((x, t_0), e) = (F_{t_0}(x, e), t_0). \tag{6.62}$$

Since $F_{t_0}(x, e) = B^e_{Ct_0}$, the above defined F has the following property

$$F(\Lambda, e) = \Lambda_e. \tag{6.63}$$

In a small neighborhood V of Λ we can therefore apply the Remark 6.8 with the above function F to obtain a function

$$R : V \times I \to \mathbb{R}^4 \times \mathbb{R}, \tag{6.64}$$

for which

$$R^s(W^s_\Lambda(\Phi^0_t), e) = W^s_{\Lambda_e}(\Phi^e_t) \tag{6.65}$$

$$R^s(\cdot, e)|_\Lambda = F(\cdot, e) \tag{6.66}$$

$$R^s(W^s_{(x,t_0)}(\Phi^0_t), e) = W^s_{F((x,t_0),e)}(\Phi^e_t). \tag{6.67}$$

From the above function R^s we will construct our map $R^s_{t_0}$.

Let $\Sigma_{t_0} = \{(x,t_0)|x \in \mathbb{R}^4\}$ be the Poincaré section on the extended phase space. The stable manifold $W^s_{B^e_{Ct_0}}(P^e_{t_0})$ of the set $B^e_{Ct_0}$ for the time 2π Poincaré map $P^e_{t_0}$ is given by

$$W^s_{B^e_{Ct_0}}(P^e_{t_0}) \times \{t_0\} = W^s_{\Lambda_e}(\Phi^e_t) \cap \Sigma_{t_0}, \tag{6.68}$$

and it's foliation is

$$W^s_x(P^e_{t_0}) \times \{t_0\} = W^s_{(x,t_0)}(\Phi^e_t) \quad \text{for all } x \in B^e_{Ct_0}. \tag{6.69}$$

On the t coordinate the flow Φ^e_t is simply a constant velocity movement in time and is independent from e. This means that we have

$$W^s_{(x,t_0)}(\Phi^e_t) \subset \Sigma_{t_0} \quad \text{for all } e \geq 0. \tag{6.70}$$

From the definition of F (6.62) and from (6.70) we have

$$W^s_{F((x,t_0),e)}(\Phi^e_t) = W^s_{(F^e_{t_0}(x,e),t_0)}(\Phi^e_t) \subset \Sigma_{t_0}. \tag{6.71}$$

From (6.70) we know that any point (y,s) from $W^s_{(x,t_0)}(\Phi^0_t)$ is in fact equal to (y,t_0). Therefore for any (y,t_0) in $W^s_{(x,t_0)}(\Phi^0_t)$ from (6.67) and (6.71) we have

$$R^s((y,t_0),e) \in W^s_{F((x,t_0),e)}(\Phi^e_t) \subset \Sigma_{t_0},$$

which means that R^s is constant on the time variable and is therefore of the form

$$R^s((y,t_0),e) = (R^s_{t_0}(y,e),t_0). \tag{6.72}$$

The above $R^s_{t_0}$ is our desired function. The set U on which $R^s_{t_0}$ is defined is given as $U \times \{t_0\} = V \cap \Sigma_{t_0}$. We must now check that all the desired properties (6.56), (6.57), (6.58) hold for our $R^s_{t_0}$. From the equations (6.69), (6.67) and (6.71), for any $x \in B_C$ we have

$$\begin{aligned}(R^s_{t_0}(W^s_x(P^0_{t_0}),e),t_0) &= R^s((W^s_x(P^0_{t_0}),t_0),e)\\ &= R^s(W^s_{(x,t_0)}(\Phi^0_t),e)\\ &= W^s_{F((x,t_0),e)}(\Phi^e_t)\\ &= W^s_{(F_{t_0}(x,e),t_0)}(\Phi^e_t)\\ &= (W^s_{F_{t_0}(x,e)}(P^e_{t_0}),t_0),\end{aligned} \tag{6.73}$$

which means that we have shown (6.58). Summing up the above over all $x \in B_C$ gives us (6.56). The equation (6.57) follows from (6.66) which for $x \in B_C$ gives

$$(R^s_{t_0}(x,e),t_0) = R^s((x,t_0),e) = F((x,t_0),e) = (F_{t_0}(x,e),t_0). \tag{6.74}$$

□

Remark 6.19

Let us finish this section by observing that an analogous result to the Lemma 6.18 holds also for the perturbed unstable manifold $W^u_{B_C}(P^0_{t_0})$ of the Poincaré map $P^0_{t_0}$. We will use the notation $R^u_{t_0}$ for the mirror function of $R^s_{t_0}$. The proof of the result is analogous to the one given for Lemma 6.18.

7
The Melnikov method

In this Chapter we will apply the procedure sketched in Chapter 5. In the first section of the chapter we will discuss how the intersections of the invariant manifolds to the Lapunov orbits behave under perturbation from the PRC3BP to the PRE3BP. In the circular problem the stable and unstable manifolds of a given Lapunov orbit intersect with one another. In the elliptic problem it will turn out that these intersections do not automatically survive. This is because in the elliptic problem the solutions are not restricted to the invariant energy manifold and therefore an additional degree of freedom appears with the perturbation. The point of intersection of the PRC3BP will usually split into two points, one associated with the stable and the other with the unstable manifold of the perturbed Lapunov orbit.

In the second section we will discuss the distances between the perturbed orbits starting from the above mentioned perturbed intersection points and the unperturbed homoclinic orbits of the PRC3BP. It will turn out that the unperturbed homoclinic orbits provide a good approximation.

In the third section of the chapter we will apply the above mentioned approximations to obtain a Melnikov type method to determine whether the stable and the unstable manifolds of the perturbed Lapunov orbits in the PRE3BP intersect transversally. The main idea of the method is that the crucial role in the transversal intersections of the perturbed problem is played by the energy. If we can detect transversal intersections in the energy level then the transversal intersections in the full phase space will follow. This is because in the unperturbed PRC3BP the stable and unstable manifolds of the Lapunov orbits intersect transversally when restricted to the constant energy manifold. This constant energy manifold is of one dimension lower than the dimension of the phase space. With the perturbation another degree of freedom appears which is connected to the energy. If thansversality in energy is detected then the rest is obtained from the one dimension lower transversality of the unperturbed problem. The Melnikov integral (from the Theorem 7.9) will measure the leading term of the energy change at a potential intersection point and will be the indicator of a transversal intersection in energy. If the integral will turn out to be zero then an intersection will occur. The above is a very rough sketch of the idea which will be fully developed in the third section of the chapter.

The main result of this chapter is the Theorem 7.9, which will provide a tool for detecting transversal intersections of invariant manifolds of invariant tori which arise from the perturbed Lapunov orbits. Such intersections will later lead directly to the existence of Arnold diffusion as was discussed in Chapter 5, Section 5.3.

The derivation of our Melnikov type integral will be performed along the orbit q^0 homoclinic to L_2 in the PRC3BP. Let us note that to develop such a result we must work under the assumption that such a homoclinic orbit exists. This happens for the values μ_k given by the Theorem 2.1. Throughout this whole chapter we will therefore assume that the following discussion is made for a mass $\mu = \mu_k$. What is more we must choose $\mu = \mu_k$ to be sufficiently small (which is equivalent to choosing a large k) in order to know that most of the Lapunov orbits around L_2 persist under perturbation from PRC3BP to the PRE3BP (see Theorem 6.16).

Before we start let us recall some of the notations. From Section 4.2 we know that inside of the $R_b(\mu, C)$ region but separated form the large mass $1-\mu$ (see Figure 2.1, Lemma 4.3 and Remark 4.5) the Hamiltonian of the PRE3BP takes form

$$H^e(q,t) = H(q) + eG(q,t) + O(e^2), \tag{7.1}$$

where $q = (x, y, p_x, p_y) \in \mathbb{R}^4$, H is the Hamiltonian of the PRC3BP (2.1), G is 2π periodic over t and is given by formula (4.50)

$$G = \frac{1-\mu}{(r_1)^3} f(x,y,\mu,t) + \frac{\mu}{(r_2)^3} f(x,y,\mu-1,t), \tag{7.2}$$

where

$$\begin{aligned} r_1^2 &= (x-\mu)^2 + y^2 \\ r_2^2 &= (x+1-\mu)^2 + y^2 \\ f(x,y,\alpha,t) &= -y\alpha[3\sin t - \sin^3 t] + x\alpha[\cos t + \cos^3 t] - \alpha^2 \cos(t). \end{aligned} \tag{7.3}$$

By Lemma 4.3 the Hamiltonian H^e generates a differential equation

$$q' = f(q) + eg(q,t) + O(e^2) \tag{7.4}$$

where

$$f(q) = J\nabla H(q) \tag{7.5}$$

$$g(q,t) = J\nabla G(q,t). \tag{7.6}$$

Let us note that for $e = 0$ the equation (7.4) is the autonomous equation of the PRC3BP i.e.

$$q' = f(q) = J\nabla H(q). \tag{7.7}$$

Let $\Sigma_{\{y=0\}}$ denote a three dimensional hyperspace in $\mathbb{R}^4$

$$\Sigma_{\{y=0\}} = \{(x, 0, p_x, p_y)|x, p_x, p_y \in \mathbb{R}\} \subset \mathbb{R}^4. \tag{7.8}$$

Let $\Sigma_{t_0} = \{(q, t)|t = t_0\} \subset \mathbb{R}^4 \times \mathbb{R}$ be the global Poincaré section and

$$P^e_{t_0} : \Sigma_{t_0-2\pi} \to \Sigma_{t_0} \tag{7.9}$$

be the time 2π shift Poincaré map for the solution of (7.4).

In general we will stick to the following convention. If a notation for a function, or a point, or a set, has a superscript e, then this will indicate that it is derived from the equation (7.4), if it has a superscript 0 then it is derived from (7.7).

7.1 The intersections of invariant manifolds of the perturbed Lapunov orbits with the section $\{y = 0\}$

From Chapter 2 we know that in the PRC3BP for any Lapunov orbit $l(c)$ the stable and unstable manifolds $W^u(l(c), P^0_{t_0})$ and $W^s(l(c), P^0_{t_0})$ intersect on $\Sigma_{\{y=0\}}$ at a point $p^0_c = (x^0_c, 0, 0, \dot{y}^0_c)$. In this section we will show how these points of intersection behave under the perturbation from the PRC3BP to the PRE3BP.

From Remark 2.6 we know that the intersection of $W^s(l(c), P^0_{t_0})$ and $W^u(l(c), P^0_{t_0})$ is not transversal, which means that we cannot simply obtain a transversal intersection of the perturbed manifolds from simple perturbation arguments. Since the intersection for the unperturbed problem is not transversal it is possible that for some t_0 the manifolds $W^s(l^e_{t_0}(c), P^e_{t_0})$ and $W^u(l^e_{t_0}(c), P^e_{t_0})$ will not intersect close to the point

$$p^0_c = (x^0_c, 0, 0, \dot{y}^0_c) \in W^s(l(c), P^0_{t_0}) \cap W^u(l(c), P^0_{t_0}) \cap \Sigma_{\{y=0\}}.$$

On the other hand we can make use of the transversality for local projections in the $x, y, \dot{x}$ coordinates which we have from Remark 2.7. This will give us the following lemma.

Lemma 7.1

For $C < C_2$ sufficiently close to C_2 and for any $\tilde{C} \in (C, C_2)$ there exists an $e_0(\tilde{C}) > 0$, such that for all $e \in [0, e_0(\tilde{C})]$ and all $c \in \mathfrak{C} \cap [C, \tilde{C}]$ for which the Lapunov orbit $l(c)$ survives under perturbation, the intersection of the invariant manifolds $W^s(l^e_{t_0}(c), P^e_{t_0})$ and $W^u(l^e_{t_0}(c), P^e_{t_0})$ with the section $\Sigma_{\{y=0\}}$ is nonempty and homeomorphic to a circle. What is more the local projections of the manifolds onto the $x, y, \dot{x}$ coordinates near to the pint p^0_c, intersect transversally at a point $(x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c})$, i.e.

$$\Pi_{x,y,\dot{x}} \left(W^s(l^e_{t_0}(c), P^e_{t_0})\right) \pitchfork \Pi_{x,y,\dot{x}} \left(W^u(l^e_{t_0}(c), P^e_{t_0})\right) \tag{7.10}$$

$$(x_{t_0\,c}^{e},0,\dot{x}_{t_0\,c}^{e})\in\Pi_{x,y,\dot{x}}\left(W^{s}(l_{t_0}^{e}(c),P_{t_0}^{e})\right)\cap\Pi_{x,y,\dot{x}}\left(W^{u}(l_{t_0}^{e}(c),P_{t_0}^{e})\right),$$

which means that in the four dimensional space we have two points

$$p_{t_0\,c}^{e\,s}=(x_{t_0\,c}^{e},0,\dot{x}_{t_0\,c}^{e},\dot{y}_{t_0\,c}^{e\,s})\in W^{s}(l_{t_0}^{e}(c),P_{t_0}^{e})\cap\Sigma_{\{y=0\}} \tag{7.11}$$
$$p_{t_0\,c}^{e\,u}=(x_{t_0\,c}^{e},0,\dot{x}_{t_0\,c}^{e},\dot{y}_{t_0\,c}^{e\,u})\in W^{u}(l_{t_0}^{e}(c),P_{t_0}^{e})\cap\Sigma_{\{y=0\}}, \tag{7.12}$$

for some $\dot{y}_{t_0\,c}^{e\,s}$ and $\dot{y}_{t_0\,c}^{e\,u}$.

Proof

This comes directly from the Remark 2.7. For $c<C_2$ sufficiently close to C_2 we have the transversality of the intersection for the unperturbed problem with $e=0$. Transversality is stable and therefore for a sufficiently small perturbation e we obtain our claim. □

For a graphical representation of the above result we can look at Figures 5.7 and 5.8. Let us also note that a more rigorous proof of the above Lemma is going to be given during the proof of Lemma 7.2.

In the following sections of the chapter we will show that under appropriate circumstances the points $p_{t_0\,c}^{e\,s}$ and $p_{t_0\,c}^{e\,u}$ are equal to one another. In order to cary out the argument we will need to define the points $p_{t_0\,c}^{e\,s}$ and $p_{t_0\,c}^{e\,u}$ not only for these $c\in\mathfrak{C}\cap[C,\tilde{C}]$ for which the Lapunov orbits persist under perturbation, but for all $c\in[C,C_2]$. The first step is to extend the definition for $c\in[C,\tilde{C}]$. This is possible due to the following lemma.

Lemma 7.2

The definition of the points $p_{t_0\,c}^{e\,s}$ and $p_{t_0\,c}^{e\,u}$ (given in Lemma 7.1) can be C^1 smoothly extended from the values c for which the Lapunov orbits persist under perturbation onto all $c\in[C,\tilde{C}]$. i.e. there exist a number $e_0(\tilde{C})>0$ and two C^1 functions

$$p_{t_0}^{s}:[C,\tilde{C}]\times[0,e_0(\tilde{C})]\to\mathbb{R}^4 \tag{7.13}$$
$$p_{t_0}^{u}:[C,\tilde{C}]\times[0,e_0(\tilde{C})]\to\mathbb{R}^4,$$

such that for all $c\in\mathfrak{C}\cap[C,\tilde{C}]$ for which Lapunov orbits survive and $e\in[0,e_0(C)]$ we will have

$$p_{t_0}^{s}(c,e)=p_{t_0\,c}^{e\,s} \tag{7.14}$$
$$p_{t_0}^{u}(c,e)=p_{t_0\,c}^{e\,u}.$$

What is more the functions can be constructed so that for all $c\in[C,\tilde{C}]$ and $e\in[0,e_0(\tilde{C})]$ we have

$$p_{t_0}^{s}(c,e)=p_{c}^{0}+O(e) \tag{7.15}$$
$$p_{t_0}^{u}(c,e)=p_{c}^{0}+O(e)$$

and the bound $O(e)$ is independent from c.

Proof

The below presented proof is both a construction of the functions $p^s_{t_0}(c,e)$ and $p^s_{t_0}(c,e)$, and also a more rigorous proof of the Lemma 7.1 at the same time. Let V be a small neighborhood of the set $\{p^0_c : c \in [C,\tilde{C}]\}$. Let $B(0,1) \subset \mathbb{R}^2$ be a disc of radius one. Let us define a local coordinate system on $W^s(l(c),P^0_{t_0})$ and $W^u(l(c),P^0_{t_0})$ around the point of intersection p^0_c in V for $c \in [C,\tilde{C}]$ by the following maps

$$\begin{aligned}\pi^s &: B(0,1)\times[C,\tilde{C}] \to \mathbb{R}^4 \\ \pi^u &: B(0,1)\times[C,\tilde{C}] \to \mathbb{R}^4\end{aligned} \tag{7.16}$$

such that

$$\begin{aligned}\pi^s((0,0),c) = \pi^u((0,0),c) &= p^0_c \\ \pi^s(B(0,1),c) &\subset W^s(l(c),P^0_{t_0})\cap V \\ \pi^u(B(0,1),c) &\subset W^u(l(c),P^0_{t_0})\cap V,\end{aligned} \tag{7.17}$$

Let us note that the functions π^s and π^u can be chosen to be smooth. This follows from point three of Theorem 6.3 which gives us such smooth parameterizations in a small neighborhood of the set B_C of the Lapunov orbits. Such parameterizations can be carried into the neighborhood V of p^0_c by an appropriate iteration of the Poincaré map $P^0_{t_0}$.

We can use a similar argument to the above to also extend the functions $R^s_{t_0}$ and $R^u_{t_0}$ (defined in Lemma 6.18 and Remark 6.19) from a small neighborhood U of B_C onto the neighborhood V of the point p^0_c. To do this we first find a point x inside of the domain U of $R^s_{t_0}$ and a number $n>0$ such that $(P^0_{t_0})^n(p^0_c)=x$, then for any y from the neighborhood V of p^0_c we can define

$$R^s_{t_0}(y,e) = (P^e_{t_0})^{-n}\left(R^s_{t_0}((P^0_{t_0})^n(y),e)\right).$$

A similar argument can be applied for to extend the domain of the function $R^u_{t_0}$ onto the neighborhood V.

We are now ready to now define the following map

$$L : B(0,1)\times B(0,1)\times[C,\tilde{C}]\times[0,e_0(\tilde{C})] \to \mathbb{R}^4 \tag{7.18}$$

as

$$\begin{aligned}L(a,b,c,e) :=&(R^s_x(\pi^s(a,c),e) - R^u_x(\pi^u(b,c),e), \\ &R^s_{\dot{x}}(\pi^s(a,c),e) - R^u_{\dot{x}}(\pi^u(b,c),e), \\ &R^s_y(\pi^s(a,c),e), \\ &R^u_y(\pi^u(b,c),e)),\end{aligned} \tag{7.19}$$

where $a,b \in B(0,1)$ and the $R^s_{t_0} = (R^s_x, R^s_y, R^s_{\dot{x}}, R^s_{\dot{y}})$ and $R^u_{t_0} = (R^u_x, R^u_y, R^u_{\dot{x}}, R^u_{\dot{y}})$.

Let us note that for all $c_0 \in [C, C_2]$

$$L((0,0),(0,0),c_0,0) = 0. \tag{7.20}$$

We will show that for any $c_0 \in [C, C_2)$

$$\det(\frac{\partial L}{\partial (a,b)}((0,0),(0,0),c_0,0)) \neq 0. \tag{7.21}$$

Since $R^s_{t_0}(\cdot,0) = R^u_{t_0}(\cdot,0) = id$, at $((a_1,a_2),(b_1,b_2),c_0,e) = ((0,0),(0,0),c_0,0))$ we can compute

$$\det\left(\frac{\partial L}{\partial (a,b)}\right) = \begin{vmatrix} \frac{\partial \pi^s_x}{\partial a_1} & \frac{\partial \pi^s_x}{\partial a_2} & -\frac{\partial \pi^u_x}{\partial b_1} & -\frac{\partial \pi^u_x}{\partial b_2} \\ \frac{\partial \pi^s_{\dot{x}}}{\partial a_1} & \frac{\partial \pi^s_{\dot{x}}}{\partial a_2} & -\frac{\partial \pi^u_{\dot{x}}}{\partial b_1} & -\frac{\partial \pi^u_{\dot{x}}}{\partial b_2} \\ \frac{\partial \pi^s_y}{\partial a_1} & \frac{\partial \pi^s_y}{\partial a_2} & 0 & 0 \\ 0 & 0 & \frac{\partial \pi^u_y}{\partial b_1} & \frac{\partial \pi^u_y}{\partial b_2} \end{vmatrix}. \tag{7.22}$$

We know that both $W^s(l(c_0), P^0_{t_0})$ and $W^u(l(c_0), P^0_{t_0})$ intersect the section $\Sigma_{y=0}$ transversally at the point $\pi^s((0,0),c_0) = \pi^u((0,0),c_0)$. Therefore there exist $i,k \in \{1,2\}$ such that $\frac{\partial \pi^s_y}{\partial a_i}((0,0),c_0) \neq 0$ and $\frac{\partial \pi^u_y}{\partial a_k}((0,0),c_0) \neq 0$. Let us assume that $\frac{\partial \pi^s_y}{\partial a_1}((0,0),c_0) \neq 0$ and $\frac{\partial \pi^u_y}{\partial b_1}((0,0),c_0) \neq 0$. We can find two real coefficients β_1 and β_2 for which at $((0,0),c_0)$ we will have

$$\begin{aligned} \beta_1 \frac{\partial \pi^s_y}{\partial a_1} + \frac{\partial \pi^s_y}{\partial a_2} &= 0 \\ \beta_2 \frac{\partial \pi^u_y}{\partial b_1} + \frac{\partial \pi^u_y}{\partial b_2} &= 0. \end{aligned} \tag{7.23}$$

Let us consider the determinant at $((0,0),c_0)$ of the following matrix

$$\begin{vmatrix} \frac{\partial \pi^s_x}{\partial a_1} & \beta_1 \frac{\partial \pi^s_x}{\partial a_1} + \frac{\partial \pi^s_x}{\partial a_2} & -\frac{\partial \pi^u_x}{\partial b_1} & -\beta_2 \frac{\partial \pi^u_x}{\partial b_1} - \frac{\partial \pi^u_x}{\partial b_2} \\ \frac{\partial \pi^s_{\dot{x}}}{\partial a_1} & \beta_1 \frac{\partial \pi^s_{\dot{x}}}{\partial a_1} + \frac{\partial \pi^s_{\dot{x}}}{\partial a_2} & -\frac{\partial \pi^u_{\dot{x}}}{\partial b_1} & -\beta_2 \frac{\partial \pi^u_{\dot{x}}}{\partial b_1} - \frac{\partial \pi^u_{\dot{x}}}{\partial b_2} \\ \frac{\partial \pi^s_y}{\partial a_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{\partial \pi^u_y}{\partial b_1} & 0 \end{vmatrix} \tag{7.24}$$

Since on the $x, \dot{x}$ plane $W^s(l(c_0))$ and $W^u(l(c_0))$ intersect transversally at $\pi^s((0,0),c_0)$ $= \pi^u((0,0),c_0)$ the determinant of the matrix

$$\begin{vmatrix} \beta_1 \frac{\partial \pi^s_x}{\partial a_1} + \frac{\partial \pi^s_x}{\partial a_2} & \beta_2 \frac{\partial \pi^u_x}{\partial b_1} + \frac{\partial \pi^u_x}{\partial b_2} \\ \beta_1 \frac{\partial \pi^s_{\dot{x}}}{\partial a_1} + \frac{\partial \pi^s_{\dot{x}}}{\partial a_2} & \beta_2 \frac{\partial \pi^u_{\dot{x}}}{\partial b_1} + \frac{\partial \pi^u_{\dot{x}}}{\partial b_2} \end{vmatrix} \neq 0, \tag{7.25}$$

which together with (7.22) and (7.24) gives (7.21).

Since $\det(\frac{\partial L}{\partial (a,b)}((0,0),(0,0),c_0,0)) \neq 0$, for every $c_0 \in [C, C_2)$ we can apply the implicit function theorem to obtain a function

$$(a_{c_0}, b_{c_0}) : [c_0 - \delta, c_0 + \delta] \times [0, e_{c_0}] \to \mathbb{R}^2 \tag{7.26}$$

such that

$$L(a_{c_0}(c,e), b_{c_0}(c,e), c, e) = 0. \tag{7.27}$$

Since the chosen interval $[C, \tilde{C}]$ is compact we can find an $e_0(\tilde{C}) > 0$ and glue the above functions to obtain

$$(a,b) : [C, \tilde{C}] \times [0, e_0(\tilde{C})] \to \mathbb{R}^2. \tag{7.28}$$

We can now define our functions $p^s_{t_0}(c,e)$ and $p^u_{t_0}(c,e)$ as

$$\begin{aligned} p^s_{t_0}(c,e) &:= R^s_{t_0}(\pi^s(a(c,e),c),e) \\ p^u_{t_0}(c,e) &:= R^u_{t_0}(\pi^u(b(c,e),c),e). \end{aligned} \tag{7.29}$$

Let us note that from our construction and from the properties (6.58) and (6.54) of $R^s_{t_0}$ and F_{t_0} we know that for all $c \in \mathfrak{C} \cap [C, \tilde{C}]$ for which Lapunov orbits survive we have

$$\begin{aligned} p^s_{t_0}(c,e) &\in R^s_{t_0}(W^s(l(c),P^0_{t_0}),e) = W^s(F_{t_0}(l(c),e),P^e_{t_0}) = W^s(l^e_{t_0}(c),P^e_{t_0}) \\ p^u_{t_0}(c,e) &\in R^u_{t_0}(W^u(l(c),P^0_{t_0}),e) = W^u(F_{t_0}(l(c),e),P^e_{t_0}) = W^u(l^e_{t_0}(c),P^e_{t_0}). \end{aligned} \tag{7.30}$$

What is more both $p^s_{t_0}(c,e)$ and $p^u_{t_0}(c,e)$ lie in $\Sigma_{\{y=0\}}$ and are equal to one another on the first three coordinates $x, y, \dot{x}$. This means that for such c we have

$$\begin{aligned} p^s_{t_0}(c,e) &= p^{e\,s}_{t_0\,c} \\ p^u_{t_0}(c,e) &= p^{e\,u}_{t_0\,c}. \end{aligned} \tag{7.31}$$

The fact that the functions $p^s_{t_0}$ and $p^u_{t_0}$ are C^1 follows again from the implicit function theorem and the fact that our function L is C^1 (for an appropriate version of the implicit function theorem to our case see for example [28, Theorem 45 and 47]).

To prove the last statement of the Lemma, since $p^s_{t_0}(c,0) = p^u_{t_0}(c,0) = p^0_c$, we can compute

$$\begin{aligned} \left|p^s_{t_0}(c,e) - p^0_c\right| &\leq e \sup\{\left|Dp^s_{t_0}(c,e)\right| : c \in [C,\tilde{C}], e \in [0, e_0(\tilde{C})]\} \\ \left|p^u_{t_0}(c,e) - p^0_c\right| &\leq e \sup\{\left|Dp^u_{t_0}(c,e)\right| : c \in [C,\tilde{C}], e \in [0, e_0(\tilde{C})]\}. \end{aligned} \tag{7.32}$$

□

In the following discussion we will also need to define the functions $p^s_{t_0}(c,e)$ and $p^u_{t_0}(c,e)$ for the energies $c \in [\tilde{C}, C_2]$. We will therefore need to extend the above constructed $p^s_{t_0}(c,e)$ and $p^u_{t_0}(c,e)$ from the set $[C,\tilde{C}] \times [0, e_0(\tilde{C})]$ onto a larger set $[C, C_2] \times [0, e_0(\tilde{C})]$. Before we define our functions $p^s_{t_0}(c,e)$ and $p^u_{t_0}(c,e)$ let us first make an important remark.

Remark 7.3

It would be impossible to construct our functions $p^s_{t_0}$ and $p^u_{t_0}$ to be C^1 on the entire closed set $[\tilde{C}, C_2] \times [0, e_0(\tilde{C})]$. It is important that we must exclude the point C_2 from the C^1 on c requirement. This is because from Theorem 3.2 we know that close to the libration point L_2 the distance between the manifolds $W^i(l(c), P^0_{t_0})$ and $W^i(l(C_2), P^0_{t_0}) = W^i(L_2, P^0_{t_0})$ for $i \in \{s, u\}$, is equal to r, where $r = \sqrt{C_2 - c}$ is the radius of the orbit $l(c)$ in the ξ, η coordinates given by the Theorem 3.2. This means that this distance is not differentiable in terms of c at C_2. Therefore we can only require that the functions are differentiable in terms of r at C_2.

Let us start with the definition of the functions $p^s_{t_0}(c,e)$ and $p^s_{t_0}(c,e)$ with the definition for $c = C_2$ and $e \in [0, e_0(\tilde{C})]$. Since the stable and unstable manifolds $W^s(L_2, P^0_{t_0})$ and $W^u(L_2, P^0_{t_0})$ of L_2 are one dimensional and intersect the section $\Sigma_{y=0}$ transversally we can define

$$\begin{aligned} p^s_{t_0}(C_2, e) &= R^s_{t_0}(W^s(L_2, P^0_{t_0}), e) \cap \Sigma_{y=0} \\ p^u_{t_0}(C_2, e) &= R^u_{t_0}(W^u(L_2, P^0_{t_0}), e) \cap \Sigma_{y=0}, \end{aligned} \tag{7.33}$$

(See Figure 7.1). Now let us define the functions for any $(c,e) \in [C, C_2] \times [0, e_0(\tilde{C})]$. For any $c \in [\tilde{C}, C_2]$ for $e < e_c$ where e_c is given by (7.26) we can define $p^s_{t_0}(c,e)$ and $p^u_{t_0}(c,e)$ in the same fashion as was done in the proof of Lemma 7.2, by the formula (7.29)

$$\begin{aligned} p^s_{t_0}(c,e) &= R^s(\pi^s(a(c,e),c),e) \\ p^u_{t_0}(c,e) &= R^u(\pi^u(b(c,e),c),e). \end{aligned} \tag{7.34}$$

For $e_c \leq e \leq e_0(\tilde{C})$ we can choose $p^s_{t_0}(c,e)$ and $p^u_{t_0}(c,e)$ to be any points such that (See Figure 7.1)

$$\begin{aligned} p^s_{t_0}(c,e) &\in R^s_{t_0}(W^s(l(c), P^0_{t_0}), e) \cap \Sigma_{y=0} \\ p^u_{t_0}(c,e) &\in R^u_{t_0}(W^u(l(c), P^0_{t_0}), e) \cap \Sigma_{y=0} \end{aligned} \tag{7.35}$$

so long as the resulting function is smooth.

Remark 7.4

Let us note that for sufficiently small c and $e_c < e$ sufficiently large the sets $R^s_{t_0}(W^s(l(c), P^0_{t_0}), e) \cap \Sigma_{y=0}$ and $R^u_{t_0}(W^u(l(c), P^0_{t_0}), e) \cap \Sigma_{y=0}$ might become disjoint (see Figure 7.1), which means that we have considerable freedom on how we choose our functions. The functions are uniquely determined though for all (c,e) such that $c \in [C, C_2)$ and $e < e_c$ and for the points (C_2, e) with $e \in [0, e_0(\tilde{C})]$.

From the construction we can see that the above defined functions are C^1 and have the following properties. First of all on the set $[C, \tilde{C}] \times [0, e_0(\tilde{C})]$ all the properties listed in Lemma 7.2 still hold. From the definition, for $c \in [C, C_2)$ and for $e = 0$ we have

$$\begin{aligned} p^s_{t_0}(c,0) &= R^s(\pi^s(a(c,0),c),0) = p^0_c \\ p^u_{t_0}(c,e) &= R^u(\pi^u(b(c,0),c),0) = p^0_c. \end{aligned} \tag{7.36}$$

For $c = C_2$ and $e = 0$ we have the same result from the following equations

$$\begin{aligned} p^s_{t_0}(C_2, 0) &= R^s_{t_0}(W^s(L_2, P^0_{t_0}), 0) \cap \Sigma_{y=0} = W^s(L_2, P^0_{t_0}) \cap \Sigma_{y=0} = p^0_{C_2} \\ p^u_{t_0}(C_2, e) &= R^u_{t_0}(W^u(L_2, P^0_{t_0}), 0) \cap \Sigma_{y=0} = W^u(L_2, P^0_{t_0}) \cap \Sigma_{y=0} = p^0_{C_2}. \end{aligned} \tag{7.37}$$

What is more our construction guarantees that for $i \in \{s, u\}$

$$p^i_{t_0}(c,e) \in W^i(l^e_{t_0}(c), P^e_{t_0}). \tag{7.38}$$

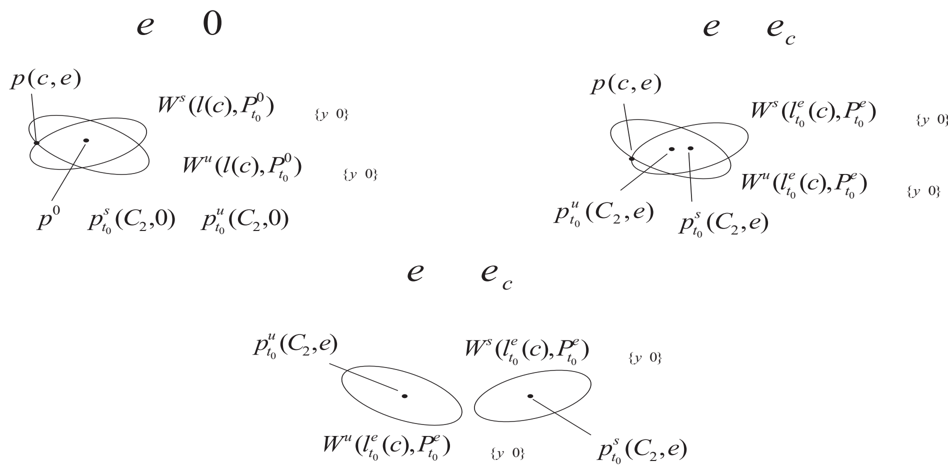

Figure 7.1 The choice of $p^s_{t_0}(c,e)$ and $p^u_{t_0}(c,e)$ for different e at $\Sigma_{\{y=0\}}$ drawn in the $x, \dot{x}$ coordinates.

Finally, for any $(c,e) \in [C, C_2] \times [0, e_0(\tilde{C})]$ we have

$$\left|p^i_{t_0}(c,e) - p^0_c\right| \le e \sup\{\left|\frac{d}{de} p^i_{t_0}(c,e)\right| : e \in [0, e_0(\tilde{C})]\} \tag{7.39}$$

which means that

$$p^i_{t_0}(c,e) = p^0_c + O(e), \tag{7.40}$$

for $i \in \{s,u\}$.

7.2 Distances between the homoclinic orbits and their perturbations

In this section we will consider the orbits of the PRE3BP which start from the points $p^s_{t_0}(c,e)$ and $p^u_{t_0}(c,e)$ which have been constructed in the previous section. It will turn out that these solutions can be approximated by the homoclinic orbits to the Lapunov orbit $l(c)$ of the PRC3BP.

Let us start this section by discussing the distance between the points p^0 and p^0_c where

$$p^0 = p^s_{t_0}(C_2, e=0) = p^u_{t_0}(C_2, 0)$$
$$p^0_c = p^s_{t_0}(c, e=0) = p^u_{t_0}(c, 0).$$

This is given by the following lemma which discusses the distance between the unperturbed orbits $q_c(t,t_0)$ and $q^0(t)$, where the orbits are homoclinic to $l(c)$ and L_2 and pass through the points p^0_c and p^0 respectively (see Definitions 2.4 and 2.5 for the definition of $q^0(t)$ and $q^0_c(t,t_0)$).

Lemma 7.5

For $c < C_2$ sufficiently close to C_2 the trajectory $q_c^0(t, t_0)$ lies close to the trajectory q^0 i.e.

$$\left|q^0(t - t_0) - q_c^0(t, t_0)\right| = O(\Delta c), \tag{7.41}$$

where $\Delta c = \sqrt{C_2 - c}$. In particular for $t = t_0$ we have

$$\left|p_c^0 - p^0\right| = O(\Delta c). \tag{7.42}$$

Proof

For c close to C_2 the large mass $1 - \mu$ is separated from the zero velocity curve by an invariant torus [10],[21]. Any orbit outside of this torus and on the right hand side of the Libration point L_2 (for example the orbit q^0 or q_c^0) is separated from both of the two masses μ and $1 - \mu$. Looking at the equations of motion of the PRC3BP (3.1) we can see that the vector field $J\nabla H$ is bounded outside of this torus and on the right of L_2 by some positive number L

$$|J\nabla H| \leq L. \tag{7.43}$$

Let us consider a box U around the Libration point L_2. Let the box in the Lapunov coordinates $\xi_1, \xi_2, \eta_1, \eta_2$, given by the Lapunov Theorem 3.2, be of the form

$$U = \{(\xi_1, \xi_2, \eta_1, \eta_2) |\, |\xi_1|\,, |\eta_1| \leq M_1, \;\; |\xi_2|\,, |\eta_2| \leq M_2\}, \tag{7.44}$$

where M_1 and M_2 are small positive numbers. Let us recall that in these coordinates the Lapunov orbits are given by (see (3.39))

$$l(c)(t) = l_r(t) = (0, re^{ta_2(0,ir^2)+i\phi}, 0, ire^{-ta_2(0,ir^2)-i\phi}), \tag{7.45}$$

where $c = h(r)$ (see Section 3.2 for more details and (3.111) for the definition of h) and that by Lemma 3.16 we know that

$$\operatorname{dist}(l(c), L_2) = O(\Delta c). \tag{7.46}$$

Let x_0 be the first intersection of $q^0(t)$ with the boundary of U

$$x_0 \in W^u(L_2) \cap \partial U. \tag{7.47}$$

There exists a $T_0 < t_0$ such that

$$x_0 = q^0(T_0 - t_0). \tag{7.48}$$

Let x_c be the first intersection of $q_c^0(t, t_0)$, starting from the neighborhood of $l(c)$ with the boundary of U, and let $T_c < t_0$ be the time of this intersection

$$x_c = q_c^0(T_c, t_0) \in \partial U. \tag{7.49}$$

In the ξ, η coordinates $(\xi_1, \xi_2, \eta_1, \eta_2)$ the points x_0 and x_c will be of the form (see equation (3.38)) $x_0 = (M_1, 0, 0, 0)^\top$, $x_c = (M_1, re^{i\phi}, 0, ire^{-i\phi})^\top$ for some

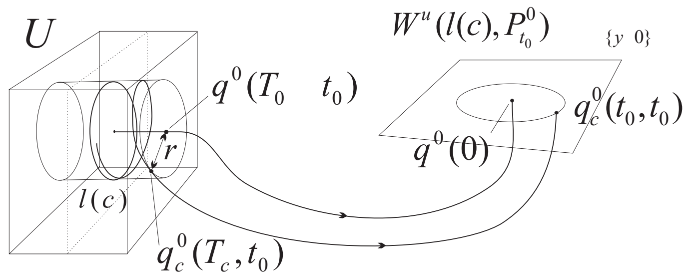

Figure 7.2 The picture of the layout of the orbits for the proof of Lemma 7.5.

$\phi \in \mathbb{R}$ and $r = h^{-1}(c)$. The formulas for $q^0(T_0 - t_0 + t)$ and $q^0_c(T_c + t, t_0)$ inside of the set U will take form

$$q^0(T_0 - t_0 + t) = \begin{pmatrix} M_1 e^{ta_1(0)} \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.50}$$

$$q^0_c(T_c + t, t_0) = \begin{pmatrix} M_1 e^{ta_1(r^2)} \\ re^{ta_2(ir^2)+i\phi} \\ 0 \\ ire^{-ta_2(ir^2)-i\phi} \end{pmatrix}. \tag{7.51}$$

Inside of the set U the distance between $q^0(T_0 - t_0 + t)$ and $q^0_c(T_c + t, t_0)$ on the ξ_2, η_2 coordinates is equal to r which is equal to $\sqrt{C_2 - c}$. On the ξ_1 coordinates inside U we have $t < 0$ and the distance can be estimated by

$$\begin{aligned} \left| M_1 e^{ta_1(0)} - M_1 e^{ta_1(r^2)} \right| &\leq M_1 \sup_{\rho \in [0,r]} \{te^{ta_1(\rho^2)}\} \left| a_1(0) - a_1(r^2) \right| \\ &\leq M_1 \sup_{\rho \in [0,R]} \{te^{ta_1(\rho^2)}\} \left| a_1(0) - a_1(r^2) \right| \\ &= O(r^2) \\ &= O(|C_2 - c|). \end{aligned} \tag{7.52}$$

This means that in the x, y, p_x, p_y coordinates this distance is going to be

$$|q^0(T_0 - t_0 + t) - q^0_c(T_c + t, t_0)| \leq \tilde{L} \Delta c, \quad \text{for } t \leq 0, \tag{7.53}$$

where $\tilde{L}$ is the Lipschitz constant of the coordinate change from ξ, η to x, y, p_x, p_y in the box U and Δc is equal to $\sqrt{C_2 - c}$.

Let us now define a function F as

$$\begin{aligned} F : V \times [t_0 - T_0 - \delta, t_0 - T_0 + \delta] &\to \mathbb{R}^4 \\ F(x, t) &= \phi_2(x, t) \end{aligned} \tag{7.54}$$

where $V \subset \partial U$ is a small neighborhood of the point x_0, δ is a small positive number and $\phi(x,t) = (\phi_1, \phi_2, \phi_3, \phi_4)(x,t)$ is the dynamical system generated by the equation of the PRC3BP

$$q' = J\nabla H(q). \tag{7.55}$$

From the choice of x_0 and T_0 we know that

$$F(x_0, t_0 - T_0) = \phi_2(x_0, t_0 - T_0) = q_2^0(0) = 0 \tag{7.56}$$

where q_2^0 is the y coordinate of our homoclinic orbit q^0. What is more, from the fact that $q^0(t)$ intersects the section $\Sigma_{\{y=0\}}$ transversally we know that

$$\frac{\partial F}{\partial t}(x_0, t_0 - T_0) = \left(q_2^0\right)'(0) \neq 0, \tag{7.57}$$

and therefore by the implicit function theorem there exists a smooth function

$$\tau : V \to \mathbb{R} \tag{7.58}$$

such that

$$\begin{aligned} F(x, \tau(x)) &= 0 \\ \tau(x_0) &= t_0 - T_0. \end{aligned} \tag{7.59}$$

Let us note that the fact that $F(x,\tau(x)) = 0$ is equivalent to

$$\phi(x, \tau(x)) \in \Sigma_{\{y=0\}}, \tag{7.60}$$

which in the case of x_c means that

$$\phi(x_c, \tau(x_c)) = q_c^0(t_0, t_0) \in \Sigma_{\{y=0\}}. \tag{7.61}$$

Combining the above with the fact that $x_c = q_c^0(T_c, t_0)$ we obtain

$$\tau(x_c) = t_0 - T_c. \tag{7.62}$$

From the fact that τ is smooth we can see that for any x from V we will have

$$\tau(x) = \tau(x_0) + O(|x - x_0|). \tag{7.63}$$

From (7.53) with $t = 0$ we know that

$$\begin{aligned} |x_0 - x_c| &= |q^0(T_0 - t_0) - q_c^0(T_c, t_0)| \\ &= O(\Delta c), \end{aligned} \tag{7.64}$$

and from (7.62) together with (7.63), (7.64) and (7.59) we can see that

$$\begin{aligned} T_c &= t_0 - \tau(x_c) \\ &= t_0 - \tau(x_0) + O(|x - x_0|) \\ &= T_0 + O(\Delta c). \end{aligned} \tag{7.65}$$

We can now use (7.64) and (7.65) to obtain an estimate

$$\begin{aligned}\left|q^0(T_0-t_0)-q_c^0(T_0,t_0)\right| &\le \left|q^0(T_0-t_0)-q_c^0(T_c,t_0)\right| \\ &\quad + |q_c^0(T_c,t_0)-q_c^0(T_0,t_0)| \\ &= O(\Delta c)+|q_c^0(T_0+O(\Delta c),t_0)-q_c^0(T_0,t_0)| \\ &\le O(\Delta c)+L\,O(\Delta c) \\ &= O(\Delta c),\end{aligned} \tag{7.66}$$

where L is the bound defined at the beginning of the proof. Using the above estimate we obtain our claim for $T_0 \le t \le t_0$ by computing

$$\begin{aligned}\left|q^0(t-t_0)-q_c^0(t,t_0)\right| &\le \left|q^0(T_0-t_0)-q_c^0(T_0,t_0)\right| e^{L(t_0-T_0)} \\ &= O(\Delta c).\end{aligned} \tag{7.67}$$

For $t \le T_0$ we can use (7.53) and (7.65) to obtain

$$\begin{aligned}\left|q^0(t-t_0)-q_c^0(t,t_0)\right| &\le |q^0(t-t_0)-q_c^0(T_c-T_0+t,t_0)| \\ &\quad + |q_c^0(T_c-T_0+t,t_0)-q_c^0(t,t_0)| \\ &= |q^0(T_0-t_0+(t-T_0))-q_c^0(T_c+(t-T_0)\,,t_0)| \\ &\quad + |q_c^0(t+O(\Delta c),t_0)-q_c^0(t,t_0)| \\ &\le O(\Delta c)+L\,O(\Delta c) \\ &= O(\Delta c).\end{aligned} \tag{7.68}$$

We have therefore proved (7.41) for $t \le t_0$. For $t \ge t_0$ the proof is analogous. □

Let us now introduce notations for the solutions of the PRE3BP starting from the points $p^s_{t_0}(c,e)$ and $p^s_{t_0}(c,e)$.

Definition 7.6

For $c \in [C, C_2]$ and $e \in [0, e_0(\tilde{C})]$ let us define the functions $q^e_{cs}(t,t_0)$ and $q^e_{cu}(t,t_0)$ to be the solutions of (7.1) starting from the points $p^s_{t_0}(c,e)$ and $p^u_{t_0}(c,e)$ at the time t_0

$$\begin{aligned} q^e_{cs}(\cdot,t_0) &: \mathbb{R} \to \mathbb{R}^4 \\ q^e_{cu}(\cdot,t_0) &: \mathbb{R} \to \mathbb{R}^4 \\ q^e_{cs}(t_0,t_0) &= p^s_{t_0}(c,e) \\ q^e_{cu}(t_0,t_0) &= p^u_{t_0}(c,e).\end{aligned} \tag{7.69}$$

In the following lemma we will show that these solutions will be close to the symmetrical homoclinic orbit $q_c^0(t,t_0)$, starting from

$$p_c^0 = p^s_{t_0}(c,0) = p^u_{t_0}(c,0)$$

at t_0, of the Lapunov orbit $l(c)$ for the unperturbed problem (see Definitions 2.4 and 2.5 for the definition of $q_c^0(t,t_0)$).

Lemma 7.7

For $c \in [C, C_2]$ and $e \in [0, e_0(\tilde{C})]$ the solutions $q^e_{cs}(t,t_0)$ and $q^e_{cu}(t,t_0)$ can be expressed as

$$q^e_{cs}(t,t_0) = q^0_c(t,t_0) + O(e) \quad \text{for } t \in [t_0, +\infty) \tag{7.70}$$

$$q^e_{cu}(t,t_0) = q^0_c(t,t_0) + O(e) \quad \text{for } t \in (-\infty, t_0]. \tag{7.71}$$

What is more the bounds $O(e)$ are uniform on the intervals $[t_0, +\infty)$ and $(-\infty, t_0]$ and independent from the choice of c.

Proof

Let us prove (7.71), the proof of (7.70) will be analogous. From (7.40) we know that

$$p^u_{t_0}(c,e) = p^0_c + O(e). \tag{7.72}$$

The above means that we already have the result for $t = t_0$

$$\begin{aligned} |q^e_{cu}(t_0,t_0) - q^0_c(t_0,t_0)| &= |p^u_{t_0}(c,e) - p^0_c| \\ &= O(e). \end{aligned} \tag{7.73}$$

Using the differential equation (7.4) for $t \leq t_0$ we have

$$\begin{aligned} |q^e_{cu}(t,t_0) - q^0_c(t,t_0)| &\leq |q^e_{cu}(t_0,t_0) - q^0_c(t_0,t_0)| \\ &\quad + \int_t^{t_0} |f(q^0_c(s,t_0)) - f(q^e_{cu}(s,t_0))| ds \\ &\quad + e\int_t^{t_0} |g(q^e_{cu}(s,t_0)), s)| + O(e^2)(t_0 - t) \\ &\leq |q^e_{cu}(t_0,t_0) - q^0_c(t_0,t_0)| \\ &\quad + \int_t^{t_0} L|q^0_c(s,t_0)) - q^e_{cu}(s,t_0)| ds \\ &\quad + (eM + O(e^2))(t_0 - t) \end{aligned} \tag{7.74}$$

where f and g are given by (7.5), (7.6), L is the bound for f defined at the beginning of the proof of Lemma 7.5 and M is the bound for g outside of the torus separating the larger mass $1-\mu$ and on the right of the libration point L_2. Applying the Gronwall Lemma 1.10 with

$$\begin{aligned} u(t) &= L \\ c(t) &= (eM + O(e^2))(t_0 - t), \end{aligned} \tag{7.75}$$

we obtain an estimate

$$\begin{aligned} |q^e_{cu}(t,t_0) - q^0_c(t,t_0)| &\leq |q^e_{cu}(t_0,t_0) - q^0_c(t_0,t_0)| \int_t^{t_0} L ds \\ &\quad + \int_t^{t_0} \left(eM + O(e^2)\right) \exp \int_t^s L d\tau ds \\ &= O(e)L(t_0 - t) + \frac{1}{L}\left(\exp L\,(t_0 - t) - 1\right)\left(eM + O(e^2)\right). \end{aligned} \tag{7.76}$$

From the above it is clear that if we fix a time T then for $T \leq t \leq t_0$ we will have

$$|q^e_{cu}(t,t_0) - q^0_c(t,t_0)| \leq O(e). \tag{7.77}$$

If we choose T sufficiently small then we will end up in a neighborhood of the periodic orbit $l(c)$ and the quasi periodic invariant torus $l^e_{t_0}(c)$. From Theorem 6.16 we know that $l(c)$ and $l^e_{t_0}(c)$ are $O(e)$ close. What is more both of them belong to normally hyperbolic sets and therefore in a fixed neighborhood U' the movement along the orbits from their unstable manifolds is dominated by the exponential terms and by the rotation. We can choose sufficiently small T so that for $t < T$ $q^0_c(t,t_0)$ is contained in this neighborhood. The same will be true for $q^e_{cu}(t,t_0)$ for $t < T$ and e sufficiently small. By the KAM Theorem 6.13 we know that the rotation on $l^e_{t_0}(c)$ is $O(e)$ conjugate to the rotation on $l(c)$ and therefore for $t < T$ we will have

$$|q^e_{cu}(t,t_0) - q^0_c(t,t_0)| \leq O(e). \tag{7.78}$$

We have proved (7.71). The proof of (7.70) is analogous.

Let us note that in the above argument the main tool used was the normal hyperbolicity of the set of Lapunov orbits, together with the KAM theorem and Gronwall estimates. Through the course of the proof the choice of c did not play an important role and therefore the bound $O(e)$ is uniform for all c. □

We will finish the section with yet another approximation of the orbits $q^e_{cs}(t,t_0)$ and $q^e_{cs}(t,t_0)$. This approximation will play an important role in the proof of the Melnikov method in the following section.

Lemma 7.8

For $c \in [C, C_2]$ and $e \in [0, e_0(\tilde{C})]$ the solutions $q^e_{Cs}(t,t_0)$ and $q^e_{Cu}(t,t_0)$ can be expressed as

$$q^e_{cs}(t,t_0) = q_c(t,t_0) + eQ^s_c(t,t_0) + o(e) \quad \text{for } t \in [t_0, t_0 + 2\pi] \tag{7.79}$$
$$q^e_{cu}(t,t_0) = q_c(t,t_0) + eQ^u_c(t,t_0) + o(e) \quad \text{for } t \in [t_0 - 2\pi, t_0], \tag{7.80}$$

and

$$Q^s_c(t,t_0) = Q_s(t,t_0) + O(\Delta c) \quad \text{for } t \in [t_0, t_0 + 2\pi] \tag{7.81}$$
$$Q^u_c(t,t_0) = Q_u(t,t_0) + O(\Delta c) \quad \text{for } t \in [t_0 - 2\pi, t_0], \tag{7.82}$$

where $\Delta c = \sqrt{C_2 - c}$ and the bounds are independent from t_0. What is more $Q_s(t,t_0)$ and $Q_u(t,t_0)$ are bounded for $t \in [t_0, +\infty)$ and $t \in (-\infty, t_0]$ respectively and satisfy the following equations

$$\dot{Q}_s(t,t_0) = Df\left(q^0(t-t_0)\right) Q_s(t,t_0) + g(q^0(t-t_0), t) \quad \text{for } t \in [t_0, \infty) \tag{7.83}$$
$$\dot{Q}_u(t,t_0) = Df\left(q^0(t-t_0)\right) Q_u(t,t_0) + g(q^0(t-t_0), t) \quad \text{for } t \in (-\infty, t_0], \tag{7.84}$$

where f and g come from (7.5) and (7.6).

Proof

We will prove (7.79) and (7.83). In the case of (7.80) and (7.84) the proof will be analogous. From the C^1 continuity of solutions with respect to initial conditions (see for example [26, Theorem 12.1]), the fact that the initial conditions $p^s_{t_0}(c,e)$ and $p^u_{t_0}(c,e)$ are C^1 implies that $q^e_{cs}(t,t_0)$ and $q^e_{cs}(t,t_0)$ are also C^1 in terms of c and e (and in terms of $r = \Delta c = \sqrt{C_2 - c}$ at C_2 - see Remark 7.3). Let us define the function $Q^s_c(t,t_0)$ as

$$Q^s_c(t,t_0) := \frac{\partial}{\partial e} q^e_{cs}(t,t_0)|_{e=0}. \tag{7.85}$$

It is clear that if we fix a $T > 0$ (in particular the following will be true for $T = 2\pi$) then for $t \in [t_0, t_0 + T]$

$$\begin{aligned} q^e_{cs}(t,t_0) &= q^0_{c,s}(t,t_0) + e\frac{\partial}{\partial e} q^e_{cs}(t,t_0)|_{e=0} + o(e) \\ &= q^0_c(t,t_0) + eQ^s_c(t,t_0) + o(e). \end{aligned} \tag{7.86}$$

We have therefore shown (7.79). In Lemma 7.7 where we have shown that

$$|q^e_{cs}(t,t_0) - q^0_c(t,t_0)| = O(e), \tag{7.87}$$

where the bound $O(e)$ is uniform on the interval $[t_0, \infty)$. This together with (7.79) gives us

$$eQ^s_c(t,t_0) + o(e) = O(e). \tag{7.88}$$

By dividing both sides by e we obtain

$$|Q^s_C(t,t_0) + o(e)/e| \leq M \tag{7.89}$$

where $o(e)$ is on the bounded interval $[t_0, T]$ and M is independent from T. Passing with e to zero gives us a uniform bound on $Q^s_C(t,t_0)$ for all $t \in [t_0, \infty)$

$$|Q^s_C(t,t_0)| \leq M. \tag{7.90}$$

Let us now define $Q_s(t,t_0)$ as

$$Q_s(t,t_0) := \frac{\partial}{\partial e} q^e_{C_2 s}(t,t_0)|_{e=0} = Q^s_{C_2}(t,t_0). \tag{7.91}$$

Clearly for $t \in [t_0, t_0 + 2\pi]$ we have

$$\begin{aligned} Q^s_c(t,t_0) &= \frac{\partial}{\partial e} q^e_{cs}(t,t_0)|_{e=0} \\ &= \frac{\partial}{\partial e} q^e_{C_2 s}(t,t_0)|_{e=0} + (\Delta c)\frac{\partial}{\partial(\Delta c)}\frac{\partial}{\partial e} q^e_{C_2 s}(t,t_0)|_{e=0} + o(\Delta c) \\ &= Q_s(t,t_0) + O(\Delta c). \end{aligned} \tag{7.92}$$

Let us note that by Remark 7.3 the fact that we are differentiating by $\Delta c = \sqrt{C_2 - c}$ at C_2 does not produce an error since Δc is simply the radius r of the Lapunov orbit $l(c)$ where $c = h(r)$ (for the definition of $h(r)$ please refer to (3.111)), so differentiating by Δc is equivalent to differentiating by r. Let us note that we have already shown that for each c the function $Q^s_c(t,t_0)$ is uniformly bounded for $t \in [t_0, \infty)$, which means that in particular this also applies to our $Q_s(t,t_0) = Q^s_{C_2}(t,t_0)$.

Let us show that the above defined $Q_s(t,t_0)$ satisfy (7.83). We can do this by computing

$$\begin{aligned}\dot{Q}_s(t,t_0) &= \frac{\partial}{\partial e}\dot{q}^e_{C_2 s}(t,t_0)|_{e=0} \\ &= \frac{\partial}{\partial e}(f(q^e_{C_2 s}(t,t_0)) + eg(q^e_{C_2 s}(t,t_0),t))|_{e=0} \\ &= Df(q^0_{C_2 s}(t,t_0))\frac{\partial}{\partial e}q^e_{C_2 s}(t,t_0)|_{e=0} + g(q^0_{C_2 s}(t,t_0),t) \\ &= Df(q^0(t-t_0))Q_s(t,t_0) + g(q^0(t-t_0),t).\end{aligned} \tag{7.93}$$

To finish off the argument let us show that the bounds can be chosen to be independent from t_0. From the fact that the PRE3BP is 2π periodic in time follows the fact that the function $q^e_{cs}(t,t_0)$ is 2π periodic in t_0. The above bounds have been obtained for a given t_0, but taking their maximum over $t_0 \in [0,2\pi]$ we obtain bounds independent from t_0. □

7.3 The Melnikov type method for finding the intersections between stable and unstable manifolds in the PRE3BP

In this section we will prove a theorem which will allow us to determine whether the stable and unstable manifolds $W^s(l^e_{t_0}(c), P^e_{t_0})$ and $W^u(l^e_{t_0}(c), P^e_{t_0})$ intersect transversally. The theorem is based on a Melnikov type approach which will allow us to detect the values of t_0 at which we will have the transversal intersection at the point $p^{e\,s}_{t_0\,c} = p^{e\,u}_{t_0\,c}$ (where the points $p^{e\,s}_{t_0\,c}$ and $p^{e\,u}_{t_0\,c}$ are defined by the Lemma 7.1). The Melnikov function $M(t_0)$, will be the leading term in the expansion over e of the difference of the energies $H(p^{e\,s}_{t_0\,c})$ and $H(p^{e\,u}_{t_0\,c})$ of the two points $p^{e\,s}_{t_0\,c}$ and $p^{e\,u}_{t_0\,c}$

$$H(p^{e\,s}_{t_0\,c}) - H(p^{e\,u}_{t_0\,c}) = eM(t_0) + O(e\Delta c) + o(e). \tag{7.94}$$

Showing that the energies $H(p^{e\,s}_{t_0\,c})$ and $H(p^{e\,s}_{t_0\,c})$ are equal to one another at some value t_0 will be the decisive factor for the existence of the transversal intersection of the manifolds $W^s(l^e_{t_0}(c), P^e_{t_0})$ and $W^u(l^e_{t_0}(c), P^e_{t_0})$. Let us observe that from the fact that the points $p^{e\,s}_{t_0\,c}$ and $p^{e\,u}_{t_0\,c}$ are 2π periodic in t_0 we can expect our function to turn out to be 2π periodic. It will turn out that if the function $M(t_0)$ will have a simple zero at $t_0 = T$ i.e.

$$\begin{aligned} M(T) &= 0 \\ \frac{dM}{dt_0}(T) &\neq 0, \end{aligned} \tag{7.95}$$

then the transversal intersection in energy and the transversal intersection of the manifolds $W^s(l^e_{t_0}(c), P^e_{t_0})$ and $W^u(l^e_{t_0}(c), P^e_{t_0})$ will follow.

The following theorem is the main result of this chapter and also one of the key results for proving the existence of Arnold diffusion in the PRE3BP.

Theorem 7.9

If

$$M(t_0) = \int_{-\infty}^{+\infty} \{H, G\}(q^0(t - t_0), t)dt \tag{7.96}$$

has simple zeros then for $C < C_2$ sufficiently close to C_2 and any $\tilde{C} \in (C, C_2)$ there exists an $e_0(\tilde{C})$ such that for all $e \in [0, e_0(\tilde{C})]$ and all $c \in \mathfrak{C} \cap [C, \tilde{C}]$ for which $l(c)$ survive under perturbation, the manifolds $W^s(l^e_{t_0}(c), P^e_{t_0})$ and $W^u(l^e_{t_0}(c), P^e_{t_0})$ intersects transversally.

Proof

Let us first recall that from Lemma 7.1 we already know that the projection onto the $x, y, \dot{x}$ coordinates gives a transversal intersection

$$\Pi_{x,y,\dot{x}} \left(W^s(l^e_{t_0}(c), P^e_{t_0})\right) \pitchfork \Pi_{x,y,\dot{x}} \left(W^u(l^e_{t_0}(c), P^e_{t_0})\right) \tag{7.97}$$

and that

$$(x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}) \in \Pi_{x,y,\dot{x}} \left(W^s(l^e_{t_0}(c), P^e_{t_0})\right) \cap \Pi_{x,y,\dot{x}} \left(W^u(l^e_{t_0}(c), P^e_{t_0})\right).$$

We also know that

$$p^{e\,s}_{t_0\,c} = (x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^{e\,s}_{t_0\,c}) \in W^s(l^e_{t_0}(c), P^e_{t_0}) \cap \Sigma_{\{y=0\}} \tag{7.98}$$

$$p^{e\,u}_{t_0\,c} = (x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^{e\,u}_{t_0\,c}) \in W^u(l^e_{t_0}(c), P^e_{t_0}) \cap \Sigma_{\{y=0\}}. \tag{7.99}$$

We will try to find our intersection at a point $(x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^e_{t_0\,c})$. This means that in order to prove that $W^s(l^e_{t_0}(c), P^e_{t_0})$ and $W^u(l^e_{t_0}(c), P^e_{t_0})$ intersect in the full four-dimensional space we shall have to show that for some t_0

$$(x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^{e\,s}_{t_0\,c}) = (x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^{e\,u}_{t_0\,c}). \tag{7.100}$$

From the equations (2.1) and (2.5) we know that $\dot{y}$ can be computed from the formula

$$\dot{y} = \pm\sqrt{2\Omega(x, y) - \dot{x}^2 - 2H}. \tag{7.101}$$

We know that for $e = 0$ we have

$$\dot{y}^{e\,s}_{t_0\,c} = \dot{y}^{e\,u}_{t_0\,c} \quad \text{for } e = 0 \tag{7.102}$$

and therefore for sufficiently small e from the fact that

$$H(x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^{e\,s}_{t_0\,c}) = H(x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^{e\,u}_{t_0\,c})$$

by using (7.101) we can conclude that $\dot{y}^{e\,s}_{t_0\,c} = \dot{y}^{e\,u}_{t_0\,c}$. Since

$$\begin{aligned} (x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^{e\,s}_{t_0\,c}) &= p^{e\,s}_{t_0\,c} = q^e_{cs}(t_0, t_0) \\ (x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^{e\,u}_{t_0\,c}) &= p^{e\,u}_{t_0\,c} = q^e_{cu}(t_0, t_0), \end{aligned} \tag{7.103}$$

from the above argument we can see that in order to check whether

$$(x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^{e\,s}_{t_0\,c}) = (x^e_{t_0\,c}, 0, \dot{x}^e_{t_0\,c}, \dot{y}^{e\,u}_{t_0\,c}) \tag{7.104}$$

we can compute

$$d(t_0) = H(q^e_{cu}(t_0, t_0)) - H(q^e_{cs}(t_0, t_0)) \tag{7.105}$$

and find a t_0 such that

$$d(t_0) = 0. \tag{7.106}$$

Let $\cdot$ denote the scalar product and let Δ_s and Δ_u denote the following functions

$$\begin{aligned} \Delta_s(t, t_0) &:= \nabla H(q^0(t - t_0)) \cdot Q_s(t, t_0) \\ \Delta_u(t, t_0) &:= \nabla H(q^0(t - t_0)) \cdot Q_u(t, t_0). \end{aligned} \tag{7.107}$$

We can use Lemmas 7.8 and 7.5, and the fact that $H = H_c$ is constant along the solution q^0_c to compute

$$\begin{aligned} H(q^e_{cs}(t_0, t_0)) &= H(q_c(t_0, t_0)) + e\nabla H(q_c(t_0, t_0)) \cdot Q^s_c(t_0, t_0) + o(e) \\ &= H_c + e\nabla H(q^0(0) + O(\Delta c)) \cdot (Q_s(t_0, t_0) + O(\Delta c)) + o(e) \\ &= H_c + e\nabla H(q^0(0)) \cdot Q_s(t_0, t_0) + O(e\Delta c) + o(e) \\ &= H_c + e\Delta_s(t_0, t_0) + O(e\Delta c) + o(e), \end{aligned} \tag{7.108}$$

and similarly

$$\begin{aligned} H(q^e_{cu}(t_0, t_0)) &= H(q_c(t_0, t_0)) + e\nabla H(q_c(t_0, t_0)) \cdot Q^u_c(t_0, t_0) + o(e) \\ &= H_c + e\nabla H(q^0(0) + O(\Delta c)) \cdot (Q_u(t_0, t_0) + O(\Delta c)) + o(e) \\ &= H_c + e\nabla H(q^0(0)) \cdot Q_u(t_0, t_0) + O(e\Delta c) + o(e) \\ &= H_c + e\Delta_u(t_0, t_0) + O(e\Delta c) + o(e). \end{aligned} \tag{7.109}$$

In order to compute $d(t_0)$ we will investigate the evolution of $\Delta_s(t, t_0)$ and $\Delta_u(t, t_0)$ in time, and later compute

$$d(t_0) = e[\Delta_u(t_0, t_0) - \Delta_s(t_0, t_0)] + O(e\Delta c) + o(e). \tag{7.110}$$

Let us first concentrate on the term $\Delta_s(t, t_0)$. We know that

$$f = J\nabla H \tag{7.111}$$

and therefore we can rewrite our term as

$$\begin{aligned} \Delta_s(t, t_0) &= \nabla H(q^0(t - t_0)) \cdot Q_s(t, t_0) \\ &= -Jf(q^0(t - t_0)) \cdot Q_s(t, t_0). \end{aligned} \tag{7.112}$$

Using the equation (7.83) from Lemma 7.8, we can compute

$$\begin{aligned} -\frac{d}{dt}\left(\Delta_s(t, t_0)\right) &= JDf(q^0(t - t_0))\dot{q}^0(t - t_0) \cdot Q_s(t, t_0) + Jf(q^0(t - t_0)) \cdot \dot{Q}_s(t, t_0) \\ &= JDf(q^0(t - t_0))f(q^0(t - t_0)) \cdot Q_s(t, t_0) \\ &\quad + Jf(q^0(t - t_0)) \cdot Df\left(q^0(t - t_0)\right) Q_s(t, t_0) \\ &\quad + Jf(q^0(t - t_0)) \cdot g(q^0(t - t_0), t) \\ &= Jf(q^0(t - t_0)) \cdot g(q^0(t - t_0), t) \end{aligned} \tag{7.113}$$

The last equality comes from the fact that for any $p, q \in \mathbb{R}^4$

$$JDf(q^0(t-t_0))p \cdot q + Jp \cdot Df\left(q^0(t-t_0)\right) q = 0 \tag{7.114}$$

with $p = f(q^0(t-t_0))$ and $q = Q_s(t,t_0)$. The equation (7.114) follows from the

Lemma 7.10

Let $x' = f(x)$, $x \in \mathbb{R}^{2n}$ be a Hamiltonian ODE and $\varphi(t,x)$ the induced flow. Let ω be a standard symplectic form ($w(p,q) = (Jp|q) = (Jp) \cdot q$). Then for any $x \in \mathbb{R}^{2n}$, and $p, q \in \mathbb{R}^{2n}$ holds

$$\omega(Df(x)p, q) + \omega(p, Df(x))q) = 0 \tag{7.115}$$

We will prove the Lemma 7.10 after completing this proof.

So far we have shown that for $t \geq t_0$

$$-\frac{d}{dt}\left(\Delta_s(t,t_0)\right) = Jf(q^0(t-t_0)) \cdot g(q^0(t-t_0),t). \tag{7.116}$$

We can now rewrite the above in terms of the functions H and G.

$$\begin{aligned} -\frac{d}{dt}\left(\Delta_s(t,t_0)\right) &= Jf(q^0(t-t_0)) \cdot g(q^0(t-t_0),t) \\ &= -\nabla H(q^0(t-t_0)) \cdot J\nabla G(q^0(t-t_0),t) \\ &= -\{H,G\}(q^0(t-t_0),t). \end{aligned} \tag{7.117}$$

In a similar fashion to the above computations for $t \leq t_0$ we can compute

$$\frac{d}{dt}\left(\Delta_u(t,t_0)\right) = \{H,G\}(q^0(t-t_0),t). \tag{7.118}$$

We will now compute $\Delta_s(t_0,t_0)$ and $\Delta_u(t_0,t_0)$. From (7.117) we know that

$$\Delta_s(+\infty,t_0) - \Delta_s(t_0,t_0) = \int_{t_0}^{+\infty} \{H,G\}(q^0(t-t_0),t)dt. \tag{7.119}$$

Since $\lim_{t\to+\infty} q^0(t-t_0) = L_2$ and $f(L_2) = 0$ and the fact that $Q_s(t,t_0)$ is bounded

$$\Delta_s(+\infty,t_0) = \lim_{t\to+\infty} Jf(q^0(t-t_0)) \cdot Q_s(t,t_0) = 0 \tag{7.120}$$

and therefore

$$-\Delta_s(t_0,t_0) = \int_{t_0}^{+\infty} \{H,G\}(q^0(t-t_0),t)dt. \tag{7.121}$$

Analogous computations give

$$\Delta_u(t_0,t_0) = \int_{-\infty}^{t_0} \{H,G\}(q^0(t-t_0),t)dt. \tag{7.122}$$

Clearly

$$\begin{aligned} d(t_0) &= e[\Delta_u(t_0,t_0) - \Delta_s(t_0,t_0)] + O(e\Delta c) + o(e) \\ &= eM(t_0) + O(e\Delta c) + o(e). \end{aligned} \tag{7.123}$$

Similar computations lead to the fact that also

$$\begin{aligned}\frac{\partial}{\partial t_0}d(t_0) &= e\frac{\partial}{\partial t_0}[\Delta_u(t_0,t_0)-\Delta_s(t_0,t_0)]+O(e\Delta c)+o(e) \\ &= e\frac{\partial}{\partial t_0}M(t_0)+O(e\Delta c)+o(e).\end{aligned} \tag{7.124}$$

Since the proof of (7.124) is a bit cumbersome in notation we will show (7.124) at the very end of the proof, after presenting the discussion which leads to the transversal intersections.

Let us now discuss that the fact that $M(t_0)$ has simple zeros implies transversal intersections. For C sufficiently close to C_2 for all $c\in[C,C_2]$ the term Δc is going to be small. If we choose $e_0(\tilde{C})$ sufficiently small (in particular, it will have to be small enough to satisfy the requirements of Lemma 7.2) then for all $c\in[C,C_2]$ and $e\in[0,e_0(\tilde{C})]$ from the fact that $M(t_0)$ has simple zeros we will know that there exists a τ_0 for which $d(t_0)$ is a simple zero. This by (7.101) implies that $\dot{y}^{e\,s}_{t_0\,c}-\dot{y}^{e\,u}_{t_0\,c}$ also has a simple zero. This means that for $c\in\mathfrak{C}\cap[C,\tilde{C}]$ for which $l(c)$ survives the perturbation and all $e\in[0,e_0(\tilde{C})]$ the invariant manifolds $W^s(l^e_{\tau_0}(c),P^e_{\tau_0})$ and $W^u(l^e_{\tau_0}(c),P^e_{\tau_0})$ have a nonempty intersection at the point $\mathbf{p}_{\tau_0}:=p^{e\,s}_{\tau_0\,c}=p^{e\,u}_{\tau_0\,c}$. We would like to show that this intersection is transversal i.e.

$$\mathrm{span}(T_{\mathbf{p}_{\tau_0}}W^s(l^e_{\tau_0}(c),P^e_{\tau_0}),T_{\mathbf{p}_{\tau_0}}W^u(l^e_{\tau_0}(c),P^e_{\tau_0}))=\mathbb{R}^4. \tag{7.125}$$

From our construction (Lemma 7.1 in particular) we know that the local projection onto the $x,y,\dot{x}$ coordinates of $W^u(l^e_{\tau_0}(c),P^e_{\tau_0})$ and $W^s(l^e_{\tau_0}(c),P^e_{\tau_0})$ intersect transversally, which means that

$$\{\dot{y}=0\}\subset\mathrm{span}(T_{\mathbf{p}_{\tau_0}}W^s(l^e_{\tau_0}(c),P^e_{\tau_0}),T_{\mathbf{p}_{\tau_0}}W^u(l^e_{\tau_0}(c),P^e_{\tau_0})), \tag{7.126}$$

where $\{\dot{y}=0\}$ is a short hand notation for the vector space $\{(x,y,\dot{x},\dot{y})|\dot{y}=0\}\subset\mathbb{R}^4$. To prove transversality it is therefore sufficient to show that we do not have

$$\mathrm{span}(T_{\mathbf{p}_{\tau_0}}W^s(l^e_{\tau_0}(c),P^e_{\tau_0}),T_{\mathbf{p}_{\tau_0}}W^u(l^e_{\tau_0}(c),P^e_{\tau_0}))\subset\{\dot{y}=0\}. \tag{7.127}$$

Let us suppose that the above is the case. Let us define a small interval $I=(\tau_0-\delta,\tau_0+\delta)$ around τ_0, for some small $\delta>0$. If (7.127) was the case, then this would imply that in the extended phase space, which includes the time variable t_0, we would have (see Figure 7.3)

$$\begin{aligned}&T_{\mathbf{p}_{\tau_0}}\{\left(W^s(l^e_{t_0}(c),P^e_{t_0}),t_0\right)|t_0\in I\}\subset\mathrm{span}\left(\{\dot{y}=0\}\times\{\tau_0\},(J\nabla H^e(\mathbf{p}_{\tau_0},\tau_0),1)\right)\\ &T_{\mathbf{p}_{\tau_0}}\{\left(W^u(l^e_{t_0}(c),P^e_{t_0}),t_0\right)|t_0\in I\}\subset\mathrm{span}\left(\{\dot{y}=0\}\times\{\tau_0\},(J\nabla H^e(\mathbf{p}_{\tau_0},\tau_0),1)\right),\end{aligned} \tag{7.128}$$

which in particular would mean that

$$\frac{d\left(\dot{y}^{e\,s}_{t_0\,c}\right)}{dt_0}(\tau_0)=\frac{d\left(\dot{y}^{e\,u}_{t_0\,c}\right)}{dt_0}(\tau_0). \tag{7.129}$$

This cannot be so because we know that $\dot{y}^{e\,s}_{t_0\,c}-\dot{y}^{e\,u}_{t_0\,c}$ has a simple zero at τ_0. Hence the intersection of the manifolds at $\mathbf{p}_{\tau_0}$ must be transversal.

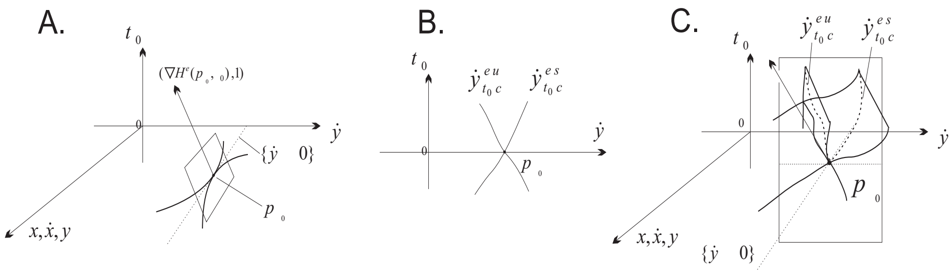

Figure 7.3 The picture of W^s and W^u in the extended phase space if we assume that $T_{\mathbf{p}_{\tau_0}}W^s, T_{\mathbf{p}_{\tau_0}}W^u \subset \{\dot{y}=0\}$ (A.), in contrast with the simple zero of $\dot{y}^{es}_{t_0c} - \dot{y}^{eu}_{t_0c}$ (B.), and the situation which we have when W^s and W^u intersect transversally (C.).

Now let us finish the proof by showings (7.124). The method is analogous to the one used earlier on in the equations (7.108) and (7.109) for the derivation of $M(t_0)$. We know that

$$d(t_0) = H(q^e_{cu}(t_0,t_0)) - H(q^e_{cs}(t_0,t_0)). \tag{7.130}$$

Let us compute $\frac{\partial}{\partial t_0}H(q^e_{cs}(t_0,t_0))$. We know that the point $p^s_{t_0}(c,0)$ is the point of intersection of $W^s(l(c),P^0_{t_0})$ with $W^u(l(c),P^0_{t_0})$ and that it is independent from the choice of t_0. This means that

$$\frac{\partial}{\partial t_0}p^s_{t_0}(c,0) = 0 \quad \text{for all } c \in [C, C_2]. \tag{7.131}$$

Using (7.131) and the fact that the function $p^s_{t_0}(c,e)$ is smooth with respect to c, e

and t_0 we can compute

$$\begin{aligned}
\frac{\partial}{\partial t_0} H(q^e_{cs}(t_0,t_0)) &= \frac{\partial}{\partial t_0} H(p^s_{t_0}(c,e)) \qquad (7.132)\\
&= \nabla H(p^s_{t_0}(c,e)) \cdot \frac{\partial}{\partial t_0} p^s_{t_0}(c,e)\\
&= \nabla H(p^s_{t_0}(c,e)) \cdot \left(\frac{\partial}{\partial t_0} p^s_{t_0}(c,0) + e \frac{\partial}{\partial e} \frac{\partial}{\partial t_0} p^s_{t_0}(c,e)|_{e=0} + o(e) \right)\\
&= \nabla H(p^s_{t_0}(c,0) + O(e)) \cdot \left(e \frac{\partial}{\partial e} \frac{\partial}{\partial t_0} p^s_{t_0}(c,e)|_{e=0} + o(e) \right)\\
&= e \nabla H(p^s_{t_0}(c,0)) \cdot \left(\frac{\partial}{\partial e} \frac{\partial}{\partial t_0} p^s_{t_0}(c,e)|_{e=0} \right) + o(e)\\
&= e \nabla H(p^s_{t_0}(C_2,0) + O(\Delta c)) \cdot \left(\frac{\partial}{\partial e} \frac{\partial}{\partial t_0} p^s_{t_0}(C_2,e)|_{e=0} + O(\Delta c) \right)\\
&\quad + o(e)\\
&= e \nabla H(p^s_{t_0}(C_2,0)) \cdot \left(\frac{\partial}{\partial t_0} \frac{\partial}{\partial e} p^s_{t_0}(C_2,e)|_{e=0} \right) + o(e) + O(e\Delta c)\\
&= e \nabla H(q^0(0)) \cdot \left(\frac{\partial}{\partial t_0} \frac{\partial}{\partial e} q^e_{C_2 s}(t_0,t_0)|_{e=0} \right) + o(e) + O(e\Delta c)\\
&= e \nabla H(q^0(0)) \cdot \frac{\partial}{\partial t_0} Q_s(t_0,t_0) + o(e) + O(e\Delta c)\\
&= e \frac{\partial}{\partial t_0} \Delta_s(t_0,t_0) + o(e) + O(e\Delta c).
\end{aligned}$$

Analogous computations to the above give us

$$\frac{\partial}{\partial t_0} H(q^e_{cu}(t_0,t_0)) = e \frac{\partial}{\partial t_0} \Delta_s(t_0,t_0) + o(e) + O(e\Delta c),$$

which gives (7.124). □

Now to finish off the proof we are left with proving Lemma 7.10.

Proof (proof of Lemma 7.10)

We know that ω is an invariant for $\varphi(t,\cdot)$. This means that for any t holds

$$\omega\left(\frac{\partial}{\partial x}\varphi(t,x)p, \frac{\partial}{\partial x}\varphi(t,x)q \right) = \omega(p,q). \qquad (7.133)$$

After we differentiate both sides with respect to t and set $t=0$ we obtain

$$\begin{aligned}
\omega\left(\frac{\partial}{\partial t}\frac{\partial}{\partial x}\varphi(t,x)p, \frac{\partial}{\partial x}\varphi(t,x)q \right) + \omega\left(\frac{\partial}{\partial x}\varphi(t,x)p, \frac{\partial}{\partial t}\frac{\partial}{\partial x}\varphi(t,x)q \right) &= 0\\
\omega\left(\frac{\partial}{\partial x}f(\varphi(t,x))p, \frac{\partial}{\partial x}\varphi(t,x)q \right) + \omega\left(\frac{\partial}{\partial x}\varphi(t,x)p, \frac{\partial}{\partial x}f(\varphi(t,x))q \right) &= 0\\
\omega\left(\frac{\partial}{\partial x}f(x)p, q \right) + \omega\left(p, \frac{\partial}{\partial x}f(x)q \right) &= 0.
\end{aligned}$$

□

Let us finish the chapter by noting that the Theorem 7.9 automatically gives us the intersections of invariant manifolds of neighboring Lapunov orbits.

Remark 7.11

If the Melnikov integral has a simple zero then there exists a $\kappa > 0$ such that for two energies c_1 and c_2 from $[C, \tilde{C}]$ for which the Lapunov orbits persist under perturbation and such that

$$|c_1 - c_2| < \kappa e \tag{7.134}$$

the manifolds $W^s(l^e_{t_0}(c_i), P^e_{t_0})$ and $W^u(l^e_{t_0}(c_j), P^e_{t_0})$ intersect transversally for $i, j \in \{1, 2\}$.

Proof

The proof of this fact is identical to the proof of the above Theorem. The argument for the transversal intersections of projections is the same. Using an identical argument to the derivation of (7.123) we obtain the distance between the manifolds being equal to

$$\begin{aligned} d(t_0) &= H_{c_i} - H_{c_j} + eM(t_0) + O(e\Delta C) + o(e) \\ &= \frac{1}{2}(c_j - c_i) + eM(t_0) + O(e\Delta C) + o(e). \end{aligned} \tag{7.135}$$

For $|c_1 - c_2| = \kappa e$ and κ sufficiently small we will have the transversal intersection. □

8
Computation of the Melnikov function.

From Theorem 7.9 of the previous chapter we know that the transversality of the intersection of the invariant manifolds $W^s(l^e_{t_0}(c), P^e_{t_0})$ and $W^u(l^e_{t_0}(c), P^e_{t_0})$ can be determined by computing the Melnikov integral (7.96)

$$M(t_0) = \int_{-\infty}^{+\infty} \{H, G\}(q^0(t), t + t_0)dt. \tag{8.1}$$

If the integral has simple zeros then the manifolds intersect transversally for sufficiently small e. This chapter is devoted to showing that M has a simple zero for $t_0 = 0$.

In the first Section of the chapter we will discuss a number of properties of the Melnikov integral which can be deduced from simple symmetry arguments. The Melnikov integral is computed along the homoclinic orbit q^0. From the fact that q^0 is symmetric we will deduce that the function $M(t_0)$ is equal to zero for $t_0 = 0$.

To check that this zero is a simple zero we will have to check that $\frac{dM}{dt_0}(0) \neq 0$. For this we will have to compute an appropriate integral along the homoclinic orbit q^0. It will turn out that for sufficiently small μ the part of q^0 which is far from the libration point L_2^μ has negligible influence on the integral. This will mean that in order to compute $\frac{dM}{dt_0}(0)$ we can focus on the neighborhood of L_2. The above can be intuitively explained as follows. The Melnikov function $M(t_0)$ describes the change of the values of the Hamiltonian H, which is equivalent to the change of the energy level. For $e = 0$ this fluctuation does not exist because all solutions have constant energy. This changes when $e \neq 0$ because the Hamiltonian for the elliptic problem is no longer autonomous. For $e \neq 0$ the small mass μ is no longer fixed in the point $(\mu - 1, 0)$ but oscillates around it. This forced oscillation has an impact on the behavior of the third, massless body and is the cause for the change of its energy level. If the mass μ is small, then it makes sense that the impact on the massless particle is significant only in the vicinity of the small mass μ. The above argument will be derived rigorously in Section 8.2 of the chapter.

8.1 Symmetry properties of the Melinikov function.

In this section we will apply simple arguments based on the symmetry properties to show that the Melnikov integral $M(t_0)$ (8.1) is equal to zero for $t_0 = 0$. In order to compute the integral (8.1) we will first derive the Poisson bracket $\{H, G\}$. Let us recall that the Hamiltonian H of the PRC3BP is given by the formula (2.1)

$$H(x, y, p_x, p_y) = \frac{(p_x + y)^2 + (p_y - x)^2}{2} - \Omega(x, y) \tag{8.2}$$

where

$$\Omega(x, y) = \frac{x^2 + y^2}{2} + \frac{1 - \mu}{\sqrt{(x - \mu)^2 + y^2}} + \frac{\mu}{\sqrt{(x + 1 - \mu)^2 + y^2}}. \tag{8.3}$$

The function G from the Hamiltonian (4.1) of the PRE3BP $H^e = H + eG + O(e^2)$ is given by the formula (4.50)

$$G(x, y) = \frac{1 - \mu}{(r_1)^3} f(x, y, \mu, t) + \frac{\mu}{(r_2)^3} f(x, y, \mu - 1, t), \tag{8.4}$$

where

$$f(x, y, \alpha, t) = -y\alpha[3 \sin t - \sin^3 t] + x\alpha[\cos t + \cos^3 t] - \alpha^2 \cos(t), \tag{8.5}$$

and

$$\begin{aligned} r_1^2 &= (x - \mu)^2 + y^2 \\ r_2^2 &= (x + 1 - \mu)^2 + y^2. \end{aligned} \tag{8.6}$$

Let us note that G is independent from the variables p_x and p_y, so the formula for the Poisson bracket $\{H, G\}$ simplifies and takes form

$$\begin{aligned} \{H, G\}(x, y, p_x, p_y) &= \frac{\partial H}{\partial x}\frac{\partial G}{\partial p_x} - \frac{\partial H}{\partial p_x}\frac{\partial G}{\partial x} + \frac{\partial H}{\partial y}\frac{\partial G}{\partial p_y} - \frac{\partial H}{\partial p_y}\frac{\partial G}{\partial y} \\ &= -\frac{\partial H}{\partial p_x}\frac{\partial G}{\partial x} - \frac{\partial H}{\partial p_y}\frac{\partial G}{\partial y} \\ &= -(p_x + y)\frac{\partial G}{\partial x} - (p_y - x)\frac{\partial G}{\partial y}. \end{aligned} \tag{8.7}$$

Let us also note that since

$$\begin{aligned} \dot{x} &= p_x + y \\ \dot{y} &= p_y - x \end{aligned} \tag{8.8}$$

we can also rewrite (8.7) as

$$\{H, G\} = -\dot{x}\frac{\partial G}{\partial x} - \dot{y}\frac{\partial G}{\partial y}. \tag{8.9}$$

Let us compute

$$\frac{\partial G}{\partial x} = \frac{\mu\,(1-\mu)}{(r_1)^3}[\cos t + \cos^3 t] - 3\frac{(x-\mu)\,(1-\mu)}{(r_1)^5} f(x,y,\mu,t) \qquad (8.10)$$
$$- \frac{\mu(1-\mu)}{(r_2)^3}[\cos t + \cos^3 t] - 3\frac{(x+1-\mu)\,\mu}{(r_2)^5} f(x,y,\mu-1,t)$$

and

$$\frac{\partial G}{\partial y} = -\frac{\mu\,(1-\mu)}{(r_1)^3}[3\sin t - \sin^3 t] - \frac{3y\,(1-\mu)}{(r_1)^5} f(x,y,\mu,t) \qquad (8.11)$$
$$+ \frac{\mu\,(1-\mu)}{(r_2)^3}[3\sin t - \sin^3 t] - \frac{3y\mu}{(r_2)^5} f(x,y,\mu-1,t).$$

Lemma 8.1

The Poisson bracket $\{H,G\}$ is R-antisymmetric i.e.

$$\{H,G\}(R(x,y,p_x,p_y,t)) = -\{H,G\}(x,y,p_x,p_y,t), \qquad (8.12)$$

where R is the symmetry of the PRC3BP defined in Section 2.2

$$R(x,y,p_x,p_y,t) = (x,-y,-p_x,p_y,-t). \qquad (8.13)$$

Proof

Let us first observe that from the definition of f, r_1 and r_2 it is clear that

$$\begin{aligned} f(x,-y,\alpha,-t) &= f(x,y,\alpha,t) \\ r_1(x,-y) &= r_1(x,y) \\ r_2(x,-y) &= r_2(x,y). \end{aligned} \qquad (8.14)$$

From this fact and from the equations (8.10) and (8.11) we can see that

$$\frac{\partial G}{\partial x}(R(x,y,p_x,p_y,t)) = \frac{\partial G}{\partial x}(x,y,p_x,p_y,t) \qquad (8.15)$$

and

$$\frac{\partial G}{\partial y}(R(x,y,p_x,p_y,t)) = -\frac{\partial G}{\partial y}(x,y,p_x,p_y,t). \qquad (8.16)$$

From the above and from (8.7) we have

$$\{H,G\}(R(x,y,p_x,p_y,t)) = -\{H,G\}(x,y,p_x,p_y,t). \qquad (8.17)$$

□

Lemma 8.2

The Melnikov integral is equal to zero for $t_0 = 0$

$$M(0) = \int_{-\infty}^{+\infty} \{H,G\}(q^0(t),t)dt = 0. \qquad (8.18)$$

Proof

The proof will be based on the previous Lemma and the fact that the homoclinic orbit q^0 is symmetric see Theorem 2.1, i.e.

$$R(q^0(t),t) = \left(q^0(-t),-t\right). \tag{8.19}$$

We can split the Melnikov integral into two parts

$$M(0) = \int_{-\infty}^{0} \{H,G\}(q^0(t),t)dt + \int_{0}^{+\infty} \{H,G\}(q^0(t),t)dt. \tag{8.20}$$

Using the symmetry of q^0 and Lemma 8.1 we can compute

$$\begin{aligned} \int_{-\infty}^{0} \{H,G\}(q^0(t),t)dt &= \int_{0}^{+\infty} \{H,G\}(q^0(-t),-t)dt \\ &= \int_{0}^{+\infty} \{H,G\}(R(q^0(t),t))dt \\ &= -\int_{0}^{+\infty} \{H,G\}(q^0(t),t)dt \end{aligned} \tag{8.21}$$

and therefore $M(0) = 0$. □

The above result is a big step in proving Theorem 1.1. We now know that for $t_0 = 0$ the Melnikov integral $M(t_0)$ is equal to zero and in order to prove the Theorem all we have to do now is to check that this zero is a simple zero.

Lemma 8.3

The Melnikov function $M(t_0)$ has a simple zero at $t_0 = 0$ if

$$M_t(0) \neq 0 \tag{8.22}$$

where

$$M_t(0) = \int_{-\infty}^{+\infty} \{H,G_t\}(q^0(t),t)dt \tag{8.23}$$

$$G_t(x,y) = \frac{1-\mu}{(r_1)^3} f_t(x,y,\mu,t) + \frac{\mu}{(r_2)^3} f_t(x,y,\mu-1,t) \tag{8.24}$$

and

$$f_t(x,y,\alpha,t) = -y\alpha\left[3\cos^3 t\right] + x\alpha[-4\sin t + 3\sin^3 t] + \alpha^2 \sin(t). \tag{8.25}$$

Proof

The orbit q^0 is the homoclinic orbit to the Libration point L_2. Let us note that the velocity $\dot{x}$ and $\dot{y}$ of $q^0(t)$ exponentially tends to zero as t tends to plus infinity and minus infinity. What is more the partial derivatives of G on $q^0(t)$ are uniformly bounded. This means that the integral over

$$\int_{-\infty}^{+\infty} |\{H,G\}|(q^0(t),t+t_0)dt = \int_{-\infty}^{+\infty} |\dot{x}\frac{\partial G}{\partial x} + \dot{y}\frac{\partial G}{\partial y}|(q^0(t),t+t_0)dt,$$

is convergent. Since the function G_t is independent from p_x and p_y we can compute $\{H, G_t\}$ as

$$\begin{aligned}\{H, G_t\} &= -(p_x + y)\frac{\partial G_t}{\partial x} - (p_y - x)\frac{\partial G_t}{\partial y}. \\ &= -\dot{x}\frac{\partial G_t}{\partial x} - \dot{y}\frac{\partial G_t}{\partial y}\end{aligned} \tag{8.26}$$

the same can be said about the integral of $|\{H, G_t\}|$ along q^0. This means that the above $M_t(0)$ can be obtained by direct computation of $\frac{d}{dt_0}M(t_0)|_{t_0=0}$. □

Let us now compute the components of the Poisson bracket $\{H, G_t\}$ from the formula (8.26)

$$\begin{aligned}\frac{\partial G_t}{\partial x} = &\frac{\mu\,(1-\mu)}{(r_1)^3}[-4\sin t + 3\sin^3 t] - 3\frac{(x-\mu)\,(1-\mu)}{(r_1)^5}f_t(x,y,\mu,t) \\ &- \frac{\mu(1-\mu)}{(r_2)^3}[-4\sin t + 3\sin^3 t] - 3\frac{(x+1-\mu)\,\mu}{(r_2)^5}f_t(x,y,\mu-1,t)\end{aligned} \tag{8.27}$$

and

$$\begin{aligned}\frac{\partial G_t}{\partial y} = &-\frac{\mu\,(1-\mu)}{(r_1)^3}[3\cos^3 t] - \frac{3y\,(1-\mu)}{(r_1)^5}f_t(x,y,\mu,t) \\ &+ \frac{\mu\,(1-\mu)}{(r_2)^3}[3\cos^3 t] - \frac{3y\mu}{(r_2)^5}f_t(x,y,\mu-1,t).\end{aligned} \tag{8.28}$$

Using these formulas we shall prove the following Lemma

Lemma 8.4

The Poisson bracket $\{H, G_t\}$ is R-symmetric i.e.

$$\{H, G_t\}(R(x, y, p_x, p_y, t)) = \{H, G_t\}(x, y, p_x, p_y, t), \tag{8.29}$$

where

$$R(x, y, p_x, p_y, t) = (x, -y, -p_x, p_y, -t). \tag{8.30}$$

Proof

The proof is analogous to the proof of Lemma 8.1. The function f_t is antisymmetric

$$f_t(x, -y, \alpha, -t) = -f_t(x, y, \alpha, t).$$

From this fact and from the equations (8.27) and (8.28) we can see that

$$\frac{\partial G_t}{\partial x}(R(x, y, p_x, p_y, t)) = -\frac{\partial G_t}{\partial x}(x, y, p_x, p_y, t) \tag{8.31}$$

and

$$\frac{\partial G_t}{\partial y}(R(x, y, p_x, p_y, t)) = \frac{\partial G_t}{\partial y}(x, y, p_x, p_y, t). \tag{8.32}$$

Using the above and (8.26) we can see that

$$\{H, G_t\}(R(x, y, p_x, p_y, t)) = \{H, G_t\}(x, y, p_x, p_y, t). \tag{8.33}$$

□

The fact that the function $\{H, G_t\}$ is symmetric will allow us to rewrite the the formula for $M_t(0)$ using the following

Lemma 8.5

The Melnikov integral $M_t(0)$ is equal to

$$M_t(0) = 2\int_{-\infty}^{0} \{H, G_t\}(q^0(t), t)dt. \tag{8.34}$$

Proof

For the proof we will use the fact that q^0 is symmetric and Lemma 8.4. We can compute

$$\begin{aligned} \int_0^{+\infty} \{H, G_t\}(q^0(t), t)dt &= \int_0^{+\infty} \{H, G_t\}(R(q^0(t), t))dt \\ &= \int_0^{+\infty} \{H, G_t\}(q^0(-t), -t)dt \\ &= \int_{-\infty}^{0} \{H, G_t\}(q^0(t), t)dt \end{aligned} \tag{8.35}$$

which gives us

$$\begin{aligned} M_t(0) &= \int_{-\infty}^{0} \{H, G_t\}(q^0(t), t)dt + \int_0^{+\infty} \{H, G_t\}(q^0(t), t)dt. \\ &= 2\int_{-\infty}^{0} \{H, G_t\}(q^0(t), t)dt. \end{aligned} \tag{8.36}$$

□

8.2 Computation of the Melnikov integral $M_t(0)$ for small μ.

In this section we will compute the integral $M_t(0)$ along $q^0(t)$ and show that for the PRE3BP with a sufficiently small mass μ_k the integral is not equal to zero. The idea behind the computation is the following. We will divide the integral into two parts. The first will be the part which is connected with the fragment of $q^0(t)$ which is in a small neighborhood of L_2. The second part of the integral will be connected with the fragment of $q^0(t)$ which lies far from L_2. We will use the results outlined in Section 2.4 (Remark 2.9 in particular) and approximate the first part of the integral with an integral over an appropriate orbit of the Hill's problem. We will show that the second part of the integral for sufficiently small μ_k is negligible.

Let t_μ denote the time at which the solution $q^0(t)$ disembarks from the section $y = -\mu^{1/4}$ (see Figure 8.1). We divide the Melnikov integral $M_t(0)$ into two parts

$$\begin{aligned} M_t(0) &= 2\int_{-\infty}^{0} \{H, G_t\}(q^0(t), t)dt \\ &= 2\int_{-\infty}^{t_\mu} \{H, G_t\}\left(q^0(t), t\right) dt + 2\int_{t_\mu}^{0} \{H, G_t\}\left(q^0(t), t\right) dt. \end{aligned} \tag{8.37}$$

Let us first measure the scale of the first part. We will start with a Remark concerning the bounds on the values of $q^0(t)$ for $t \in (-\infty, t_\mu]$ which will be useful in our future estimations.

Remark 8.6

By the definition of the time t_μ at $t = t_\mu$ the y value of $q^0(t)$ is $y = -\mu^{1/4}$. When computed in the Hill's coordinates (2.27)

$$\begin{aligned} x_H &= \mu^{-1/3}(x + 1 - \mu) \\ y_H &= \mu^{-1/3} y. \end{aligned} \tag{8.38}$$

we have

$$y_H = -\mu^{-1/3}\mu^{1/4} = -\mu^{-1/12}. \tag{8.39}$$

This means that for the time interval $(-\infty, t_\mu]$ the value of y_H of $q^0(t)$ is contained in the interval $[-\mu^{-1/12}, 0]$. By Remark 2.9, for the time interval $(-\infty, t_\mu]$ the homoclinic orbit $q^0(t)$ lies close to the invariant manifold $W^u_H(L^H_2)$. This means that the x_H coordinate of $q^0(t)$ oscillates close to the zero velocity curve of the Hill's problem $x_H = 3^{1/6}$, and is therefore bounded (see Figure 2.5). Again by Remark 2.9 we also know that the velocity $\dot{x}$ and $\dot{y}$ of $q^0(t)$ is $O(\mu^{1/3})$ on the time interval $(-\infty, 0]$.

What is more from Remark 2.9 we know that the orbit $q^0(t)$ from L_2 down to the section $y = -\mu^{1/4}$ can be approximated by the orbit $q^H(t)$ on the unstable manifold $W^u_H(L^H_2)$ (See Remark 2.9 for the definition of $q^H(t)$).

In order to perform our estimations let us write out the Poisson bracket $\{H, G_t\}$ in full form and assign a number to each term. This will help us in the future, since some of the terms are more important than others and numbering them will help us in pointing them out. From the equations (8.26) (8.27) and (8.28) we can write out

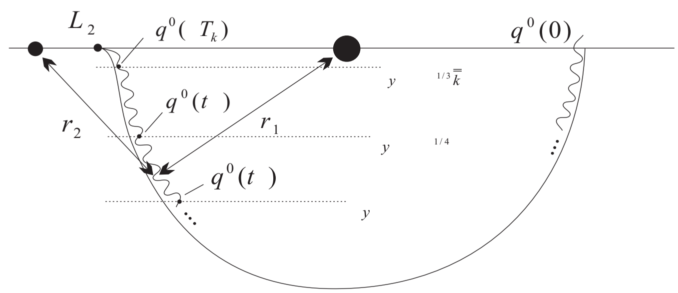

Figure 8.1 The orbit $q^0(t)$ and it's intersections with the sections $\{y = -\mu^{1/3}\overline{\overline{k}}\}$, $\{y = -\mu^{1/4}\}$, and $\{y = -\kappa\}$.

$$\{H, G_t\} = -\dot{x}\frac{\partial G_t}{\partial x} - \dot{y}\frac{\partial G_t}{\partial y}$$

$$= -\dot{x}\frac{\mu(1-\mu)}{(r_1)^3}[-4\sin t + 3\sin^3 t] \tag{8.40}$$

$$+ \dot{x}3\frac{(x-\mu)(1-\mu)}{(r_1)^5}f_t(x, y, \mu, t) \tag{8.41}$$

$$+ \dot{x}\frac{\mu(1-\mu)}{(r_2)^3}[-4\sin t + 3\sin^3 t] \tag{8.42}$$

$$+ 3\dot{x}\frac{(x+1-\mu)\mu}{(r_2)^5}f_t(x, y, \mu-1, t) \tag{8.43}$$

$$+ \dot{y}\frac{\mu(1-\mu)}{(r_1)^3}[3\cos^3 t] \tag{8.44}$$

$$+ \dot{y}\frac{3y(1-\mu)}{(r_1)^5}f_t(x, y, \mu, t) \tag{8.45}$$

$$- \dot{y}\frac{\mu(1-\mu)}{(r_2)^3}[3\cos^3 t] \tag{8.46}$$

$$+ \dot{y}\frac{3y\mu}{(r_2)^5}f_t(x, y, \mu-1, t). \tag{8.47}$$

Remark 8.7

Let us note that the integral of each of the terms (8.40),...,(8.47) along $q^0(t)$ on the time interval $(-\infty, t_\mu]$ is finite. This comes from the fact that each of the terms is multiplied by $\dot{x}$ or $\dot{y}$ which for $q^0(t)$ exponentially tend to zero as t tends to minus infinity.

Lemma 8.8

From L_2 down to the section $\{y = -\mu^{1/4}\}$ the Poisson bracket $\{H, G_t\}$ on $q^0(t)$

takes form

$$\{H, G_t\} = 9\left(\frac{y_H\dot{y}_H + x_H\dot{x}_H}{\left(x_H^2 + y_H^2\right)^{5/2}}\right)\left(\sin^3 t - \sin t\right) + O(\mu^{1/4}) \tag{8.48}$$

where x_H and y_H are the Hill's coordinates

Proof

In order to estimate the integral close to the libration point L_2 we will switch to the Hill's coordinates (8.38) (see also Section 2.4). In these coordinates we can write

$$\begin{aligned} r_2 &= \sqrt{(x+1-\mu)^2 + y^2} \\ &= \mu^{1/3}\sqrt{x_H^2 + y_H^2}, \end{aligned} \tag{8.49}$$

and

$$\begin{aligned} r_1 &= \sqrt{(x-\mu)^2 + y^2} \\ &= \sqrt{(\mu^{1/3}x_H - 1)^2 + \left(\mu^{1/3}y_H\right)^2}. \end{aligned} \tag{8.50}$$

As μ decreases, the distance from the large mass r_1 tends to one. This means that since by Remark 8.6 the $x, y, \dot{x}, \dot{y}$ coordinates of $q^0(t)$ are bounded and since $f_t(x, y, \mu, t) = O(\mu)$, for the terms (8.40), (8.41), (8.44) and (8.45) in $\{H, G_t\}$, which contain r_1, we have

$$\begin{aligned} \dot{x}\frac{\mu\,(1-\mu)}{\left(r_1\right)^3}[-4\sin t + 3\sin^3 t] &= O(\mu) \\ 3\dot{x}\frac{(x-\mu)\,(1-\mu)}{\left(r_1\right)^5}f_t(x, y, \mu, t) &= O(\mu) \\ \dot{y}\frac{\mu\,(1-\mu)}{\left(r_1\right)^3}[3\cos^3 t] &= O(\mu) \\ \dot{y}\frac{3y\,(1-\mu)}{\left(r_1\right)^5}f_t(x, y, \mu, t) &= O(\mu). \end{aligned} \tag{8.51}$$

This implies that these terms are negligible for small μ. Let us note that from Remark 2.9 points 2 and 3 we know that $x_H, \dot{x}_H$ are bounded and that $\sqrt{x_H^2 + y_H^2} \geq 3^{-1/3}/2$. For the terms (8.42) and (8.43) from $\{H, G_t\}$, which contain r_2 and $\dot{x}$, we therefore have the following estimates

$$\begin{aligned} \dot{x}\frac{\mu(1-\mu)}{(r_2)^3}[-4\sin t + 3\sin^3 t] &= \left(\mu^{1/3}\dot{x}_H\right)\frac{\mu(1-\mu)}{\mu\left(x_H^2 + y_H^2\right)^{3/2}}[-4\sin t + 3\sin^3 t] \\ &= O(\mu^{1/3}), \end{aligned} \tag{8.52}$$

$$
\begin{aligned}
\dot{x}3\frac{(x+1-\mu)\,\mu}{(r_2)^5}f_t(x,y,\mu-1,t) &= \left(\mu^{1/3}\dot{x}_H\right)3\frac{\left(\mu^{1/3}x_H\right)\mu}{\mu^{\frac{5}{3}}\left(x_H^2+y_H^2\right)^{5/2}}f_t(x,y,\mu-1,t) \\
&= \frac{3x_H\dot{x}_H}{\left(x_H^2+y_H^2\right)^{5/2}}f_t(x,y,\mu-1,t) \qquad (8.53) \\
&= \frac{9x_H\dot{x}_H}{\left(x_H^2+y_H^2\right)^{5/2}}\left(\sin^3 t-\sin t\right)+O(\mu^{1/4}).
\end{aligned}
$$

The last equality in (8.53) comes from the fact that that for $t \in (-\infty, t_\mu]$ on $q^0(t)$ we have $|\mu^{1/3}y_H| = |y| \le \mu^{1/4}$ which gives

$$
\begin{aligned}
f_t(x,y,\mu-1,t) &= -\mu^{1/3}y_H(\mu-1)\left[3\cos^3 t\right] \\
&\quad + \left(\mu^{1/3}x_H-1+\mu\right)(\mu-1)[-4\sin t+3\sin^3 t] \\
&\quad + (\mu-1)^2\sin(t) \qquad (8.54) \\
&= 3\left(\sin^3 t-\sin t\right)+O(\mu^{1/4}).
\end{aligned}
$$

Similarly, for the terms (8.46) and (8.47) from $\{H, G_t\}$, which contain r_2 and $\dot{y}$, we have

$$
\begin{aligned}
\dot{y}\frac{\mu\,(1-\mu)}{(r_2)^3}[3\cos^3 t] &= \left(\mu^{1/3}\dot{y}_H\right)\frac{\mu\,(1-\mu)}{\mu\left(x_H^2+y_H^2\right)^{3/2}}[3\cos^3 t] \\
&= O(\mu^{1/4}), \qquad (8.55)
\end{aligned}
$$

$$
\begin{aligned}
\dot{y}\frac{3y\mu}{(r_2)^5}f_t(x,y,\mu-1,t) &= \left(\mu^{1/3}\dot{y}_H\right)\frac{3\mu^{1/3}y_H\mu}{\mu^{5/3}\left(x_H^2+y_H^2\right)^{5/2}}f_t(x,y,\mu-1,t) \\
&= \frac{3y_H\dot{y}_H}{\left(x_H^2+y_H^2\right)^{5/2}}f_t(x,y,\mu-1,t) \qquad (8.56) \\
&= \frac{9y_H\dot{y}_H}{\left(x_H^2+y_H^2\right)^{5/2}}\left(\sin^3 t-\sin t\right)+O(\mu^{1/4})
\end{aligned}
$$

Putting the estimations (8.51), (8.53) and (8.56) together we obtain our formula (8.48). ☐

Now we will show that the second part of the integral (8.37) is small for sufficiently small μ.

Lemma 8.9

The integral of the Poisson bracket $\{H, G_t\}$ along the orbit $q^0(t)$ on the interval $[t_\mu, 0]$ is $O(\mu^{1/4})$ i.e.

$$
\int_{t_\mu}^{0}\{H,G_t\}\left(q^0(t),t\right)dt \le O(\mu^{1/4}). \qquad (8.57)
$$

Proof

First of all let us choose a small number $\kappa > 0$ which is independent from μ and such that $\mu^{1/4} < \kappa$. Let t_κ denote the time in which $q^0(t)$ reaches the section $\{y = -\kappa\}$ (See Figure 8.1). We will first estimate the integral

$$\int_{t_\mu}^{t_\kappa} \{H, G_t\} \left(q^0(t), t\right) dt. \tag{8.58}$$

For the time interval $[t_\mu, t_\kappa]$, the largest value of $\{H, G_t\}$ is associated with the terms (8.42), (8.46), (8.43) and (8.47). This is because for large negative times t, the key role is played by the fact that r_2 is small (the bounds on the other integrals coming directly from the fact that $\dot{x}, \dot{y} = O(\mu^{1/3})$, $f_t(x, y, \mu, t) = O(\mu)$ and $|t_\mu - t_\kappa| < |T_k| = O(\mu^{-1/3})$). We will therefore show that these terms are small.

Let us start with the terms (8.42), (8.46) which are the simpler in obtaining our bounds. For $t \in [t_\mu, t_\kappa]$ we have (see Figure 8.1)

$$|r_2| \geq |y| \geq \mu^{1/4}. \tag{8.59}$$

From Remark 8.6 we know that for $t \in [t_\mu, t_\kappa]$

$$\begin{aligned} \dot{x} &= O(\mu^{1/3}) \\ \dot{y} &= O(\mu^{1/3}). \end{aligned} \tag{8.60}$$

Combining (8.59) and (8.60) gives us the following bound for the term (8.42)

$$\left| \int_{t_\mu}^{t_\kappa} \dot{x} \frac{\mu(1-\mu)}{(r_2)^3} [-4 \sin t + 3 \sin^3 t] dt \right| \leq |t_\mu - t_\kappa| O(\mu^{1/3}) \frac{\mu(1-\mu)}{\left(\mu^{1/4}\right)^3} = O(\mu^{1/4}). \tag{8.61}$$

The bound on (8.46) is identical.

Let us now obtain our bounds for the term (8.43). First let us recall the formulas (2.15) and (2.16) for $q^0(t)$ given by Theorem 2.1, which we can rewrite in our case as

$$d(t) = \mu^{1/3}(\frac{2}{3}N(\infty) - 3^{1/6} + M(\infty) \cos(t + T_k) + R_1(t, \mu)), \tag{8.62}$$

$$\alpha(t) = -\pi + \mu^{1/3}(N(\infty)(t + T_k) + 2M(\infty) \sin(t + T_k) + R_2(t, \mu)). \tag{8.63}$$

where $d(t)$ is the distance from the zero velocity curve, $\alpha(t)$ is the angle coordinate, the functions R_i have the property that

$$\max_{t \in [t_\mu, 0]} |R_i(t, \mu)| \overset{\mu \to 0}{\to} 0 \quad \text{for } i \in \{1, 2\}, \tag{8.64}$$

and $-T_k$ is the time at which the orbit $q^0(t)$ intersects the section $\{y = -\mu^{1/3}\overline{\overline{k}}\}$ (See Remark 2.9 and Figure 8.1). From Remark 8.6 we know that for $t = t_\mu$ the y coordinate of $q^0(t)$ is equal to $-\mu^{1/4}$. From Section 2.4 we know that from the section $\{y = -\mu^{1/3}\overline{\overline{k}}\}$ downwards the formulas (8.62), (8.63) start to take effect. This means that we can apply them to obtain our orbit from the section $\{y = -\mu^{1/4}\}$ down to $\{y = \kappa\}$. Since the distance of the orbit $q^0(t)$ is close to the zero velocity curve and the distance of the zero velocity curve from the origin is close to one

$$|q^0(t)| \approx 1, \tag{8.65}$$

(See Figure 8.1). What is more we know that

$$y(t_\mu) = -\mu^{1/4}. \tag{8.66}$$

Both $\alpha(t_\mu)$ and $\alpha(t)$ for $t \in [t_\mu, t_\kappa]$ are very close to $-\pi$, which means that we have

$$|\sin\alpha(t) - \sin\alpha(t_\mu)| > \frac{1}{2}\left|\alpha(t) - \alpha(t_\mu)\right|. \tag{8.67}$$

Combining (8.65), (8.66) and (8.67) gives us an estimate

$$\begin{aligned}|y(t)| &= |y(t_\mu) + (y(t) - y(t_\mu))\,| \\ &= |-\mu^{1/4} + \big(|q^0(t)|\sin\alpha(t) - |q^0(t_\mu)|\sin\alpha(t_\mu)\big)\,| \\ &\geq |-\mu^{1/4} + \frac{1}{2}\left(\alpha(t) - \alpha(t_\mu)\right)|.\end{aligned} \tag{8.68}$$

Since for $t \in [t_\mu, t_\kappa]$ we have $\alpha(t) - \alpha(t_\mu) < 0$ (see Figure 8.1 and formula (8.63)) this means that

$$r_2 = \sqrt{(x+1-\mu)^2 + y^2} \geq |y| \geq \mu^{1/4} + \frac{1}{2}|\alpha(t) - \alpha(t_\mu)|. \tag{8.69}$$

We also know that since T_k is $O(\mu^{-1/3})$ we have

$$|t_\kappa - t_\mu| \leq |t_\mu| \leq |T_k| \leq c\mu^{-1/3}, \tag{8.70}$$

where $c > 0$ is a constant independent from μ. Finally we always know that

$$\begin{aligned} r_2 &= \sqrt{(x+1-\mu)^2 + y^2} \geq |x+1-\mu| \\ r_2 &= \sqrt{(x+1-\mu)^2 + y^2} \geq |y|. \end{aligned} \tag{8.71}$$

Using the fact that $f_t(x(t), y(t), \mu - 1, t)$ is bounded together with the equations (8.60) and (8.71) we have the following estimate for the integral over (8.43)

$$\int_{t_\mu}^{t_\kappa} \left| 3\dot{x}(t)\frac{(x(t)+1-\mu)\,\mu}{(r_2)^5} f_t(x(t), y(t), \mu-1, t)\right| dt \leq M \int_{t_\mu}^{t_\kappa} \left|\mu^{1/3}\frac{\mu}{(r_2)^4}\right| dt \tag{8.72}$$

for some $M > 0$ independent from μ. We can now use (8.69) to obtain

$$\begin{aligned} M\int_{t_\mu}^{t_\kappa}\left|\frac{\mu^{4/3}}{(r_2)^4}\right| dt &\leq M\int_{t_\mu}^{t_\kappa}\left|\frac{\mu^{4/3}}{\left(\mu^{1/4} + \frac{1}{2}|\alpha(t) - \alpha(t_\mu)|\right)^4}\right| dt \qquad (8.73) \\ &\leq M\int_{t_\mu}^{t_\kappa}\left|\frac{\mu^{4/3}}{\left(\mu^{1/4} + \frac{1}{2}\mu^{1/3}(N(\infty)\,(t - t_\mu) - 4M(\infty))\right)^4}\right| dt \\ &\leq M\int_{t_\mu}^{t_\kappa}\left|\frac{\mu^{4/3}}{\left(\frac{1}{2}\mu^{1/4} + \frac{1}{2}\mu^{1/3}N(\infty)\,(t - t_\mu)\right)^4}\right| dt \qquad (8.74) \end{aligned}$$

We can now use (8.70) to obtain

$$M \int_{t_\mu}^{t_\kappa} \left| \frac{\mu^{4/3}}{\left(\mu^{1/4} + \mu^{1/3} N(\infty)\,(t - t_\mu)\right)^4} \right| dt \leq M \int_0^{c\mu^{-1/3}} \left| \frac{\mu^{4/3}}{\left(\mu^{1/4} + \mu^{1/3} N(\infty) t\right)^4} \right| dt = O(\mu^{1/4}). \tag{8.75}$$

The estimation of the integral of (8.47) over the time interval $[t_\mu, t_\kappa]$ is analogous.

We have shown that

$$\int_{t_\mu}^{t_\kappa} \{H, G_t\} \left(q^0(t), t\right) dt = O(\mu^{1/4}). \tag{8.76}$$

Now all we have to do is to estimate

$$\int_{t_\kappa}^{0} \{H, G_t\} \left(q^0(t), t\right) dt. \tag{8.77}$$

This is the easiest part. For all times $t > t_\kappa$ since $y(t) \leq -\kappa$, we know that (See Figure 8.1)

$$\begin{aligned} r_1(t) &\geq |y(y)| > \kappa, \\ r_2(t) &> \frac{1}{2}, \end{aligned} \tag{8.78}$$

which means that all the terms (8.40),...,(8.47) in $\{H, G_t\}$ are $O(\mu)$. We also know that $|t_\kappa| < |T_k| = O(\mu^{-1/3})$ which means that we can estimate the integral by

$$\int_{t_\kappa}^{0} \{H, G_t\} \left(q^0(t), t\right) dt \leq O(\mu^{2/3}). \tag{8.79}$$

□

From Lemmas 8.8 and 8.9 and Remark 8.7, when computing the integral $M_t(0)$ we have

$$\begin{aligned} M_t(0) &= 2 \int_{-\infty}^{0} \{H, G_t\}(q^0(t), t) dt \\ &= 2 \int_{-\infty}^{t_\mu} \{H, G_t\} \left(q^0(t), t\right) dt + 2 \int_{t_\mu}^{0} \{H, G_t\} \left(q^0(t), t\right) dt \\ &= 18 \int_{-\infty}^{t_\mu} \left(\frac{y_H(t) \dot{y}_H(t) + x_H(t) \dot{x}_H(t)}{\left(x_H^2(t) + y_H^2(t)\right)^{5/2}} \right) \left(\sin^3 t - \sin t\right) dt + O(\mu^{1/4}) \end{aligned} \tag{8.80}$$

We would like to estimate the above integral over a homoclinic orbit $q^0(t) = q^0_{\mu_k}(t)$ for small values of μ_k. Let us first introduce some notations and a lemma

$$\begin{aligned} \Gamma(x_H, y_H, \dot{x}_H, \dot{y}_H) &:= \frac{y_H \dot{y}_H + x_H \dot{x}_H}{\left(x_H^2 + y_H^2\right)^{5/2}} \\ M_k &:= 18 \int_{-\infty}^{t_\mu} \Gamma(q^0_{\mu_k}(t)) \left(\sin^3 t - \sin t\right) dt. \end{aligned} \tag{8.81}$$

Lemma 8.10

The integral M_k tends to

$$\lim_{k\to\infty} M_{2k} = M_H \tag{8.82}$$
$$\lim_{k\to\infty} M_{2k+1} = -M_H,$$

where

$$M_H = 18\int_{-\infty}^{\infty} \Gamma(q^H(t))\left(\sin^3 t - \sin t\right) dt, \tag{8.83}$$

and $q^H(t)$ is the orbit lying on the unstable manifold $W^u(L_2^H)$ to the libration point L_2^H of the Hill's problem (See (2.40) for the definition of $q^H(t)$).

Proof

The homoclinic orbit $q^0(t)$ is dependent on the choice of μ_k. In our previous discussion we have omitted the subscript μ_k in order to simplify notations, but for the purpose of this proof it is important that we distinguish between different homoclinic orbits for different μ_k, therefore we will use the notation $q^0_{\mu_k}(t)$. We can compute

$$\begin{aligned} M_k &= 18\int_{-\infty}^{t_\mu} \Gamma(q^0_{\mu_k}(t))\left(\sin^3 t - \sin t\right) dt \\ &= 18\int_{-\infty}^{t_\mu + k\pi} \Gamma(q^0_{\mu_k}(t-k\pi))\left(\sin^3 (t-k\pi) - \sin(t-k\pi)\right) dt \\ &= (-1)^k 18\int_{-\infty}^{t_\mu + k\pi} \Gamma(q^0_{\mu_k}(t-k\pi))\left(\sin^3 t - \sin t\right) dt \end{aligned} \tag{8.84}$$

From Remarks 2.2 and 2.9 we know that

$$\lim_{k\to\infty} |T_k - k\pi| = 0 \tag{8.85}$$
$$q^H(t) - q^0_{\mu_k}(t - T_k) = O(\mu_k^{1/12}),$$

for all $t \in (-\infty, t_\mu + T_k]$ where T_k is the time it takes the homoclinic orbit to L_2 to reach $\{y = 0\}$ from the initial condition $y_H = -\overline{\overline{k}}$ (See Figure 8.1). From (8.85) we have

$$q^0_{\mu_k}(t - k\pi) \overset{k\to\infty}{\to} q^H(t). \tag{8.86}$$

Let us note that if we choose some $T > 0$ then the above convergence on the interval $t \in (-\infty, T]$ follows from the fact that for small t the function $\Gamma(q^0_{\mu_k}(t - k\pi))$ is dominated by the exponential convergence to zero and the fact that on $(-\infty, T]$ the functions $q^0_{\mu_k}(t - k\pi)$ converge uniformly to $q^H(t)$. This means that

$$\int_{-\infty}^{T} \Gamma(q^0_{\mu_k}(t-k\pi))\left(\sin^3 t - \sin t\right) dt - \int_{-\infty}^{T} \Gamma(q^H(t))\left(\sin^3 t - \sin t\right) dt \overset{k\to\infty}{\to} 0. \tag{8.87}$$

In order to show the convergence (8.82) all we need to show is that the functions $\Gamma(q^0_{\mu_k}(t - k\pi))$ are uniformly bounded on the interval $[T, +\infty)$ by some integrable function $g(t)$, which will allow us to apply the Lebesgue limit theorem. This can be accomplished using a similar argument to the one used in the proof of Lemma 8.9 for obtaining a bound on (8.43) (compare with the derivation of (8.75)). We can start with

$$\begin{aligned} \left|\Gamma(q^0_{\mu_k}(t-k\pi))\right| &= \left|\left(\frac{y_H\dot{y}_H + x_H\dot{x}_H}{(r_2)^5}\right)(t-k\pi)\right| \\ &\le \left(\frac{|y_H|\,|\dot{y}_H| + |x_H|\,|\dot{x}_H|}{|y_H|)^5}\right)(t-k\pi) \\ &\le \left(\frac{|\dot{y}_H| + |\dot{x}_H|}{|y_H|^4}\right)(t-k\pi). \end{aligned} \tag{8.88}$$

These estimations come from the fact that $|r_2| > |y_H| > |x_H|$. From the fact that

$$y_H(t) = \mu^{-1/3}y(t) = |q^0_{\mu_k}(t)|\sin\alpha(t), \tag{8.89}$$

we can follow with the estimates using the bound $|q^0_{\mu_k}(t-k\pi)| > 1/2$, the formula (8.63) for α, and the fact that $N(\infty) - 2M(\infty) > 0$ [21], which gives

$$\begin{aligned} \left(\frac{|\dot{y}_H| + |\dot{x}_H|}{|y_H|^4}\right)(t-k\pi) &\le \left|\frac{M}{(\mu^{-1/3}|q^0_{\mu_k}(t-k\pi)|\sin\alpha(t-k\pi))^4}\right| \\ &\le \left|\frac{M}{(\mu^{-1/3}\frac{1}{2}\left[\alpha(t-k\pi)+\pi\right])^4}\right| \\ &\le \left|\frac{M}{(\mu^{-1/3}\frac{1}{2}\mu^{1/3}\left[(N(\infty)-2M(\infty))\,(t+T_k-k\pi)\right])^4}\right| \\ &\le \left|\frac{\tilde{M}}{t^4}\right|. \end{aligned} \tag{8.90}$$

for some large $\tilde{M} \in \mathbb{R}$. This means that we can choose our $g(t)$ as $\tilde{M}/t^4$ and the convergence (8.82) is now the consequence of the Lebesgue Theorem (see for example [28, Theorem 38]). □

Using the above Lemma and (8.80), for $\mu = \mu_k$ we have

$$|M_t(0)| = |M_k + O(\mu_k^{1/4})| \overset{k\to\infty}{\to} |M_H|. \tag{8.91}$$

If we can show that M_H is nonzero then for sufficiently small μ_k we will have $M_t(0) \neq 0$. The numerical computation of (8.83) using Maple give an approximate solution

$$M_H = 2.06, \tag{8.92}$$

which by (8.91) means that for sufficiently small μ_k we will have

$$|M_t(0)| \approx 2.06. \tag{8.93}$$

We therefore know that for sufficiently small μ_k the integral $M_t(0)$ cannot be equal to zero, which means that $M(t_0)$ has a simple zero at $t_0 = 0$, which in turn by Theorem 7.9 proves the transversal intersection of the perturbed stable and unstable manifolds of the Lapunov orbits which survive under perturbation.

9
Proof of the main Theorem.

Let us reformulate our main Theorem 1.1 into a rigorous form.

Theorem 9.1

For sufficiently small $\mu \in \{\mu_k\}_{k=2}^{\infty}$ there exists an interval of energies $[C, \tilde{C}]$ close to the energy of the libration point L_2 and a subset $\mathfrak{C}$ of $[C, \tilde{C}]$ with complement measure of order $e^{1/2}$, such that for any $c_1, c_2 \in \mathfrak{C}$ the Lapunov orbits $l(c_1)$ and $l(c_2)$ associated which the energies c_1 and c_2 survive under a small perturbation $e > 0$. This means that they are perturbed into one dimensional invariant tori $l_e(c_1)$ and $l_e(c_2)$ of the time 2π shift along a trajectory map $P^e_{t_0}$

$$P^e_{t_0} : \Sigma_{t_0} \to \Sigma_{t_0+2\pi}, \tag{9.1}$$

of the elliptic problem (4.1), where $\Sigma_s = \{(x, y, p_x, p_y, t) \in \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} | t = s\}$.

What is more there exists an interval I in $[C, \tilde{C}]$ of a measure of order $e^{1/2}$, such that for any c_1, $c_2 \in \mathfrak{C} \cap I$ there exists a transition chain between $l_e(c_1)$ and $l_e(c_2)$. By a transition chain we mean that there exists a number $N > 0$ and energies $c_1 = C_1 < C_2 < ... < C_N = c_2$, such that the Lapunov orbits $l(C_i)$ survive under perturbation $e > 0$ and are perturbed into invariant tori $l_e(C_i)$ for $i = 1, \ldots, N$. In addition to this the stable manifold $W^s(l_e(C_i), P^e_{t_0})$ intersects with the unstable manifold $W^u(l_e(C_{i+\delta}), P^e_{t_0})$ for all $i = 1, \ldots, N$ and $\delta \in \{-1, 0, 1\}$.

Proof

From the previous Chapter we know that for t_0 equal to zero the Melnikov integral $M(t_0)$ has a simple zero. There we have also shown that for sufficiently small μ_k the derivative of $M(t_0)$ at zero will be close to

$$|\frac{\partial}{\partial t_0} M(0)| \approx 2.06. \tag{9.2}$$

Having fixed a small μ_k we can therefore find an interval $[C, \tilde{C}]$ so that we have both the results of the KAM Theorem 6.16, which ensures the survival of the perturbed Lapunov orbits $l(c)$ for c from the Cantor set $\mathfrak{C}$, and of the Theorem 7.9 and the Remark 7.11, which ensures the transversal intersections of stable and unstable manifolds of neighboring tori.

What is left to show is that for any $e > 0$ there exists an interval $I \subset [C, \tilde{C}]$ of the measure of order $e^{1/2}$, for which the set of energies $\mathfrak{C} \cap I$ for which Lapunov orbits persist under perturbation has gaps smaller than κe, where κ is the parameter from Remark 7.11. This and the Remark 7.11 will allow us to construct transition chains between any two energies from $\mathfrak{C} \cap I$.

The fact that such an interval exists will follow from the result of Poschel [27], who has proved that the complement of the Cantor set $\mathfrak{C} \subset [C, \tilde{C}]$ is of the measure $O(e^{1/2})$. We wish to show that in $[C, \tilde{C}]$ there exists an interval I of a measure of order $e^{1/2}$, such that $\mathfrak{C} \cap I$ does not contain gaps greater than κe. To see this let us divide the the interval $[C, \tilde{C}]$ into n equal parts. If on every interval the set $\mathfrak{C}$ contains gaps larger than κe, then from the fact that the measure of the complement of $\mathfrak{C}$ is $O(e^{1/2})$ (let us say that this $O(e^{1/2})$ is equal to $Me^{1/2}$ for some $M > 0$) the number of such intervals n must satisfy

$$n\kappa e \leq Me^{1/2}, \tag{9.3}$$

which means that $n \leq \frac{M}{\kappa e^{1/2}}$. If we divide the interval $[C, \tilde{C}]$ into a slightly larger number $\tilde{n}$ of equal intervals then at least one of them (this will be our interval I) cannot contain a gap larger than κe. The size of such an interval is equal to

$$\frac{|\tilde{C} - C|}{\tilde{n}} \approx \frac{|\tilde{C} - C|}{n} = e^{1/2} \frac{|\tilde{C} - C|\kappa}{M}, \tag{9.4}$$

which is clearly a measure of the order $e^{1/2}$. This concludes our proof. □

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