Summary of Academic Accomplishments

1. *Name*: Maciej J. Capiński

2. *Degrees*:

2000 MSc in mathematics, with distinction, Institute of Mathematics, Jagiellonian University, Kraków, Poland,

2005 PhD in mathematics, with distinction, Institute of Mathematics, Jagiellonian University, Kraków, Poland, titled "Arnold diffusion in the planar restricted elliptic three body problem", supervised by Piotr Zgliczyński.

3. *Employment history*:

- September 2004 - September 2005, lecturer at University of Hull, UK,
- September 2005 - September 2006, assistant at Institute of Mathematics, Jagiellonian University, Kraków,
- September 2006 - September 2007, assistant at Faculty of Applied Mathematics, AGH, Kraków,
- January 2008 - May 2008, January 2009, visiting professor at the African Institute for Mathematical Sciences, Muizenberg, Cape Town, South Africa,
- September 2010 - December 2010, visiting professor, University of Texas at Austin, United States,
- August 2012 - March 2013, visiting professor, Georgia Institute of Technology, United States,
- **current position:** Since September 2007, associate professor (adiunkt) at Faculty of Applied Mathematics, AGH, Kraków.

4. *Academic accomplishment fulfilling the requirement of article 16 A.2 from 14 March 2003, concerning academic degrees (Dz. U. nr 65, poz. 595 ze zm.):*

a) *Title:*

Topological and Geometric Methods for Normally Hyperbolic Invariant Manifolds in Dynamical Systems

b) *List of publications:*

The following publications constitute a series on which the habilitation is based:

[C] M.J. Capiński, *Covering relations and the existence of topologically normally hyperbolic invariant sets*, Discrete Contin. Dyn. Syst. Ser. A 23 (3) (2009) 705–725.

[CZ1] M.J. Capiński, P. Zgliczyński, *Cone conditions and covering relations for topologically normally hyperbolic manifolds*, Discrete Contin. Dyn. Syst. 30 (2011) 641–670.

[CR] M.J. Capiński, P. Roldán, *Existence of a center manifold in a practical domain around L1 in the restricted three-body problem*, SIAM J. Appl. Dyn. Syst. 11 (1) (2012) 285–318.

[CS] M.J. Capiński, C. Simó, *Computer assisted proof for normally hyperbolic manifolds*, Nonlinearity 25 (2012) 1997-2026.

[CZ2] M.J. Capiński, P. Zgliczyński, *Geometric proof for normally hyperbolic invariant manifolds*, J. Differential Equations 259 (2015) 6215–6286.

c) *Overview of academic accomplishments and applications:*

Below we discuss the results. We start with the motivation. We recall the classical normally hyperbolic manifold theorem and use it to pose questions, which were the main motivation for the research. Section 2 places the work in the context of current developments in the literature. In section 3 we present the main theoretical results, and finish with applications presented in section 4.

1 Motivation

The theory of normally hyperbolic invariant manifolds has been developed by Hirsch, Pugh, Shub [16] in the setting of discrete dynamical systems and by Fenichel [6] for flows. The theory has found numerous applications. It is used for the study of global perturbations, resonance and diffusion, and for investigation of geometric properties of dynamical systems. It is frequently applied by physicists, engineers, and mathematicians working on nonlinear problems from a geometric viewpoint.

In our discussion we shall focus mainly on the theory for discrete dynamical systems. All of the results though can also be applied to ODEs, by considering time shift maps along the flow.

We start with the definition of a normally hyperbolic manifold:

Definition 1. Let $\Lambda \subset \mathbb{R}^n$ be a manifold, invariant under $f : \mathbb{R}^n \to \mathbb{R}^n$, i.e., $f(\Lambda) = \Lambda$, where f is a C^r-diffeomorphism, $r \geq 1$. We say that Λ is a normally hyperbolic invariant manifold if there exists a constant $C > 0$, rates

$$0 < \lambda < \mu^{-1} < 1 \tag{1}$$

and a splitting for every $\mathbf{x} \in \Lambda$

$$\mathbb{R}^n = E^u_{\mathbf{x}} \oplus E^s_{\mathbf{x}} \oplus T_{\mathbf{x}}\Lambda$$

such that

$$\begin{aligned} v \in E^u_{\mathbf{x}} &\Leftrightarrow \|Df^n(\mathbf{x})v\| \leq C\lambda^{|n|} \|v\|, \quad n \leq 0, \\ v \in E^s_{\mathbf{x}} &\Leftrightarrow \|Df^n(\mathbf{x})v\| \leq C\lambda^{n} \|v\|, \quad n \geq 0, \\ v \in T_{\mathbf{x}}\Lambda &\Leftrightarrow \|Df^n(\mathbf{x})v\| \leq C\mu^{|n|} \|v\|, \quad n \in \mathbb{Z}. \end{aligned}$$

The $E^s_{\mathbf{x}}$ is the stable (contracting) eigenspace, and $E^u_{\mathbf{x}}$ is unstable (expanding) eigenspace. Intuitively, an invariant manifold Λ is normally hyperbolic, if the dynamics on Λ is weaker than the dynamics in the stable and unstable directions. This is reflected in the assumption (1), which we will refer to as the *rate condition*. (This condition is also frequently referred to as the *spectral gap condition* throughout the literature.)

In the sequel, we will assume that Λ is compact or that f is uniformly C^r in a neighbourhood of Λ.

Given a normally hyperbolic invariant manifold we define its unstable and stable manifold as

$$\begin{aligned} W^u(\Lambda, f) &= \{\mathbf{y} \in \mathbb{R}^n | d(f^n(\mathbf{y}), \Lambda) \leq C_{\mathbf{y}}\lambda^{|n|}, n \leq 0\}, \\ W^s(\Lambda, f) &= \{\mathbf{y} \in \mathbb{R}^n | d(f^n(\mathbf{y}), \Lambda) \leq C_{\mathbf{y}}\lambda^{n}, n \geq 0\}. \end{aligned}$$

The manifolds $W^u(\Lambda, f)$, $W^s(\Lambda, f)$ are foliated by

$$\begin{aligned} W^u(\mathbf{x}, f) &= \{\mathbf{y} \in \mathbb{R}^n | \|f^n(\mathbf{y}) - f^n(\mathbf{x})\| \leq C_{\mathbf{x},\mathbf{y}}\lambda^{|n|}, n \leq 0\}, \\ W^s(\mathbf{x}, f) &= \{\mathbf{y} \in \mathbb{R}^n | \|f^n(\mathbf{y}) - f^n(\mathbf{x})\| \leq C_{\mathbf{x},\mathbf{y}}\lambda^{n}, n \geq 0\}. \end{aligned}$$

We define

$$l = \min\left\{r, \frac{|\ln \lambda|}{\ln \mu}\right\}.$$

Theorem 2. *[16] Let $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}^n$ be a family of C^r diffeomorphisms with $r \geq 1$. Assume that Λ is a normally hyperbolic invariant manifold for f_0 with rates λ, μ. Then there exists an $\varepsilon_0 > 0$ such that for $|\varepsilon| < \varepsilon_0$ there exist C^l normally hyperbolic manifolds Λ_ε invariant under f_ε, with rates λ, μ. The manifolds $W^u(\Lambda_\varepsilon, f_\varepsilon)$, $W^s(\Lambda_\varepsilon, f_\varepsilon)$ are C^l and the fibers $W^u(\mathbf{x}, f_\varepsilon)$, $W^s(\mathbf{x}, f_\varepsilon)$ are C^r. The manifolds Λ_ε are C^l close to Λ. The same holds for the stable/unstable manifolds and their fibers.*

Remark 3. The statement of Theorem 2 written out in [16] is weaker in terms of regularity of the manifold. Theorem 2 reflects the current development in the theory, and its modern survey can be found in [22].

The results of the habilitation address the following questions:

1. Theorem 2 relies on the fact that the dynamics on Λ is weaker than the associated contraction and expansion. (This is reflected in the rate condition (1) in the definition of a normally hyperbolic manifold.) Is it possible to obtain a version of the theorem without assuming (1)?

2. Theorem 2 needs the existence of Λ as one of its assumptions. Is it possible to provide a non-perturbative version of the theorem, which would not need to assume existence of a manifold?

3. Theorem 2 establishes the existence of stable/unstable manifolds and fibers, but does not provide information about their size and positioning. Is it possible to obtain explicit estimates on size and position of $W^u(\Lambda_\varepsilon, f_\varepsilon)$, $W^s(\Lambda_\varepsilon, f_\varepsilon)$ and $W^u(\mathbf{x}, f_\varepsilon)$, $W^s(\mathbf{x}, f_\varepsilon)$?

4. Theorem 2 does not give explicit bounds on the size of the perturbation under which the manifolds persist. Is it possible to provide explicit estimates for the parameter ε_0 from Theorem 2?

We now make several comments on above questions to place our results in their context.

The first question is subtle. There are examples [17] in which in the absence of rate conditions (1) an invariant manifold can be destroyed, under arbitrarily small perturbation, to a set which is not even a topological manifold. This does not mean though that the manifold vanishes or that it is completely destroyed. This problem has been studied by Floer in [11]. He has shown that if the perturbation is small enough, then we have an invariant set with a cohomology ring that contains the cohomology of the unperturbed manifold as a subring. We take a different approach in [C]. We establish existence of an invariant set that contains a graph over Λ. The advantage of our method is that it does not need to assume the existence of a normally hyperbolic manifold and to use perturbation theory. The method from [2] can also be applied in a perturbative setting to determine an explicit size of the perturbation of the system under which the invariant set persists. This is the subject of results presented in section 3.1.

The second question is partially answered in [C]. The method is not perturbative, and ensures existence of an invariant set within a given neighbourhood, provided that certain assumptions, referred to later as "covering conditions", are satisfied. These assumptions though do not ensure that the invariant set is a manifold. In Section 3.2 we present the results from [CZ1], where by adding additional assumptions, which are later referred to as "cone conditions", it can be determined that the invariant set is indeed a manifold, with normally hyperbolic-type properties. The paper [CZ1] does not address the issue of regularity of the manifold. The regularity can be established by considering additional assumptions, which we refer to as the "rate conditions". This is the subject of the paper [CZ2], and the results are presented in section 3.3. All results mentioned above are not perturbative, and give explicit estimates on where the invariant manifold is positioned.

The paper [CZ2] gives also results which ensure explicit bounds on the positioning and slopes of $W^u(\Lambda_\varepsilon, f_\varepsilon)$, $W^s(\Lambda_\varepsilon, f_\varepsilon)$ and $W^u(\mathbf{x}, f_\varepsilon)$, $W^s(\mathbf{x}, f_\varepsilon)$, thus answering the third question.

All of the methods mentioned above can also be applied in a perturbative setting. If we consider a family of maps, then as long as assumptions hold for each map, we can established existence of the manifolds. This way one can obtain explicit estimates on the size of perturbations under which the manifolds will survive, thus answering question 4.

Our objective was to develop a method that can be applier in practical setting. The emphasis is on developing tools, which can be applied to produce computer assisted proofs. The papers [CS] and [CR] contain examples of applications of the methodology. The paper [CS] considers the driven logistic map. This is one of the classical examples which demonstrates that seemingly simple numerical simulations can lead to incorrect conclusions (here the false evidence points to chaotic dynamics, whereas in truth the system has a smooth attractor). This is a good example to test the theory, since due to numerical difficulty it poses a real challenge. The paper [CR] gives a proof of existence and explicit bounds for a centre manifold in the restricted three body problem (R3BP). The R3BP is one of the classical examples in dynamical systems whose history goes back 300 years to the works of Laplace, Newton and Poincaré [19, 20, 21]. The problem has been studied extensively. A method can prove its worth by producing new results in the field.

2 Literature overview

In this section we place our results in the context of current developments in the field.

In the standard approach to the proof of various invariant manifold theorems, all considerations are done in suitable function spaces or sequences spaces. Moreover, the results are perturbative and the existence of an invariant manifold for an unperturbed map (or ODE) is usually assumed [5, 6, 7, 8, 9, 16, 22]. Typically these proofs do not give any computable bounds for the size of perturbation for which the invariant manifold exists.

Our results are in similar spirit to a number of results for establishing invariant manifolds that have recently emerged, which assume that there exists a manifold that is 'approximately' invariant, and provide conditions that ensure the existence of a true invariant manifold within a given neighbourhood. In [1] Bates, Lu and Zeng present such approach within a context of semiflows, which makes their method general and applicable to infinite dimensional systems and PDEs. Compared to [1] our results is more explicit. Contrary to [1], where main theorems require that some constants are sufficiently small depending on other constants, in our main theorem we just have several explicit inequalities. In [4, 10, 12, 13, 14, 15] Calleja, Celletti, Haro, de la Llave, Figueras, Fontich and Sire provide a framework and results for establishing existence of whiskered tori with quasi periodic dynamics, which is suitable for computer assisted validation.

All above proofs are based on constructions in suitable function spaces. In contrast, in our method the whole proof is made in the phase space. This method is not entirely new. For example, a similar approach is adapted in the proof of Jones [18] in the context of slow-fast system of ODEs. Jones though considered a perturbation of a normally hyperbolic invariant manifold. A topological approach has been applied by Berger and Bounemoura [2], where persistence and smoothness of invariant manifolds is established using geometric and topological methods. Their result relies though on a perturbation of a normally hyperbolic invariant manifold. It is a later result to [C, CZ1] and uses similar methodology. The improvement is that it establishes C^1 regularity of the manifolds. Another topological approach is that of Floer [11], who used the Conley index to study the problem. His results are also perturbative.

In the current state of development there seem to be two methods that can be applied to produce computer assisted proofs: the above mentioned works of Calleja, Celletti, Haro, de la Llave, Figueras, Fontich and Sire [4, 10, 12, 13, 14, 15] and the method which is the topic of this presentation. The former is based on a parametrisation method, which involves solving a fixed point equation in a suitable function space. In order to do so, it is required to expand the map (or vector field) to a high order Taylor or Fourier representation and to perform manipulations on such series. This can be numerically costly. In contrast, our method does not required such expansions and can be applied without the need of estimating higher order derivatives. It can also be applied in the absence of higher order differentiability of the considered manifolds (which is the usual setting close to their breakdown). The tradeoff is that the parametrisation method can produce more accurate enclosures. The two methods seem to complement each other, the first focusing on the tools from functional analysis, and the second on geometric and topological construction.

3 Theoretical framework

In this section we give the main theoretical results, which are based on the papers [C,CZ1,CZ2].

3.1 Topologically normally hyperbolic sets [C]

In this section we relax the assumptions of the normally hyperbolic invariant manifold theorem (Theorem 2) to consider topological hyperbolicity.

Before we present our results we need two auxiliary definitions. Let $\overline{B_n}$ denote a closed ball of radius R centred at zero in $\mathbb{R}^n$. (Throughout the discussion R will be fixed.)

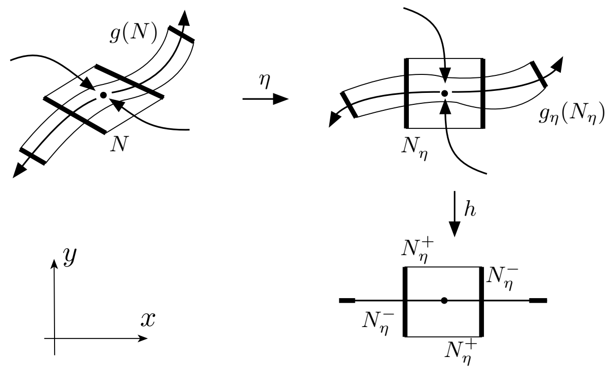

Figure 1: An h-set N, and a covering relation $N \overset{g}{\Longrightarrow} N$, in the setting of a hyperbolic fixed point.

Definition 4. [23] An h-set, is an object consisting of the following data

1. N - a compact subset of $\mathbb{R}^k$,
2. $u, s \in \{0, 1, 2, 3, \ldots\}$, such that $u + s = k$,
3. a homeomorphism $\eta_N : \mathbb{R}^k \to \mathbb{R}^k = \mathbb{R}^u \times \mathbb{R}^s$ such that

$$\eta_N(N) = \overline{B_u} \times \overline{B_s}.$$

We consider the following subsets of N_η,

$$\begin{aligned} N_\eta &= \overline{B_u} \times \overline{B_s}, \\ N_\eta^- &= \partial\overline{B_u} \times \overline{B_s}, \\ N_\eta^+ &= \overline{B_u} \times \partial\overline{B_s}, \\ N^- &= \eta_N^{-1}(N_\eta^-), \\ N^+ &= \eta_N^{-1}(N_\eta^+). \end{aligned}$$

Definition 5. [23] Assume that N, M are h-sets. Let $g : N \to \mathbb{R}^k$ be a continuous map and let $g_\eta = \eta_M \circ g \circ \eta_N^{-1} : N_\eta \to \mathbb{R}^u \times \mathbb{R}^s$. We say that N *g-covers* M, which we denote as

$$N \overset{g}{\Longrightarrow} M,$$

if the following conditions are satisfied:

1. There exists a continuous homotopy $h : [0,1] \times N_\eta \to \mathbb{R}^u \times \mathbb{R}^s$ such that the following conditions hold true

$$\begin{aligned} h_0 &= g_\eta, \\ h([0,1], N_\eta^-) \cap M_\eta &= \emptyset, \\ h([0,1], N_\eta) \cap M_\eta^+ &= \emptyset. \end{aligned}$$

2.1. If $u > 0$, then there exists a linear map $A : \mathbb{R}^u \to \mathbb{R}^u$, such that

$$\begin{aligned} &h_1(x,y) = (Ax, 0), \quad \text{where } x \in \mathbb{R}^u \text{ and } y \in \mathbb{R}^s, \\ &A(\partial B_u) \subset \mathbb{R}^u \backslash \overline{B_u}. \end{aligned}$$

2.2. If $u = 0$, then

$$h_1(y) = 0, \quad \text{for } \ y \in N_\eta.$$

The idea behind Definition 5 is that the coordinate $x \in \mathbb{R}^u$ is the direction of topological expansion and $y \in \mathbb{R}^s$ is the coordinate of topological contraction. (The notations u, s stand for "unstable" and "stable", respectively.)

To provide some intuition for the Definitions 4 and 5, let us illustrate the setting in the case of a hyperbolic fixed point (see Figure 1). I such a case we can take $N_\eta = M_\eta$ to be a small box surrounding the fixed point. The homotopy h corresponds to a projection onto the unstable coordinate in the linearized coordinates of hyperbolic expansion and contraction. The homeomorphism η is the local change of coordinates around the fixed point.

Let us note that the class of functions satisfying Definition 5 is broader than those having a hyperbolic invariant set. In particular, Definition 5 does not require the function to be differentiable.

We now set up the notations needed for topological normal hyperbolicity. Let D be a compact set in $\mathbb{R}^k$. Let us assume that there exists a neighbourhood U of D and a homeomorphism $\phi : U \to \mathbb{R}^k$, such that

$$\phi(D) = \Lambda \times N,$$

where $N = \overline{B_u} \times \overline{B_s}$ and Λ is a compact $c = k - u - s$ dimensional sub-manifold, without a boundary, in $\mathbb{R}^k$.

We consider a homeomorphism

$$f : U \to U.$$

We will look for an invariant set in the interior of D, which projects onto the manifold $\phi^{-1}(\Lambda, 0, 0)$. The exact meaning of this statement will be made clear in the formulation of the result.

A point $\mathbf{x}$ in $\Lambda \times N$ will be represented as $\mathbf{x} = (\lambda, x, y)$, where λ, x, and y correspond to Λ, $\overline{B_u}$ and $\overline{B_s}$ coordinates respectively. For a given point $\lambda \in \Lambda$ we will use the notations f_λ and, f_λ^{-1} for functions

$$f_\lambda, f_\lambda^{-1} : N \to \mathbb{R}^{u+s},$$

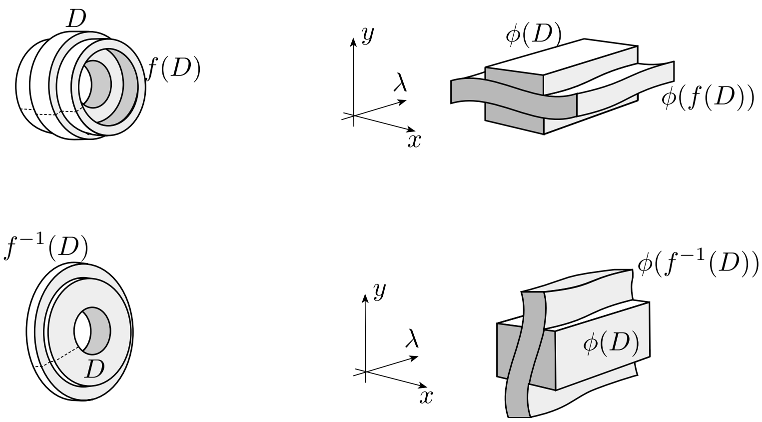

Figure 2: Example of a function which satisfies covering conditions.

defined as

$$f_\lambda(x,y) := \pi_{x,y} \circ \phi \circ f \circ \phi^{-1}(\lambda, x, y),$$
$$f_\lambda^{-1}(x,y) := \pi_{x,y} \circ \phi \circ f^{-1} \circ \phi^{-1}(\lambda, x, y),$$

where $\pi_{x,y}$ is the projection onto the x, y coordinates. In line with Definition 5 we will adapt a notation in which x will be the unstable and y will be the stable coordinate (in the topological sense of covering relations) for the maps f_λ.

Definition 6. If for any $\lambda \in \Lambda$

$$N \stackrel{f_\lambda}{\Longrightarrow} N, \tag{2}$$

and

$$N \stackrel{f_\lambda^{-1}}{\Longrightarrow} N. \tag{3}$$

Then we say that f *satisfies covering conditions*.

Remark 7. For (3) we make a natural assumption that the roles of the stable and unstable directions are reversed with respect to (2). The coordinates x become the stable coordinates and y the unstable coordinates for the maps f_λ^{-1}.

We give a simple example of a function which satisfies covering conditions in Figure 2. In this example $D \subset \mathbb{R}^3$ and $\Lambda = \mathbb{S}^1$. The cut from the left plot is depicted in grey on the right. Thus, the front and back face of $\phi(D)$ on the right hand plots coincide.

The following theorem is the main result from [C]. It gives a tool for the establishing an invariant set for the map f.

Theorem 8. *[C, Theorem 3.1] If $f : U \to U$ is a homeomorphism which satisfies covering conditions, then for any $\lambda \in \Lambda$ the set*

$$K_\lambda := \{\mathbf{x} \in D | f^m(\mathbf{x}) \in D \text{ for all } m \in \mathbb{Z}, \text{ and } \mathbf{x} = \phi^{-1}(\lambda, x, y) \text{ for some } x \in \overline{B_u}, y \in \overline{B_s}\}$$

is nonempty and lies in the interior of D.

It is important to note that in Definition 6 we do not assume any expansion or contraction properties for the coordinate λ. In contrast to condition (1), the dynamics on this coordinate can be much stronger than the dynamics on x, y. What is only needed is that the x and y are coordinates of topological expansion and contraction, respectively.

3.2 Topologically normally hyperbolic manifolds [CZ1]

In this section we extend the results from section 3.1, by adding additional assumptions, which ensure that the invariant set obtained in Theorem 8 is a C^0 normally hyperbolic manifold, with associated stable and unstable manifolds.

Let Λ be a c-dimensional manifold without boundary and let N be an h-set. To formulate the results we will make an additional assumption that

$$\Lambda = (\mathbb{R}/\mathbb{Z})^c, \tag{4}$$

i.e. Λ is a c-dimensional torus and that

$$D = \Lambda \times N = \Lambda \times \overline{B_u} \times \overline{B_s} \subset \Lambda \times \mathbb{R}^u \times \mathbb{R}^s. \tag{5}$$

We will also assume that the radius R of the balls $\overline{B_u}, \overline{B_s}$ in N satisfies $R < \frac{1}{4}$.

The results in [CZ1] are written in more generality in the context of vector bundles. Here we simplify the setting to focus on the main features, without going into technical details.

We consider two functions

$$Q_h, Q_v : \mathbb{R}^c \times \mathbb{R}^u \times \mathbb{R}^s \to \mathbb{R},$$

defined as

$$Q_h(\lambda, x, y) = -\gamma \|\lambda\|^2 + \alpha \|x\|^2 - \beta \|y\|^2,$$
$$Q_v(\lambda, x, y) = -\gamma \|\lambda\|^2 - \alpha \|x\|^2 + \beta \|y\|^2,$$

where $\alpha, \beta, \gamma \in \mathbb{R}$ and $\alpha, \beta, \gamma > 0$. These functions can be used to define cones. In Figure 3 we see the level sets of Q_h and Q_v for $0 < c_1 < c_2$. The sets $\{Q_h \geq 0\}$ and $\{Q_v \geq 0\}$ consist of cones starting from the origin; we refer to these sets as horizontal and vertical cones, respectively.

We consider an invertible map

$$f : \Lambda \times \mathbb{R}^u \times \mathbb{R}^s \to \Lambda \times \mathbb{R}^u \times \mathbb{R}^s.$$

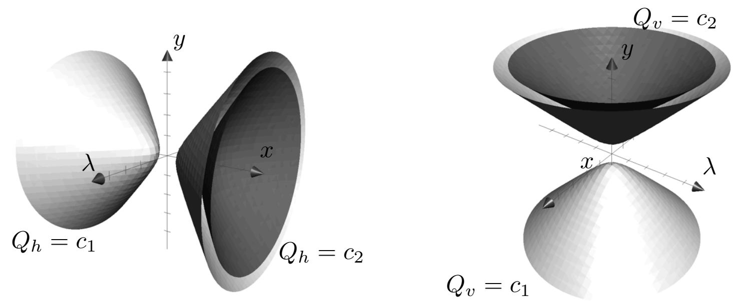

Figure 3: The level sets $\{Q_h = c_1\}$, $\{Q_v = c_1\}$ (in light gray) and $\{Q_h = c_2\}$, $\{Q_v = c_2\}$ (in dark gray), for $0 < c_1 < c_2$.

Definition 9. We say that f *satisfies* (Q, m) *cone conditions*, if for any $\mathbf{x}_1, \mathbf{x}_2 \in D$ such that $Q(\mathbf{x}_1 - \mathbf{x}_2) \geq 0$,

$$Q(f(\mathbf{x}_1) - f(\mathbf{x}_2)) > mQ(\mathbf{x}_1 - \mathbf{x}_2). \tag{6}$$

The intuition behind above definition is as follows. Let us consider $Q = Q_h$ and $m > 1$. The condition $Q_h(\mathbf{x}_1 - \mathbf{x}_2) \geq 0$ states that $\mathbf{x}_2$ lies within the horizontal cone centred at $\mathbf{x}_1$. For such two points (6) implies that we have expansion in terms on the level sets of Q_h. Intuitively this means that $\mathbf{x}_1$ and $\mathbf{x}_2$ will be pushed apart in the horizontal direction.

We now write out the main result from [CZ1].

Theorem 10. *[CZ1, Theorem 4.7] If f satisfies covering conditions and there exists an $m > 1$ such that f satisfies (Q_h, m) cone conditions and f^{-1} satisfies (Q_v, m) cone conditions, then:*

1. *There exists a continuous function*

   $$\chi : \Lambda \to \mathrm{int} D,$$

   such that $\pi_\lambda \chi(\lambda) = \lambda$, and

   $$\chi(\Lambda) = \{\mathbf{x} \in D : f^n(\mathbf{x}) \in D \textit{ for all } n \in \mathbb{Z}\}.$$

2. *There exist C^0 submanifolds W^u, W^s such that*

   $$W^u \cap W^s = \chi(\Lambda).$$

   W^u consists of all points whose backward iterations converge to $\chi(\Lambda)$, and W^s consists of all points whose forward iterations converge to $\chi(\Lambda)$.

The results from [CZ1] are written in a more general setting than above. They treat the case of a vector bundle over a base which is a compact manifold without boundary (and not only a trivial bundle over a torus, which was discussed here). This involves setting up the assumptions using the atlas of the manifold, which makes them more involved than above. This also allows more flexibility, allowing for instance to have different Q_h and Q_v in different local maps. In such general setting, the definition of the covering condition also needs to be written in terms of local maps, making it somewhat more complicated than Definition 6.

The paper [CZ1] also demonstrates how covering conditions and cone conditions can be verified. It is shown that they can be validated based on the estimates on the derivatives of f and f^{-1}. The method is applied to a family of rotating Hénon maps, producing an existence proof of normally hyperbolic invariant curves for the maps over a given range of parameters. This example is presented in more detail in section 4.3.

3.3 Geometric proof for normally hyperbolic invariant manifolds [CZ2]

The drawback of the method [CZ1] is that the manifolds obtained in Theorem 10 are known only to be continuous. This is to be expected, since the proof is based on C^0 assumptions on the map. A natural question is if by adding additional assumptions one can obtain higher order regularity. This is the topic of this section.

As in (4), we assume that Λ is a torus, and consider a set D of the form (5). We now consider a C^{k+1} map, for $k \geq 1$,

$$f : D \to \Lambda \times \mathbb{R}^u \times \mathbb{R}^s.$$

We will write f as $f = (f_\lambda, f_x, f_y)$, where f_λ, f_x, f_y stand for projections onto Λ, $\mathbb{R}^u$ and $\mathbb{R}^s$, respectively.

Remark 11. The results from [CZ2] do not require f to be invertible. This is an improvement compared to [CZ1], which required invertibility.

We consider a constant $R_\Lambda < \frac{1}{2}$ and assume that the radius R of the balls $\overline{B_u}$, $\overline{B_s}$ is small enough so that $R < \frac{1}{2}R_\Lambda$. For a point $\mathbf{x} \in D$ we will use the notation

$$P(\mathbf{x}) = \{\mathbf{z} \in D \mid \|\pi_\lambda \mathbf{x} - \pi_\lambda \mathbf{z}\| \leq R_\Lambda/2\}.$$

To formulate our result, we need to define a number of constants. First we state the following auxiliary definition:

Definition 12. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Let $\|x\|$ be any norm on $\mathbb{R}^n$, then we define

$$m(A) = \max\{L \in \mathbb{R} : \|Ax\| \geq L\|x\| \text{ for all } x \in \mathbb{R}^n\}.$$

For an interval matrix $\mathbf{A} \subset \mathbb{R}^{k\times n}$ we set

$$m(\mathbf{A}) = \inf_{A\in\mathbf{A}} m(A).$$

Let $L \in \left(\frac{2R}{R_\Lambda}, 1\right)$, and let us define the following constants:

$$\mu_{s,1} = \sup_{\mathbf{x}\in D}\left\{\left\|\frac{\partial f_y}{\partial y}(\mathbf{x})\right\| + \frac{1}{L}\left\|\frac{\partial f_y}{\partial(\lambda,x)}(\mathbf{x})\right\|\right\},$$
$$\mu_{s,2} = \sup_{\mathbf{x}\in D}\left\{\left\|\frac{\partial f_y}{\partial y}(\mathbf{x})\right\| + L\left\|\frac{\partial f_{(\lambda,x)}}{\partial y}(\mathbf{x})\right\|\right\},$$

$$\xi_{u,1} = \inf_{\mathbf{x}\in D}\left\{m\left(\frac{\partial f_x}{\partial x}(\mathbf{x})\right) - \frac{1}{L}\left\|\frac{\partial f_x}{\partial(\lambda,y)}(\mathbf{x})\right\|\right\},$$
$$\xi_{u,1,P} = \inf_{\mathbf{x}\in D} m\left[\frac{\partial f_x}{\partial x}(P(\mathbf{x}))\right] - \frac{1}{L}\sup_{\mathbf{x}\in D}\left\|\frac{\partial f_x}{\partial(\lambda,y)}(\mathbf{x})\right\|,$$
$$\xi_{u,2} = \inf_{\mathbf{x}\in D}\left\{m\left(\frac{\partial f_x}{\partial x}(\mathbf{x})\right) - L\left\|\frac{\partial f_{(\lambda,y)}}{\partial x}(\mathbf{x})\right\|\right\},$$

$$\mu_{cs,1} = \sup_{\mathbf{x}\in D}\left\{\left\|\frac{\partial f_{(\lambda,y)}}{\partial(\lambda,y)}(\mathbf{x})\right\| + L\left\|\frac{\partial f_{(\lambda,y)}}{\partial x}(\mathbf{x})\right\|\right\},$$
$$\mu_{cs,2} = \sup_{\mathbf{x}\in D}\left\{\left\|\frac{\partial f_{(\lambda,y)}}{\partial(\lambda,y)}(\mathbf{x})\right\| + \frac{1}{L}\left\|\frac{\partial f_x}{\partial(\lambda,y)}(\mathbf{x})\right\|\right\},$$

$$\xi_{cu,1} = \inf_{\mathbf{x}\in D}\left\{m\left(\frac{\partial f_{(\lambda,x)}}{\partial(\lambda,x)}(\mathbf{x})\right) - L\left\|\frac{\partial f_{(\lambda,x)}}{\partial y}(\mathbf{x})\right\|\right\},$$
$$\xi_{cu,1,P} = \inf_{\mathbf{x}\in D} m\left[\frac{\partial f_{(\lambda,x)}}{\partial(\lambda,x)}(P(\mathbf{x}))\right] - L\sup_{\mathbf{x}\in D}\left\|\frac{\partial f_{(\lambda,x)}}{\partial y}(\mathbf{x})\right\|,$$
$$\xi_{cu,2} = \inf_{\mathbf{x}\in D}\left\{m\left(\frac{\partial f_{(\lambda,x)}}{\partial(\lambda,x)}(\mathbf{x})\right) - \frac{1}{L}\left\|\frac{\partial f_y}{\partial(\lambda,x)}(\mathbf{x})\right\|\right\}.$$

These constants can be thought of to measure the strength of the map in given projections, in terms of selected coordinates. We need them to define an analog of the rate condition (1) from Definition 1.

Definition 13. We say that f *satisfies rate conditions of order* $k \geq 1$ if $\xi_{u,1}$, $\xi_{u,1,P}$, $\xi_{u,2}$, $\xi_{cu,1}$, $\xi_{cu,1,P}$, $\xi_{cu,2}$ are strictly positive, holds

$$\mu_{s,1} < 1 < \xi_{u,1,P},$$

$$\frac{\mu_{cs,1}}{\xi_{u,1,P}} < 1, \qquad \frac{\mu_{s,1}}{\xi_{cu,1,P}} < 1,$$
$$\frac{\mu_{cs,2}}{\xi_{u,1}} < 1, \qquad \frac{\mu_{s,1}}{\xi_{cu,2}} < 1,$$

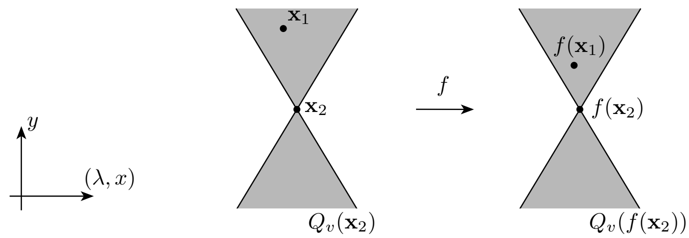

Figure 4: A function which satisfies backward cone conditions.

and for all j satisfying $k \geq j \geq 1$,

$$\frac{(\mu_{cs,1})^{j+1}}{\xi_{u,2}} < 1, \qquad \frac{\mu_{s,2}}{(\xi_{cu,1})^{j+1}} < 1.$$

We see that Definition 13 has more inequalities than the standard rate condition (1) from Definition 1. The reason for this is that we work in a non-perturbative setting. Our results will imply the existence of an invariant manifold inside of the set D. To this end, we measure various rates of contraction and expansion on the entire set D. We can not measure them along the eigenspaces of the manifold (as is done in Definition 1), since we we do not assume its existence.

To formulate the main result of [CZ2] we need one more definition.

Definition 14. Let $Q_v(\mathbf{x}) \subset D$ be a set defined as

$$Q_v(\mathbf{x}) = \{(\lambda, x, y) : \|(\lambda, x) - \pi_{\lambda,x}\mathbf{x}\| \leq 1/L \, \|y - \pi_y \mathbf{x}\|\}.$$

We say that f *satisfies backward cone conditions* if the following condition is fulfilled:

If $\mathbf{x}_1, \mathbf{x}_2, f(\mathbf{x}_1), f(\mathbf{x}_2) \in D$ and $f(\mathbf{x}_1) \in Q_v(f(\mathbf{x}_2))$ then $\mathbf{x}_1 \in Q_v(\mathbf{x}_2)$.

The sets $Q_v(\mathbf{x})$ are cones originating from $\mathbf{x}$ (see Figure 4). We refer to them as vertical cones. Intuitively, a function satisfies backward cone conditions if images of two points can be vertically aligned in terms of Q_v, only provided that the points themselves are vertically aligned. This property is depicted in Figure 4.

An assumption that a function satisfies backward cone condition might seem artificial. In [CZ2] though it is shown that it is necessary to establish existence of a normally hyperbolic manifold, and that without it the invariant sets can have a Cantor type structure.

We can now formulate our main theorem:

Theorem 15. *[CZ2, Theorem 16] If f satisfies covering conditions, rate conditions of order k and backward cone conditions, then there exists a normally* *hyperbolic invariant manifold* $\Lambda^* \subset D$*, with associated stable* $W^s = W^s(\Lambda^*, f)$ *and unstable* $W^u = W^u(\Lambda^*, f)$ *manifolds.*

The W^s*,* W^u *and* Λ^* *are* C^k *manifolds, which are graphs of* C^k *functions*

$$\begin{aligned} w^s &: \Lambda \times \overline{B_s} \to \overline{B_u}, \\ w^u &: \Lambda \times \overline{B_u} \to \overline{B_s}, \\ \chi &: \Lambda \to \overline{B_u} \times \overline{B_s}, \end{aligned}$$

meaning that

$$\begin{aligned} W^s &= \left\{ (\lambda, w^s(\lambda, y), y) : \lambda \in \Lambda, y \in \overline{B_s} \right\}, \\ W^u &= \left\{ (\lambda, x, w^u(\lambda, y)) : \lambda \in \Lambda, x \in \overline{B_u} \right\}, \\ \Lambda^* &= \left\{ (\lambda, \chi(\lambda)) : \lambda \in \Lambda \right\}. \end{aligned}$$

Moreover, $f_{|W^u}$ *is an injection,* w^s *and* w^u *are Lipschitz with constants* L*, and* χ *is Lipschitz with the constant* $\frac{\sqrt{2}L}{\sqrt{1-L^2}}$*. The manifolds* W^s *and* W^u *intersect transversally, and*

$$W^s \cap W^u = \Lambda^*.$$

The manifolds W^s *and* W^u *are foliated by invariant fibers* $W^s(\mathbf{x}, f)$ *and* $W^u(\mathbf{x}, f)$*. The* $W^s(\mathbf{x}, f)$ *and* $W^u(\mathbf{x}, f)$ *are graphs of* C^k *functions*

$$\begin{aligned} w^s_{\mathbf{x}} &: \overline{B_s} \to \Lambda \times \overline{B_u}, \\ w^u_{\mathbf{x}} &: \overline{B_u} \to \Lambda \times \overline{B_s}, \end{aligned}$$

meaning that

$$\begin{aligned} W^s(\mathbf{x}, f) &= \left\{ (w^s_{\mathbf{x}}(y), y) : y \in \overline{B_s} \right\}, \\ W^u(\mathbf{x}, f) &= \left\{ (\pi_\lambda w^u_{\mathbf{x}}(x), x, \pi_y w^u_{\mathbf{x}}(x)) : x \in \overline{B_u} \right\}. \end{aligned}$$

The functions $w^s_{\mathbf{x}}$ *and* $w^u_{\mathbf{x}}$ *are Lipschitz with constants* $1/L$*.*

Theorem 15 can be used to establish the existence of a normally hyperbolic invariant manifold inside of an explicitly given set D. It also gives the bounds on its slope and on the slopes of the stable/unstable manifolds and the stable/unstable fibers. It also establishes that the stable/unstable manifolds (and fibers) are represented as graphs over a given domain. In other words, it states how far they extend and not only that they exist within some small, unspecified neighbourhood of the manifold. The theorem is non-perturbative and can be applied in the setting where a priori we do not know if the manifolds exists. It also establishes the smoothness of these manifolds.

4 Examples of applications

In this section we give three applications of the methods presented in section 3. The presented examples are taken from [CS,CR,CZ2].

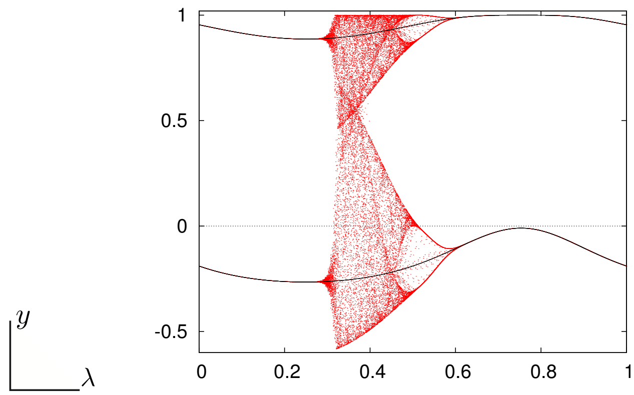

Figure 5: Misleading numerical plot of the attractor for T, obtained using `double precision` (consisting of points), and the true invariant curves computed with 128bit accuracy.

4.1 Driven logistic map [CS]

Consider a driven logistic map

$$T : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1 \times \mathbb{R},$$

defined as

$$T(\lambda, x) = (\lambda + \alpha, 1 - (a_0 + \varepsilon \sin(2\pi\lambda))x^2).$$

This differs from the well-known logistic map in the fact that the parameter a has been replaced by $a_0 + \varepsilon \sin(2\pi\lambda)$ and λ has a quasiperiodic dynamics. Concretely, we consider the parameter values $a_0 = 1.31$, $\varepsilon = 0.3$ and $\alpha = \frac{g}{200}$, where g is the golden mean $g = (\sqrt{5} - 1)/2$. Hence the dynamics on the base of the skew-product is slow. Numerical simulations in `double precision` (say, with mantissa of 52 binary digits) suggest that the map possesses a chaotic global attractor (see Figure 5). When the same simulations are done with multiple precision, one can guess that the attractor consists of two invariant curves (see Figure 5). The same example was considered for other values of α and in a non-rigorous way in [3] to illustrate that one has to be careful with the arithmetics in simulations.

In [CS] it is proved that T does in fact possess a contracting invariant manifold, which consists of two smooth curves. This means in particular that the plot obtained using `double precision` (Figure 5) does not shows the true dynamics. The method presented i sections 3.2 gives an estimate for the position of these curves with 10^{-3} accuracy. The proof is conducted with computer assistance, using rigorous, interval arithmetic based computations. This required some fine-tuning of the method, due to the fact that the example is numerically hard. It defies the standard numerical simulation, producing a false evidence

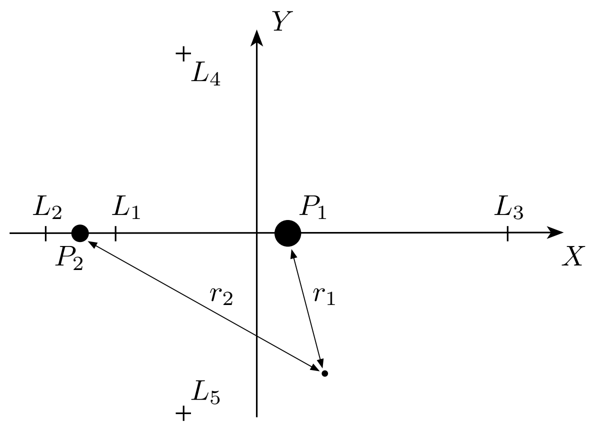

Figure 6: The Sun has the mass $1-\mu$ and is fixed at $P_1 = (\mu, 0)$. The Earth with the mass μ is fixed at $P_2 = (\mu - 1, 0)$. The third massless particle moves in the XY plane.

of chaos, and the true dynamics is revealed only when performing numerics in multiple precision. Validating the true result in interval arithmetic turned out to be nontrivial.

4.2 Center manifold in the planar restricted three body problem [CR]

The problem is defined as follows: two main bodies rotate in the plane about their common center of mass on circular orbits under their mutual gravitational influence. A third body moves in the same plane of motion as the two main bodies, attracted by the gravitation of previous two but not influencing their motion. The problem is to describe the motion of the third body.

We refer to the two large rotation bodies as the *primaries*. In [CR] the chosen primaries are the Sun and the Earth. The third body can be regarded as a satellite or a spaceship of negligible mass.

A convenient choice of coordinates for the problem is to consider a framework which rotates with the primaries. This way the primaries can be positioned on the X axis (Figure 6). It is also customary to rescale the masses μ_1 and μ_2 of the primaries so that $\mu_1 + \mu_2 = 1$. After such rescaling the distance between the primaries is 1. In the Sun-Earth system, the smaller mass is $\mu_2 = \mu = 3.040423398444176 \times 10^{-6}$ and the larger is $\mu_1 = 1 - \mu$. We use a convention in which in the rotating coordinates the Sun is located to the right of the origin at $P_1 = (\mu, 0)$, and the Earth is located to the left at $P_2 = (\mu - 1, 0)$.

The equations of motion of the third body take Hamiltonian form if we consider positions X, Y and momenta $P_X = \dot{X} - Y$, $P_Y = \dot{Y} + X$. The

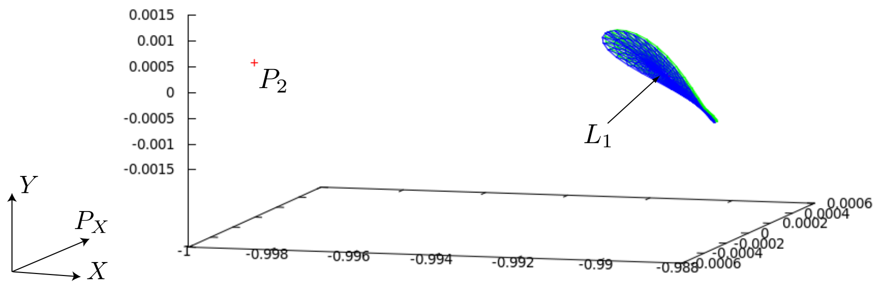

Figure 7: The projection onto X, Y, P_X coordinates of the region containing the center manifold. The centre manifold is a two dimensional disk, which lies between the blue and green surfaces.

Hamiltonian is

$$H = \frac{1}{2}(P_X^2 + P_Y^2) + YP_X - XP_Y - \frac{1-\mu}{r_1} - \frac{\mu}{r_2},$$

where r_1, r_2 denote the distances from the third body to the larger and the smaller primary, respectively (see Figure 6)

$$r_1^2 = (X-\mu)^2 + Y^2,$$
$$r_2^2 = (X-\mu+1)^2 + Y^2.$$

The vector field describing the motion of the massless particle is given by

$$F = J\nabla H, \tag{7}$$
$$J = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}, \quad \mathrm{id} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The restricted three body problem in a rotating frame has five equilibrium points. Three of them, denoted L_1, L_2 and L_3, lie on the X-axis and are usually called the ‘collinear’ equilibrium points (see Figure 6). In [CR] the main focus of investigation is in the neighbourhood of the point L_1.

Since L_1 is a saddle-center fixed point, the centre manifold theorem ensures that there exists a centre manifold associated to it. The centre manifold theorem though is local in nature and does not specify how far the manifold extends. In [CR] the method described in section 3.2 was used to prove that the centre manifold extends over a given neighbourhood around L_1. Figure 7 shows the region throughout which the manifold spans. The objective was to show that the method can produce a macroscopic region.

The method from section 3.2 is for maps, and the restricted three body problem is determined by a Hamiltonian ODE. To apply the method, the conditions which ensure covering and cone conditions were derived directly from the estimates on the vector field. To extend the manifold as far as possible from L_1, in [CS] the problem is considered in suitable coordinates. These were chosen to be the change of coordinates that puts (7) in a normal form. Thus, the proof consisted of merging the theory of normal forms together with rigorous, interval arithmetic based, computer assisted computations, to validate assumptions of Theorem 10.

4.3 Rotating Hńeon map [CZ2]

We consider a one dimensional torus (circle) Λ and a family of rotating Hénon maps $f_\varepsilon : \Lambda \times \mathbb{R}^2 \to \Lambda \times \mathbb{R}^2$,

$$f_\varepsilon(\lambda, x, y) = (\lambda + c + \varepsilon x \cos(2\pi\lambda)), 1 + y - ax^2 + \varepsilon \cos(2\pi\lambda), bx). \tag{8}$$

We take $a = 0.68$, $b = 0.1$ and an arbitrary constant $c \in \mathbb{R}$. We investigate the existence and smoothness of the normally hyperbolic invariant manifold and its associated stable/unstable manifolds for a range of parameters $\varepsilon \in [0, 1/100]$.

For $\varepsilon = 0$ the map (8) has an invariant normally hyperbolic manifold (circle)

$$\Lambda_0^* = \{(\lambda, x^*, y^*) : \lambda \in \mathbb{S}^1\},$$

where

$$x^* = \frac{-(1-b) - \sqrt{(1-b)^2 + 4a}}{2a} \approx -2.043\,3,$$
$$y^* = bx^* \approx -0.204\,33.$$

The question considered in [CZ2] is for which $\varepsilon > 0$ the manifold persists, and what can be said about its regularity. It is proved that the manifold persists from $\varepsilon = 0$ to $\varepsilon = \frac{1}{100}$. It is also shown that the manifold is C^k smooth. Table 1 shows the regularity of the manifold established using Theorem 15. Once again, the proof is computer assisted.

ε	k	ε	k	ε	k
$[0, 0.0001]$	737	$[0.0005, 0.001]$	73	$[0.005, 0.006]$	11
$[0.0001, 0.0002]$	368	$[0.001, 0.002]$	36	$[0.006, 0.007]$	9
$[0.0002, 0.0003]$	245	$[0.002, 0.003]$	24	$[0.007, 0.008]$	8
$[0.0003, 0.0004]$	184	$[0.003, 0.004]$	17	$[0.008, 0.009]$	7
$[0.0004, 0.0005]$	147	$[0.004, 0.005]$	14	$[0.009, 0.01]$	6

Table 1: The C^k regularity of the normally hyperbolic manifold for various parameters.

The main aim of this example was to demonstrate that Theorem 15 can be effectively used to establish estimates on the smoothness of the manifolds.

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5. *Other academic accomplishments*

5.1 Publications

Below is a list of publications, which are not a part of the habilitation:

Research papers:

[A1] M. J. Capiński, K. Wójcik, *Isolating Segments for Carathéodory Systems and Existence of Periodic Solutions*, Proc. Am. Math. Soc. 131, no. 8 (2003), 2443–2451.

[A2] M. J. Capiński and P. Zgliczyński, *Covering relations and non-autonomous perturbations of ODEs*, Discrete Contin. Dyn. Syst. Ser. A, 14 (2006), 281–293.

[A3] M. J. Capiński and P. Zgliczyński, *Transition tori in the planar restricted elliptic three-body problem*, Nonlinearity 24 (2011) 1395–1432.

[A4] M. J. Capiński, *Computer assisted existence proofs of Lyapunov orbits at L2 and transversal intersections of invariant manifolds in the Jupiter-Sun PCR3BP*. SIAM J. Appl. Dyn. Syst., 11(4) (2012), 1723–1753.

[A5] M. J. Capiński *Hedging Conditional Value at Risk with options* European Journal of Operational Research 242 (2015) 688–691.

[A6] M. J. Capiński, A. Wasieczko *Geometric Proof of Strong Stable/Un-stable Manifolds, with Application to the Restricted Three Body Problem*, Topological Methods in Nonlinear Analysis Volume 46, No. 1 (2015), 363–399.

Submitted article:

[P1] M.J. Capiński, M. Gidea, R. de la Llave, *Arnold diffusion in the planar elliptic restricted three-body problem: mechanism and numerical simulation,* http://arxiv.org/abs/1510.00591, submitted to Nonlinearity

Books:

[B1] M. J. Capiński, T. Zastawniak, *Numerical Methods in Finance with C++*, Cambridge University Press (2012).

[B2] M. J. Capiński, P. E. Kopp, *Portfolio Theory and Risk Management*, Cambridge University Press (2014).

Citation statistics:

The papers in total (including both the habilitation series as well as the remaining articles) are cited 34 times by 29 authors on the MathSciNet Mathematical Reviews. The number of citations on Web of Science is 39, with the h-index=4.

We now give an overview of the publications [A1–A6,P1] and the books [B1,B2].

5.1.1 Time dependent perturbations of ODEs [A1, A2]

The paper [A1] deals with a time periodic system, in which the right hand side satisfies Carathéodory conditions; that is, satisfies only measurability assumptions on the time variable. It extends a geometric method of isolating segments to such setting and establishes a theorem which ensures existence of periodic solutions.

The second paper [A2] considers a non periodic time dependent perturbation of an autonomous ODE with certain symbolic dynamics properties. It is shown that regardless of the fact that the perturbation is not time periodic, it can be shown that the perturbed map will have the same symbolic dynamics properties.

5.1.2 Diffusion in the three body problem [A3, A4, A6, P1]

The papers [A3, A4, A6] deal with the the restricted three body problem, which we discussed in section 4.2. The paper [A3] deals with the instability and diffusion in the three body problem. It deals with a conjecture by V.I. Arnlod, which states that a certain type of diffusion (called Arnold diffusion) is a generic property and should be observed in the n-body. Even though the conjecture dates back almost fifty years, it has not yet been proven up till this day. The paper [A3] deals with a special case where the circular restricted three body problem is perturbed to an elliptic problem, where the primaries rotate on ellipses instead of circles. The paper also considers a number of additional conditions on the types of masses considered, and demonstrates a mechanism that leads to diffusion in such setting. This is close to proving the Arnold conjecture, except for two facts: 1) some of the assumptions of the theorems underlying the diffusion mechanism have not been proven in [A3], but are only supported by numerical evidence. 2) The mechanism of diffusion can not overcome distances which would be independent of the perturbation. It is apparent though that such mechanism, with more careful treatment of the underlying geometry, should lead to the proof of the conjecture.

It is in fact the case that the theory for the establishing of normally hyperbolic invariant manifolds, which was the main theme of the habilitation, has been motivated by the need for overcoming some of the problems in the constructions towards a proof of Arnold's conjecture.

To this end, some of the methods that stem from the results [CZ1, CZ2] have been tested on the restricted three body problem. The first paper [A4] applies the methodology to produce a computer assisted proof for transversal intersections of invariant manifolds in the restricted three body problem. Such intersections are the main geometric features that are needed in the proof of diffusion. The method proved efficient and established sharp bounds on the manifolds and on their intersections. The paper [A6] shows a method for establishing homoclinic orbits in the restricted three body problem. This, again, is one of the required steps towards a proof of diffusion.

In most recent work [P1], the construction from [A3] has been combined with a shadowing mechanism that allows for diffusion over distances independent of the perturbation. This result is not fully rigorous and is based on some numerical simulation. The construction though is performed in a way that would allow for rigorous, computer assisted verification using the techniques from [CZ1, CZ2]. This would result in a full proof of the Arnold diffusion.

The work is ongoing and the proof of the Arnold conjecture seems in sight.

5.1.3 Mathematical finance [B1, B2, A5]

The books [B1,B2] are a part of the "Mastering mathematical finance" series at Cambridge University Press. The series cover the core topics and the most common electives offered in master's programs in mathematical/quantitative finance. The books are closely coordinated but each is self-contained, so that they can be used efficiently in combination but also individually.

The book [B1] is driven by concrete computational problems in quantitative finance. It is designed to provide aspiring quant developers with the needed numerical techniques and programming skills. The book starts from scratch in terms of programming skills, without requiring previous experience of C++. Beginning with straightforward option pricing on binomial trees, the book gradually progresses towards more advanced topics, including nonlinear solvers, Monte Carlo techniques for path-dependent derivative securities, finite difference methods for partial differential equations, and American option pricing by solving a linear complementarity problem. The book is intended as a preparation for work as an entry-level quant programmer and to give readers the confidence to progress to more advanced skill sets involving C++ design patterns as applied in finance.

The book [B2] is designed to give the needed tools to the readers for handling risk assessments in modern finance. It provides a treatment of the scope and limitations of mean-variance portfolio theory and introduces popular modern risk measures. The discussion of risk measure such as Value at Risk (VaR) and its more robust generalisations, including Average Value at Risk (AVaR), brings recent developments in risk measures within range of some undergraduate courses and includes a novel discussion of reducing VaR and AVaR by means of hedging techniques. The chapters from [B2] dealing with AVaR reduction using put options have lead to the methods, which were later compared with dynamic hedging strategies, which was the topic of [A5].

5.2 Grants and awards

- Awarded four times with the Polish Ministry of Education Scholarship: 1997, 1998, 1999, 2000,
- S. Batory Trust scholarship for the Hillary term 1999/2000 at Oxford University,
- Polish Ministry of Education PhD research grant 1 P03A 002 26, 2003-2005,

- Member of the European Union grant POKL 04.01.01 00-364/08, for the development of interactive online studies in mathematical finance, 2009,
- Member in the Polish State Ministry of Science and Information Technology grant N201 543238, 2009-2012,
- Kościuszko Foundation grant for an academic six month visit to University of Texas at Austin 2010,
- Head of the Polish State Ministry of Science and Information Technology grant “Diffusion in Hamiltonian systems” NCN 2012/05/B/ST1/00355, 2012-2015,
- Chancellor of AGH University awards for:
  - academic accomplishments: 2010 - 3rd degree, 2012 - 2nd degree, 2014 - 1st degree,
  - teaching and study development: 2015 - 1st degree,
- Takliński[1] 2015 award for achievements in developing new teaching modules, methods, materials and books.

5.3 Student supervision

I am an assistant supervisor of a PhD student Anna Wasieczko, who is writing her thesis “Geometric Methods for Strong Stable and Unstable Manifolds in Dynamical Systems” at Faculty of Mathematics, AGH University of Science and Technology.

I am currently involved in a joint grant proposal for the ITN Marie Curie[2] grant. The title of the call is “Global dynamics in hamiltonian systems”. The objective is to create a university interlinked environment for PhD students. Each of the participants is to supervise a PhD student, and the aim is for the students to be involved in the activities of other departments. I am amongst the following participants: Alessandra Celletti, Alfonso Sorrentino (Tor Vergata University, Rome), Jean Pierre Marco (Pierre and Marie Curie University, Paris), Jacques Féjoz (Université Paris-Dauphine), Christos Efthymiopoulos (Academy of Athens), Vered Rom-Kedar (Weizmann Institute, Rehovot), Ernest Fontich, Angel Jorba (Barcelona University), Piotr Zgliczyński (Jagiellonian University), Heinz Hanssmann (Utrecht University), Tere Seara, Amadeu Delshams, Marcel Guardia (Universitat Politécnica de Catalunya). The proposal is under preparation.

I have been a referee of the PhD of Ferenc A. Bartha, “Computer-aided proofs and algorithms in analysis” at the Department of Mathematics, University of Bergen Norway.

I have also supervised over 30 MSc students.

[1] www.agh.edu.pl/en/staff-members/nagrody-i-odznaczenia/nagroda-im-prof-w-taklinskiego/

[2] https://ec.europa.eu/research/participants/portal/desktop/en/opportunities/h2020/topics/2056-msca-itn-2016.html

5.4 Peer reviews

I have been refereeing papers in the following journals:

- SIAM Journal on Applied Dynamical Systems (SIADS),
- Nonlinearity,
- Physica D: Nonlinear Phenomena,
- Communications in Nonlinear Science and Numerical Simulation,
- European Journal of Operational Research,
- Nonlinear Analysis Series A: Theory, Methods & Applications,
- Applied Numerical Mathematics.

5.5 International collaboration

Visiting positions:

- January 2008 - May 2008 and January 2009, visiting professor at the African Institute for Mathematical Sciences, Muizenberg, Cape Town, South Africa,
- September 2010 - December 2010, visiting professor, University of Texas at Austin, United States,
- August 2012 - March 2013, visiting professor at Georgia Institute of Technology, United States.

Plenary talks at conferences:

- "Geometric methods for invariant manifolds in dynamical systems" four lectures at JISD2012, http://www.ma1.upc.edu/recerca/jisd/jisd2012,
- "Computer assisted method for existence and higher order smoothness of normally hyperbolic invariant manifolds" DyToComp 2012, http://ww2.ii.uj.edu.pl/DyToComp2012,
- "Arnold diffusion in the elliptic restricted 3-body problem" GDHam15, Nuria, Spain 2015, http://gdham15.ma1.upc.edu.

upcoming invited plenary talk:

- "Geometric methods and computer assisted proofs for invariant manifolds in dynamical systems", 17th International Symposium on Scientific Computing, Computer Arithmetic, and Verified Numerical Computation SCAN'2016, Uppsala, Sweden, September 26-30, 2016.

I regard three events as my main achievements in terms of international recognition:

The first is the invitation to give a series of plenary lectures at the "Workshop on Interactions Between Dynamical Systems and Partial Differential Equations" (JISD2012) in Barcelona in 2012 (first item on the above list). These consisted of four, one and a half hour lectures, in which I was invited to present my methodology for geometric and topological proofs for invariant manifolds in dynamical systems. The workshop is an annual event. Each year four plenary speakers are chosen to present their findings. The objective is to bring together young researchers and phd students and expose them to important recent findings in the fields of dynamical systems and partial differential equations. The workshop is a popular event, with the number of participants each year oscillating around seventy. An invitation to be one of the four plenary speaks is considered as a sign of recognition amongst the community.

The second is the invitation to give a plenary talk at the "Global Dynamics in Hamiltonian Systems" (GDHam15) conference in Nuria, Spain 2015 (third item on the list of plenary talks). The conference was an event bringing together a large number of the key figures in the community of celestial mechanics and Hamiltonian systems. As a younger and aspiring academic, I considered it a privilege to have the opportunity to give a plenary talk there.

The third is the invitation to give a plenary talk at SCAN'2016 (last item on the list of plenary talks). This is a large international conference, bringing together every two years over one hundred specialists in the field of verified numerical computation. I have been chosen as one of the eight plenary speakers for the upcoming event in 2016.

Other talks:

- "Transition chains in the planar restricted elliptic three body problem" DyToComp 2006 and University of Barcelona dynamical systems seminar 2007,
- "Cone conditions and covering relations for normally hyperbolic invariant manifolds", University of Barcelona dynamical systems seminar 2008 and Foundations on Computational Dynamics conference in Hong Kong 2008,
- "A topological method for the detection of normally hyperbolic invariant manifolds" Workshop on Stability and Instability in Mechanical Systems, Barcelona 2008,
- "Finding Normally Hyperbolic Invariant Manifolds Around L_1 in the Restricted 3-Body Problem" DyToComp 2009,
- "Computer Assisted Proof for Normally Hyperbolic Invariant Manifolds" dynamical systems seminar, University of Barcelona 2009,
- "Detection of center manifolds and applications to space mission design" University of KwaZulu-Natal, Durban 2009,

- "Geometric methods for invariant manifolds in dynamical systems", University of Texas, Austin 2010,
- "Transition Tori in the Planar Restricted Elliptic Three Body Problem", University of Texas, Austin 2010,
- "Predictability leads to chaos" Northeastern Illinois University, Chicago 2010 and University of Oklahoma 2010 seminars,
- "Finding the Center Manifold Around L_1 in the Planar Restricted Three Body Problem" 8th AIMS Conference on Dynamical Systems, Dresden 2010,
- "Computer Assisted Proof for Normally Hyperbolic Invariant Manifolds" 8th AIMS Conference on Dynamical Systems, Dresden 2010,
- "Computer Assisted Proof for Fibers of Invariant Manifolds in the Planar Restricted Circular Three Body Problem" University of Barcelona 2011,
- "Computer Assisted Proof for Invariant Manifolds in the Restricted Three Body Problem" Symposium on Classical and Celestial Mechanics, Siedlce 2011 and School and Conference on Computational Methods in Dynamics, Trieste 2011,
- "Computer assisted method for higher order smoothness of normally hyperbolic invariant manifolds" Workshop on Rigorous Computations in Dynamical Systems, Uppsala 2011,
- "Computer assisted proofs for normally hyperbolic invariant manifolds", Georgia Tech, Atlanta, 2012,
- "Covering Relations and the Existence of Topologically Normally Hyperbolic Invariant Sets" International Conference on Dynamics of Differential Equations, Georgia Tech, Atlanta 2013,
- "Geometric proof of strong stable/unstable manifolds" Yeshiva University seminar, New York 2014,
- "Geometric proof of strong stable/unstable manifolds with application to the Restricted 3 Body Problem" 10th AIMS Conference on Dynamical Systems, Madrid 2014,
- "Arnold diffusion in the elliptic restricted 3-body problem" 10th AIMS Conference on Dynamical Systems, Madrid 2014 and DyToComp 2015.

Current research projects:

- Bubble Intersections of Invariant Manifolds in the Lomeli Map; with Jordi Lluis Figueras, Jason D. M. James and Warwick Tucker,

- Arnold Diffusion in the Restricted Three-Body Problem: Computer Assisted Proof; with Marian Gidea and Rafael de la Llave,
- Computer assisted Melnikov method; with Piotr Zgliczyński,
- Lorenz attractor in the Morioka-Shimizu system; with Dmitry Turaev and Piotr Zgliczyński.