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Symbolic Dynamics and Tilings of $\mathbb{R}^d$

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Abstract. Aperiodic tilings of Euclidean space can profitably be studied from the point of view of dynamical systems theory. This study takes place *via* a kind of dynamical system called a *tiling dynamical system*.

1. Introduction

In this chapter we study tilings of Euclidean space from the point of view of dynamical systems theory, and in particular, symbolic dynamics. Our goal is to show that these two subjects share many common themes and that they can make useful contributions to each other. The tilings we study are tilings of $\mathbb{R}^d$ by translations of a finite number of basic tile types called "prototiles". A good general reference on tilings is [**GS87**]. The link between tilings and dynamics will be established using a kind of dynamical system called a *tiling dynamical system*, first described by Dan Rudolph [**Rud88**], [**Rud89**]. The parts of the theory we concentrate on here are the parts most closely related to symbolic dynamics. Interestingly, these also tend to be the parts related to the theory of quasicrystals.

A quasicrystal is a solid which, like a crystal, has a regular enough atomic structure to produce sharp spots in its X-ray diffraction patterns, but unlike a crystal, has an aperiodic atomic structure. Because of this aperiodicity, quasicrystals can have "symmetries" forbidden to ordinary crystals, and these can be observed in their X-ray diffraction patterns. The first quasicrystals were discovered in 1984 by physicists at NIST (see [**SBGC84**]) who observed a diffraction pattern with 5-fold rotational symmetry. For a good mathematical introduction to quasicrystals see [**Sen95**].

The theory of quasicrystals is tied up with some earlier work on tiling problems in mathematical logic ([**Wan61**], [**Brg66**]). Central to this circle of ideas is the concept of an *aperiodic* set of prototiles. One of the most interesting aperiodic sets, which anticipated the discovery of quasicrystals, is the set of Penrose tiles, discovered in the early 1970s by Roger Penrose [**Pen74**]. Penrose tilings play a central role in the theory of tiling dynamical systems because they lie at the crossroad of the three main methods for constructing examples: local matching rules, tiling substitutions, and the projection method. As we will see, the tiling spaces constructed

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by these methods are analogous to three well known types of symbolic dynamical systems: finite type shifts, substitution systems and Sturmian systems.

The connection between tilings and symbolic dynamics goes beyond the analogies discussed above. Since tilings are (typically) multi-dimensional, tiling dynamics is part of the theory of multi-dimensional dynamical systems. We will show below that one can embed the entire theory of $\mathbb{Z}^d$ symbolic dynamics (the subject of Doug Lind's chapter in this volume) into the theory of tiling dynamical systems. It turns out that much of the complication inherent in multi-dimensional symbolic dynamics (what Lind calls "the swamp of undecidability") is closely related to the existence of aperiodic prototile sets.

Finally, one can view tiling dynamical systems as a new *type* of symbolic dynamical system. Since tilings are geometric objects, the groups that act naturally on them are continuous rather than discrete (i.e., $\mathbb{R}^d$ versus $\mathbb{Z}^d$). Because of this, one needs to define a new kind of compact metric space to replace the shift spaces studied in classical symbolic dynamics. We call this space a *tiling space*. Even in the one dimensional case (i.e., for flows) tiling spaces provide a new point of view.

In the first part of this chapter we carefully set up the basic theory of tiling dynamical systems and give complete proofs of the main results. In later sections, we switch to survey mode, giving references to access the relevant literature. Of course there are many topics we can not cover in such a short chapter.

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2. Basic definitions in tiling theory

2.1. Tiles and tilings. A set $D \subseteq \mathbb{R}^d$, $d \geq 1$, is called a *tile* if it is compact and equal to the closure of its interior. We will always assume that tiles are homeomorphic to topological balls, although in some situations it is useful to allow disconnected tiles. Tiles in $\mathbb{R}$ are closed intervals. Tiles in $\mathbb{R}^2$ are often polygons, but fractal tiles also occur frequently in examples.

A *tiling*[1] x of $\mathbb{R}^d$ is a collection of tiles that *pack* $\mathbb{R}^d$ (any two tiles have pairwise disjoint interiors) and that *cover* $\mathbb{R}^d$ (their union is $\mathbb{R}^d$). Two tiles D_1, D_2 are *equivalent*, denoted $D_1 \sim D_2$, if one is a translation of the other. Equivalence class representatives are called *prototiles*.

Definition 2.1. Let $\mathcal{T}$ be a finite set of inequivalent prototiles in $\mathbb{R}^d$. Let $X_{\mathcal{T}}$ be the set of all tilings of $\mathbb{R}^d$ by translations of the prototiles in $\mathcal{T}$. We refer to $X_{\mathcal{T}}$ as a *full tiling space*.

Broadly speaking, geometry is concerned with properties of objects that are invariant under congruence. Similarly, dynamics is generally concerned with group actions. In this chapter, we will be interested in how groups of rigid motions act on

[1] We use the lower case notation x for a tiling because we want to think of x as a point in a tiling space X.

sets of tilings. Because of this, we will distinguish between congruent tilings in $X_{\mathcal{T}}$ that sit differently in $\mathbb{R}^d$. Of central interest will be the action of $\mathbb{R}^d$ by translation.

DEFINITION 2.2. For $\mathbf{t} \in \mathbb{R}^d$ and $x \in X_{\mathcal{T}}$ let $T^{\mathbf{t}}x \in X_{\mathcal{T}}$ be the tiling of $\mathbb{R}^d$ in which each tile $D \in x$ has been shifted by the vector $-\mathbf{t}$, that is $T^{\mathbf{t}}x = \{D - \mathbf{t} : D \in x\}$. We denote this *translation action* of $\mathbb{R}^d$ on $X_{\mathcal{T}}$ by T.

The primary reason for studying T is that it is related to the long range order properties of the tilings in $X_{\mathcal{T}}$. While such properties are geometric in nature, we will gain access to them through dynamical systems theory.

2.2. Local finiteness. Let $\mathcal{T}$ be a set of prototiles. A $\mathcal{T}$-*patch* y is a finite subset $y \subseteq x$ of a tiling $x \in X_{\mathcal{T}}$ such that the union of tiles in y is connected. This union is called the *support* of y and written $\operatorname{supp}(y)$. The notion of *equivalence* extends to patches, and a set of equivalence class representatives of patches is denoted by $\mathcal{T}^*$. The subset of patches of n tiles, called the n-*patches*, is denoted by $\mathcal{T}^{(n)} \subseteq \mathcal{T}^*$.

We will impose one additional condition, called the *local finiteness condition*, on all tiling spaces $X_{\mathcal{T}}$.

DEFINITION 2.3. A tiling space $X_{\mathcal{T}}$ has *finite local complexity* if $\mathcal{T}^{(2)}$ is finite.

Equivalently, $\mathcal{T}^{(n)}$ is finite for each n. Sometimes the geometry of the tiles themselves will impose the local finiteness condition, but we usually need to add it as an extra assumption. From now on, whenever we write $\mathcal{T}$, $\mathcal{T}^*$ or $X_{\mathcal{T}}$, it will always implicitly include a choice of a finite $\mathcal{T}^{(2)}$. When working with polygonal prototiles in $\mathbb{R}^2$, a common way to achieve local finiteness to assume that all tiles meet *edge-to-edge*.

EXAMPLE 2.4. Consider the set $\mathcal{S}$ consisting of a single 1×1 square prototile. Without any local finiteness condition, fault lines exist in the tilings $x \in X_{\mathcal{S}}$ with a continuum of possible displacements. Imposition of the edge-to-edge condition means that every $x \in X_{\mathcal{S}}$ is a translation of a single *periodic* tiling. See Figure 1.

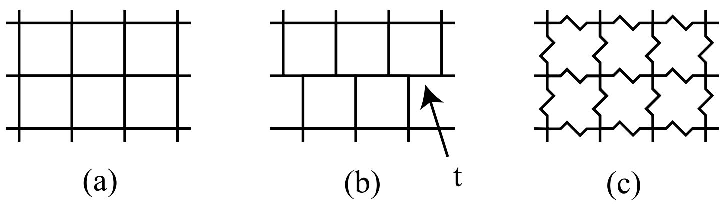

FIGURE 1. (a) Part of an edge-to-edge square tiling. (b) A square tiling with a fault having displacement t. (c) Local finiteness can always be forced geometrically by cutting "keys" on the edges of tiles.

EXAMPLE 2.5. We get more interesting square tiling examples by taking $\mathcal{S}_n$, $n > 1$, to be the set of 1×1 square prototiles *marked* with "colors" $1, 2, \ldots, n$. To do this we also need to modify our notion of equivalence so that differently colored squares are not considered to be equivalent.

Now consider the subset $X_0 \subseteq X_{\mathcal{S}_n}$ consisting of all tilings whose vertices lie on the lattice $\mathbb{Z}^d \subseteq \mathbb{R}^d$ and let T_0 be the restriction of the $\mathbb{R}^d$ shift action T to the subgroup $\mathbb{Z}^d$. It is clear that X_0 is T_0-invariant.

We will see later how this example links tilings to discrete symbolic dynamics.

EXAMPLE 2.6. Fix $n \geq 4$. Let $s = n$ for n odd, and $s = 2n$ for n even. For $0 \leq k < n$ let $\mathbf{v}_k = (\cos(2\pi k/s), \sin(2\pi k/s)) \in \mathbb{R}^2$, i.e., $\mathbf{v}_k$ is a sth root of unity, viewed as a vector in $\mathbb{R}^2$. Let $\mathcal{R}_n$ denote the set of all $\binom{n}{2}$ rhombi with translations of the vectors $\mathbf{v}_k$ as sides. Define $X_{\mathcal{R}_n}$ to be the corresponding edge-to-edge tiling space. Two examples of $x \in X_{\mathcal{R}_5}$ (one with markings) are shown in Figures 3 and 10.

2.3. The tiling topology. As we now show, finite local complexity tiling spaces have particularly nice topological properties. The tiling topology is based on a simple idea: two tilings are close if after a small translation they agree on a large ball around the origin (see [**Rad99**], [**Rob96b**], [**Rud88**], [**Sol97**]). However, the details turn out to be a little subtle.

Given $K \subseteq \mathbb{R}^d$ compact and $x \in X_{\mathcal{T}}$, let $x[[K]]$ denote set of all sub-patches $x' \subseteq x$ such that $K \subseteq \operatorname{supp}(x')$. The smallest such patch is denoted $x[K]$. For $r > 0$ let $B_r = \{\mathbf{t} \in \mathbb{R}^d : ||\mathbf{t}|| < r\}$, where $||\cdot||$ denotes the Euclidean norm on $\mathbb{R}^d$.

LEMMA 2.7. *For $x, y \in X_{\mathcal{T}}$ define*

$$\begin{aligned} d(x,y) = \inf \big(\{\sqrt{2}/2\} \cup \{0 < r < \sqrt{2}/2 : \exists\, x' \in x[[B_{1/r}]], \\ y' \in y[[B_{1/r}]], \text{ with } T^{\mathbf{t}}x' = y' \text{ for some } ||\mathbf{t}|| \leq r\}\big). \end{aligned} \tag{2.1}$$

Then d defines a metric on $X_{\mathcal{T}}$.

We call d the *tiling metric*.

PROOF. We prove only the triangle inequality. Let $0 < d(x,y) = a' \leq d(y,z) = b'$ with $a' + b' < \sqrt{2}/2$. Let $0 < \epsilon < \sqrt{2}/2 - (a' + b')$ and put $a = a' + \epsilon/2$ and $b = b' + \epsilon/2$. Then there are $x' \in x[[B_{1/a}]]$, $y' \in y[[B_{1/a}]]$, $y'' \in y[[B_{1/b}]]$ and $z'' \in z[[B_{1/b}]]$, and also $\mathbf{t}, \mathbf{s} \in \mathbb{R}^d$ with $||\mathbf{t}|| \leq a$ and $||\mathbf{s}|| \leq b$, such that $T^{\mathbf{t}}x' = y'$ and $T^{-\mathbf{s}}z'' = y''$.

Let $y_0 = y' \cap y''$, $x_0 = T^{-\mathbf{t}}y_0 \subseteq x'$ and $z_0 = T^{\mathbf{s}}y_0 \subseteq z'$. Then

$$T^{-(\mathbf{t}+\mathbf{s})}z_0 = x_0 \text{ where } ||\mathbf{t} + \mathbf{s}|| \leq a + b. \tag{2.2}$$

Letting $c = a + b$, then since $0 < a \leq b < \sqrt{2}/2$,

$$0 \leq \frac{1}{c} = \frac{1}{a+b} \leq \frac{1}{b} - a,$$

and it follows that $B_{1/c} \subseteq (B_{1/b} + \mathbf{t})$. Now $y', y'' \in y[[B_{1/b}]]$ so $x_0 \in x[[B_{1/b} + \mathbf{t}]] \subseteq x[[B_{1/c}]]$.

Combining this with (2.2), we have $d(x,z) \leq a + b = d(x,y) + d(y,z) + \epsilon$, where $\epsilon > 0$ is arbitrarily small. The triangle inequality follows.

$\square$

LEMMA 2.8. *The tiling metric d is complete.*

PROOF. Consider a Cauchy sequence x_n of tilings. Assume $d(x_{n+1}, x_n) > 0$ and let $s_n = d(x_{n+1}, x_n) + 2^{-n}$. By passing to a subsequence, we may assume s_n is decreasing and $\sum_{n=1}^{\infty} s_n < \infty$. it follows from (2.1) that for each n there exists $\mathbf{t}_n \in \mathbb{R}^d$ with $||\mathbf{t}_n|| \leq s_n$ and $x'_n \in x_n[[B_{1/s_n}]]$ such that $T^{\mathbf{t}_n} x'_n \subseteq x'_{n+1}$. Put $\mathbf{r}_n = \sum_{k=n}^{\infty} \mathbf{t}_k$. Then

$$T^{\mathbf{r}_n} x'_n = T^{\mathbf{r}_{n+1}} T^{\mathbf{t}_n} x'_n \subseteq T^{\mathbf{r}_{n+1}} x'_{n+1}.$$

This implies that $T^{\mathbf{r}_n} x'_n$ is an increasing sequence of patches, so we can define a tiling $x = \cup_n T^{\mathbf{r}_n} x'_n$. Finally, $d(x, x_n) \leq \max(||\mathbf{r}_n||, s_n) \to 0$. □

THEOREM 2.9. (Rudolph [**Rud89**]) *Suppose $X_{\mathcal{T}}$ is a finite local complexity tiling space. Then $X_{\mathcal{T}}$ is compact in the tiling metric d. Moreover, the action T of $\mathbb{R}^d$ by translation is on $X_{\mathcal{T}}$ is continuous.*

EXERCISE 1. The proof of compactness amounts to the observation that the local finiteness condition is equivalent to $X_{\mathcal{T}}$ being *totally bounded* (see [**Mun75**]). Fill in the details of this proof and also prove the continuity of T.

EXERCISE 2. Prove the following converse to Theorem 2.9: a translation invariant set X of tilings which is compact in the tiling metric (2.1) must have finite local complexity.

3. Tiling dynamical systems

3.1. Tiling spaces as symbolic dynamical systems. Throughout this chapter, a *dynamical system* will be a pair (X, T) where X is a compact metric space (the *phase space*) and T is a continuous action of a group, usually (but not always) $\mathbb{R}^d$. The study of the topological properties of dynamical systems is called *topological dynamics.* The study of the "statistical properties" of dynamical systems is called *ergodic theory.* An excellent introduction to both topological dynamics and ergodic theory is Walters[2] [**Wal82**].

Symbolic dynamics studies a special kind of dynamical system called a *symbolic dynamical system.* The classical set-up is 1-dimensional, but we describe here the general d-dimensional case (see also the chapter by D. Lind). For the group we take $\mathbb{Z}^d$, and we let $X_n = \{1, \ldots, n\}^{\mathbb{Z}^d}$, $n > 1$, with the product topology. Letting T be the *shift* action of $\mathbb{Z}^d$ on X_n, we obtain a dynamical system (X_n, T) called the *d-dimensional full shift* on *n symbols.* In some ways this example itself is too simple to be interesting, but it has very complicated subsets. A $\mathbb{Z}^d$*-symbolic dynamical system* is defined to be a pair (X, T) where X is a closed T-invariant subset $X \subseteq X_n$ called a *shift space*[3] (see [**LM95**] and [**Que87**]).

DEFINITION 3.1. Let $X_{\mathcal{T}}$ be a full d-dimensional tiling space and let T denote the translation action of $\mathbb{R}^d$. A *tiling space* X is a closed T-invariant subset $X \subseteq X_{\mathcal{T}}$. We call the pair (X, T) a *tiling dynamical system.*

Now we can precisely state our way of thinking of tiling dynamical systems as a new type of symbolic dynamical system. We think of the prototiles $D \in \mathcal{T}$ as the symbols. The full tiling space $X_{\mathcal{T}}$ corresponds to the full shift, and more general tiling spaces correspond to more general shift spaces. Like the product topology,

[2] Even though this book concentrates almost exclusively on $\mathbb{Z}$ actions, the theory goes through with very little effort to actions of $\mathbb{Z}^d$ and $\mathbb{R}^d$.

[3] A shift space is also sometimes called a subshift.

the tiling topology is compact and metric, and in both cases closeness corresponds to a good match near the origin. However, in the the case of tilings there is the possibility of a small translation, and since we want this to be continuous, the topology needs to be defined accordingly.

REMARK 3.2. The theory of tiling dynamical systems contains the theory of $\mathbb{Z}^d$ symbolic dynamics. One can show that the space X_0 constructed in Example 2.5 is homeomorphic to the symbolic full shift, and that T_0 implements the shift action on X_0.

3.2. Finite type. Let $X_{\mathcal{T}}$ be a full tiling space and let $\mathcal{F} \subseteq \mathcal{T}^*$. Let $X_{\setminus\mathcal{F}} \subseteq X_{\mathcal{T}}$ be the set of all tilings $x \in X_{\mathcal{T}}$ such that no patch y in x is equivalent to any patch in $\mathcal{F}$. We call such a set $\mathcal{F}$ a set of *forbidden patches.*

One can show that for any $\mathcal{F} \subseteq \mathcal{T}^*$, the set $X_{\setminus\mathcal{F}}$ is a tiling space (i.e., it is closed and T-invariant). Moreover, it is clear that every tiling space $X \subseteq X_{\mathcal{T}}$ is defined by a set $\mathcal{F}$ of forbidden patches. However, the set $\mathcal{F}$ is not unique!

EXERCISE 3. Prove the three statements in the previous paragraph. Hint: See [**LM95**].

In symbolic dynamics, the most important kind of shift space is a finite type shift. The following definition introduces the corresponding idea in tiling theory.

DEFINITION 3.3. A tiling space $X \subseteq X_{\mathcal{T}}$ is called a *finite type tiling space* if there exists a *finite* $\mathcal{F} \subseteq \mathcal{T}^*$ so that $X = X_{\setminus\mathcal{F}}$

The most common case is $\mathcal{F} \subseteq \mathcal{T}^{(2)}$. This is called a *local matching rule.* It is convenient to formulate this case in terms of the *allowed* 2-patches rather than forbidden ones. To accomplish this, we put $\mathcal{Q} = \mathcal{T}$, let $\mathcal{Q}^{(2)} = \mathcal{T}^{(2)}\setminus\mathcal{F}$, and write $X_{\mathcal{Q}}$ for $X_{\setminus\mathcal{F}}$. Note that imposing a local matching rule really just amounts to strengthening the local finiteness condition. Thus a full tiling space is a kind of finite type tiling space.

EXAMPLE 3.4. (The Penrose tiles) Consider the marked version $\mathcal{P}$ (shown in Figure 2) of the prototiles $\mathcal{R}_5$. The set $\mathcal{P}^{(2)}$ (which defines the matching rules)

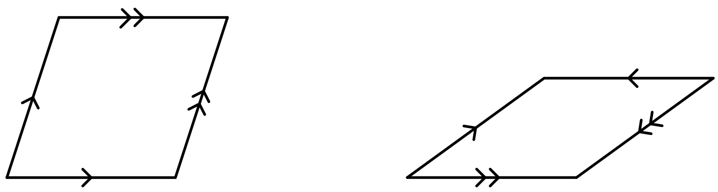

FIGURE 2. The Penrose tiles. The protoset $\mathcal{P}$ consists of the two marked tiles shown, and all rotations so that edges have have angles $2\pi n/10$. In particular, card$(\mathcal{P}) = 20$

.

imposes the requirement that the markings on any pair of adjacent tiles must match. (As we will see below, markings are often used for this purpose). We call $\mathcal{P}$, together with the matching rules, the *Penrose tiles.* Tilings $x \in X_{\mathcal{P}}$ are called *Penrose tilings.* Part of a Penrose tiling is shown in Figure 3.

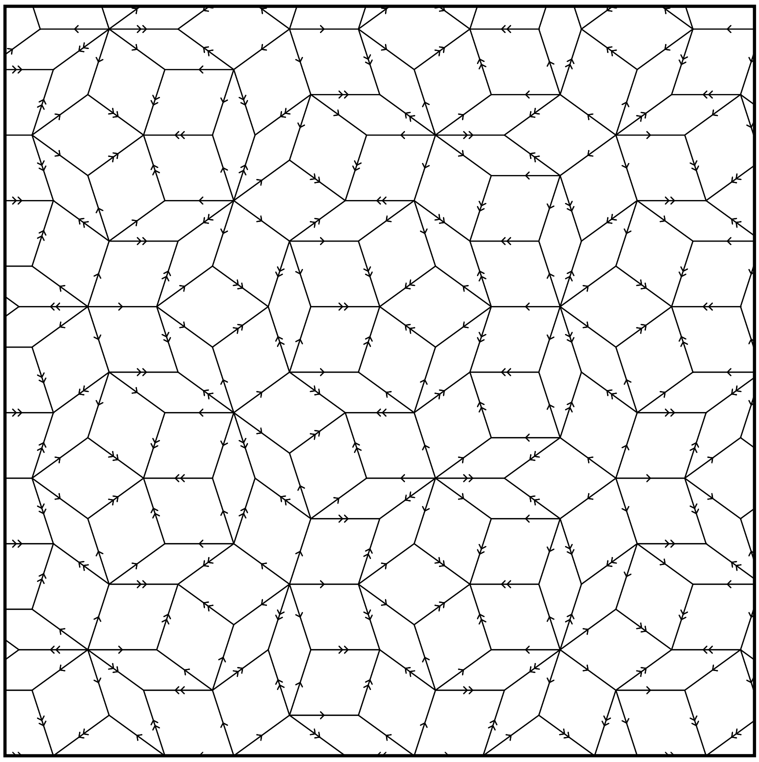

FIGURE 3. A part of a Penrose tiling.

3.3. The Tiling Problem. Suppose we are given a set $\mathcal{T}$ of prototiles and a set $\mathcal{F} \subseteq \mathcal{T}^*$ of forbidden patches. Consider the following problem:

TILING PROBLEM. Is $X_{\setminus\mathcal{F}} \neq \emptyset$?

We begin with a positive result, a version of which appeared in [**Wan61**] (see [**GS87**] for a proof).

EXTENSION THEOREM. *Let $\mathcal{T}$ be a collection of prototiles with a local finiteness condition $\mathcal{T}^{(2)}$ and let $\mathcal{F} \subseteq \mathcal{T}^*$ be a set of forbidden patches. Define $\mathcal{T}^+ \subseteq \mathcal{T}^*$ to be the set of patches that do not contain any forbidden sub-patches. Then $X_{\setminus\mathcal{F}} \neq \emptyset$ if and only for each $r > 0$ there is a patch $y \in \mathcal{T}^+$ with $B_r + \mathbf{t} \subseteq \mathrm{supp}(y)$ for some $\mathbf{t} \in \mathbb{R}^d$.*

The trouble with the Extension Theorem is that it is not constructive. To conclude that $X_{\setminus\mathcal{F}}$ is nonempty one needs to see infinitely many patches in $\mathcal{T}^+$.

This difficulty can be appreciated if one tries to tile the plane manually with Penrose tiles. There are a lot of "dead ends": patches in $y \in \mathcal{P}^+$ that do not belong to $\mathcal{P}^*$. How can we know that $\mathcal{P}^+$ doesn't have some largest patch y? Later, we will give a proof that $X_{\mathcal{P}} \neq \emptyset$, but that proof will require a new idea.

The question of whether the Tiling Problem is *decidable* was raised by Wang [**Wan61**] for the case of marked square tiles $\mathcal{S}_n$, together with a local matching rule. These are now known as Wang tiles.

When $d = 1$ there is an easy algorithm to answer the Tiling Problem: First we draw a graph G with vertex set $\mathcal{T}$ and directed edges $\mathcal{T}^{(2)}$, and let A be the $m \times m$ adjacency matrix for G, where $m = \text{card}(\mathcal{T})$. The entries of A^k give the number of paths of length k in G. If $A^{m+1} \neq \mathbf{0}$ (i.e., not identically zero) then $A^k \neq \mathbf{0}$ for any $k > m$, and we conclude $X_{\mathcal{Q}} \neq \emptyset$.

Definition 3.5. A tiling x of $\mathbb{R}^d$ is called a *periodic tiling* if its *translation group* $\Gamma_x = \{\mathbf{t} \in \mathbb{R}^d : T^{\mathbf{t}}x = x\}$ is a *lattice*: that is a subgroup of $\mathbb{R}^d$ with d linear independent generators. A tiling x is called *aperiodic* if $\Gamma_x = \{0\}$.

In the case $d = 1$ one can easily show that if $X_{\mathcal{T}} \neq \emptyset$ then there is a periodic tiling $x \in X_{\mathcal{T}}$. Wang conjectured [**Wan61**] that the same holds for $d > 1$.

Wang's Conjecture.

(1) *There is an algorithm to decide the tiling problem.*
(2) *Whenever $X_{\mathcal{T}} \neq \emptyset$, there exists a periodic tiling $x \in X_{\mathcal{T}}$.*

Wang proved that (2) implies (1). The argument goes as follows:

First suppose $X_{\mathcal{T}} \neq \emptyset$. Then there exists a periodic $x \in X_{\mathcal{T}}$. For each $n \in \mathbb{N}$, list all tiling patches $y \in \mathcal{T}^+$ with $B_n \subseteq \text{supp}(y)$ and $B_{n+1} \not\subseteq \text{supp}(y)$. We will eventually see a complete period of x. In this case the algorithm will stop and answer "yes".

Now suppose $X_{\mathcal{T}} = \emptyset$. Then it follows from the Extension Theorem that we will eventually find $n \in \mathbb{N}$ so that no $y \in \mathcal{T}^+$ has support containing B_n. In this case the algorithm will stop and answer "no".

But Wang's conjecture turns out to be false! For $d \geq 2$ Berger [**Brg66**] showed that the tiling problem is, in fact, *undecidable*. His solution included the construction of an example of a 2-dimensional finite type tiling space $X_{\mathcal{Q}}$ containing no periodic tilings.

Definition 3.6.

(1) A nonempty tiling space X is called an *aperiodic tiling space* if it contains no periodic tilings (i.e., every $x \in X$ is an aperiodic tiling).
(2) A prototile set $\mathcal{Q}$ with local matching rule $\mathcal{Q}^{(2)}$ is called an *aperiodic prototile set* if $X_{\mathcal{Q}}$ is an aperiodic tiling space.[4]

Berger's original aperiodic prototile set $\mathcal{Q}$ satisfies $\text{card}(\mathcal{Q}) > 50{,}000$ (see [**GS87**]). Later, Raphael Robinson [**rRob71**] found a simple example with $\text{card}(\mathcal{Q}) = 32$ (a picture of these tiles appears in Lind's chapter of this volume).

3.4. Counting prototiles and the "einstein" problems. The problem of finding small aperiodic sets of prototiles has been a popular one (see [**GS87**] for the history up to 1987). The exact formulation depends on how one counts prototiles. With our notion of equivalence (translation but not more general congruences), the Penrose tiles $\mathcal{P}$ consist of 20 prototiles. Counting this way, the current best example in $\mathbb{R}^2$ is a set $\mathcal{K}$ of Wang tiles due to Kari and Culik ([**Kar96**], [**Cul96**]) with $\text{card}(\mathcal{K}) = 13$. This example is particularly interesting because it is not related to any other known examples.

[4] In some literature, the term "aperiodic tiling" is reserved for tilings $x \in X_{\mathcal{Q}}$, where $\mathcal{Q}$ is an aperiodic prototile set.

It is perhaps more natural to allow congruence classes of prototiles to count only once (i.e., to allow rotations of the prototiles). With this system of counting there are just 2 Penrose tiles, and for $\mathbb{R}^2$ this is the best result so far.

The question of whether there exists an aperiodic prototile set consisting of a *single* tile has been named the "einstein problem" by Ludwig Danzer. For $d = 3$ an example of an einstein was discovered by Schmitt and Conway (see [**Sen95**]). However, it tiles in a way that is, in a certain sense, very weakly aperiodic.

There are some partial results on the 2-dimensional einstein problem as well. For any prototile set $\mathcal{T} = \{D\}$, where D is a topological disk, whenever $X_{\mathcal{T}} \neq \emptyset$ there exists a periodic tiling $x \in X_{\mathcal{T}}$ ([**Ken92**], [**Ken93**], [**GBN89**]). In other words, there is no einstein up to translation.

4. Substitution tiling spaces

4.1. Perfect decompositions. Let $L \in Gl(d, \mathbb{R})$ be an *expansive* linear transformation of $\mathbb{R}^d$. Expansive means that every eigenvalue of L lies outside the unit circle. The case $L = \lambda M$, where M is an isometry and $\lambda > 1$ is called a *similarity.*

A *perfect decomposition* (or just a *decomposition*) is a mapping $C : \mathcal{T} \to L^{-1}\mathcal{T}^*$ that (up to equivalence) satisfies the *perfect overlap* condition

$$\operatorname{supp}(C(D)) = \operatorname{supp}(D), \tag{4.1}$$

In a slight abuse of language, a decomposition is called *self-similar* if L is a similarity; in the general case it is called a *self-affine* decomposition.

The mapping $S = LC$ is called a *perfect self-similar* or *self-affine tiling substitution*[5] on $\mathcal{T}$.

In the case $d = 1$ there is no difference between a self-similar and a self-affine substitution, and any tiling substitution can be written $S = \lambda C$, where λ is a positive real number. In the self-similar case when $d = 2$, we can identify $\mathbb{R}^2$ with $\mathbb{C}$. By replacing S with S^2 we can assume L orientation preserving. Then we can regard L as multiplication by $\lambda \in \mathbb{C}$, with $|\lambda| > 1$, so that $S = \lambda C$.

Example 4.1. (Polyomino decompositions) The *chair decomposition* C_c on the set $\mathcal{C}$ of four chair prototiles is obtained by taking the decomposition pictured in Figure 4(a) and its four rotations. The *table decomposition* C_t, Figure 4(b), is

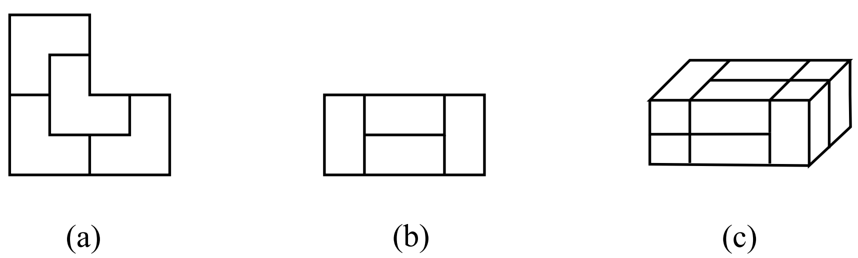

Figure 4. (a) The chair, (b) the table and (c) the 3-dimensional table.

defined on the protoset $\mathcal{D}_2$ of two "dimers" in the plane. The 3-dimensional table, Figure 4(c), is defined on the set $\mathcal{D}_3$ of 6 dimers in $\mathbb{R}^3$. The asymmetry of this

[5]This is also sometimes called an *inflation* mapping.

example makes it necessary to keep track of the prototiles' orientations. Many other polyomino examples are easily devised.

In all cases shown in Figure 4, L is a similarity with $M = Id$ and $\lambda = 2$. A non-self-similar polyomino decomposition with $L = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$ is shown in Figure 5.

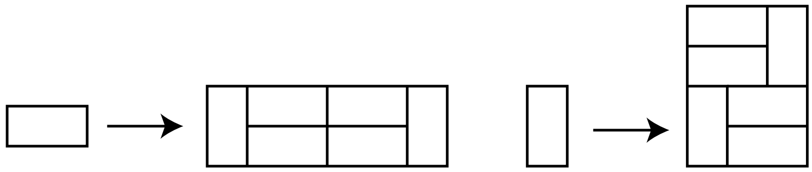

FIGURE 5. The *folding table*: a non-self-similar version of the table. Here we show the decomposition of $L\mathcal{D}_2$ into $\mathcal{D}_2$.

EXAMPLE 4.2 (Raphael Robinson's triangular Penrose tilings). This is a non-polyomino self-similar decomposition C_1 with $\lambda = (1/2)(1+\sqrt{5})$ on a set $\mathcal{P}_1$ of 40 marked triangular prototiles. Two of the prototiles are shown in Figure 6(a). The decomposition C_1 is shown in Figure 6(b). Let $X_{\mathcal{P}_1}$ denote the finite type tiling

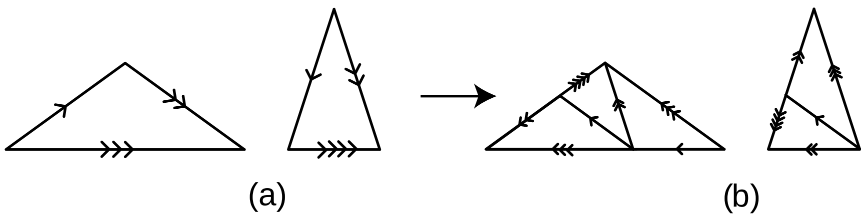

FIGURE 6. The *triangular Penrose tilings*: The protoset $\mathcal{P}_1$ consists of a finite set of rotations of the two tiles shown that is closed under decomposition.

space corresponding to the usual matching rule that the arrows on adjacent edges must match. We will show below how to use the tiling substitution $S_1 = \lambda C_1$. to prove that $X_{\mathcal{P}_1} \neq \emptyset$.

EXAMPLE 4.3 (The pinwheel tilings). The self-similar decomposition shown in Figure 7 has been studied extensively by Radin (see [**Rad94**]). Up to rotation and reflection it has a single prototile. In the decomposition, one copy of the prototile is rotated by an angle $\theta = \arctan\frac{1}{2}$, so that $\frac{\theta}{2\pi}$ is irrational. Such a rotation has infinite order and hence there is no finite prototile set invariant under this decomposition. We can get around this difficulty by modifying the definition of equivalence and the tiling metric d to allow rotations as well as translations.

The next two examples don't quite satisfy (4.1), but can nevertheless easily be accommodated. We refer to them as *imperfect decompositions* and the corresponding substitutions as *imperfect tiling substitutions* (see Definition 5.25).

EXAMPLE 4.4. (The Penrose decomposition). This decomposition applies to the marked prototile set $\mathcal{P}$ of Example 3.4. Since $C(D) \supseteq D$, this is an imperfect decomposition.

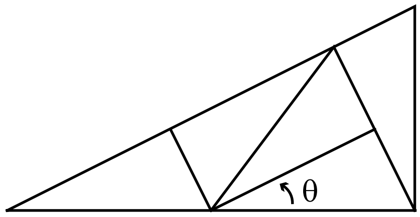

FIGURE 7. The *pinwheel* decomposition: One prototile is irrationally rotated relative to the others.

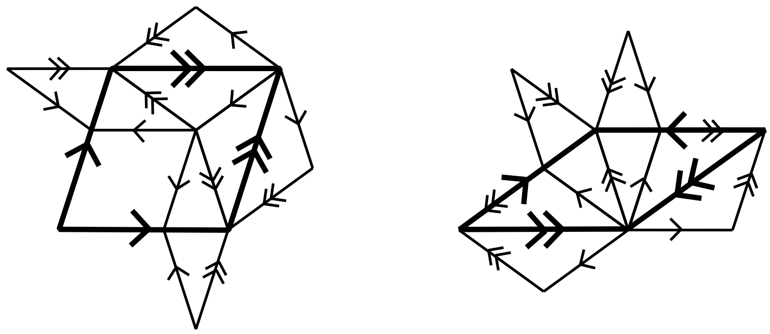

FIGURE 8. The imperfect decomposition of the rhombic Penrose tilings.

EXAMPLE 4.5. (The binary tiling decomposition). The decomposition shown in Figure 9 is from [**GL92**]. A patch of a tiling x in the corresponding substitution

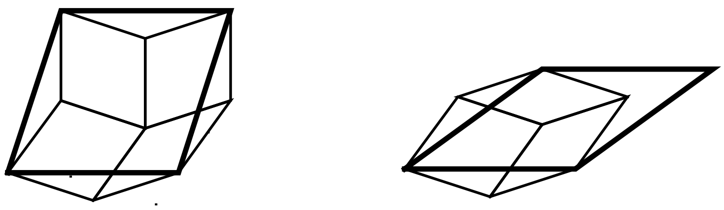

FIGURE 9. The imperfect "binary" tiling decomposition C_b is self-similar. In the corresponding tiling substitution $S_b = \lambda M C_b$ the similarity $L = \lambda M$ includes a rotation M by $2\pi/40$. Note, however, that $C_b^2(\mathcal{T}) \subseteq L^{-2}\mathcal{T}$, so there is a version of S_b^2 with no rotation.

tiling space is shown in Figure 10. A perfect version of this decomposition is shown in Figure 13.

4.2. Properties of tiling substitutions.

DEFINITION 4.6. Let $S = LC$ be a tiling substitution on $\mathcal{T}$ where $X_{\mathcal{T}}$ has finite local complexity. We say S satisfies the 2-*patch closure property* if $S(\mathcal{T}^{(2)}) \subseteq \mathcal{T}^*$.

From now on, unless we say otherwise, we will assume every tiling substitution S satisfies the 2-patch closure property. The reason[6] for this assumption is that a

[6]The need for such an assumption was pointed out to the author by Natalie Priebe Frank.

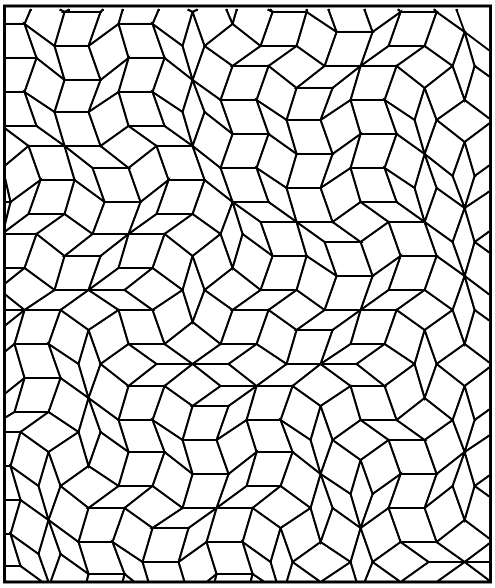

FIGURE 10. A patch of binary tiling.

tiling substitution $S : \mathcal{T} \to \mathcal{T}^*$ (satisfying 2-patch closure) can be extended to a mapping $S : X_{\mathcal{T}} \to X_{\mathcal{T}}$. This mapping performs the decomposition C, viewed as a mapping $C : X_{\mathcal{T}} \to X_{L^{-1}\mathcal{T}}$, and then applies the linear expansion L to the entire tiling. Note that C satisfies *translation invariance*

$$T^{\mathbf{t}}C = CT^{\mathbf{t}} \text{ for all } \mathbf{t} \in \mathbb{R}^d. \tag{4.2}$$

EXERCISE 4. Show that both C and S are continuous in the tiling topology and that S satisfies $T^{L\mathbf{t}}S = ST^{\mathbf{t}}$.

We call a tiling substitution $S = LC$ *invertible* if it is 1:1. In this case S has a continuous inverse $S^{-1} = L^{-1}C^{-1}$ on $S(X_{\mathcal{T}})$. Then C^{-1} is a continuous and translation invariant mapping, which is called a *composition*.

Let us denote $\mathcal{T} = \{D_1, \ldots, D_n\}$. We define the *structure matrix* A of the tiling substitution $S = LC$ to be $n \times n$ matrix with entries $A_{i,j}$ equal to the number of prototiles equivalent to D_i that occur in $S(D_j)$. A tiling substitution is called *primitive* if $A^k > 0$ for some $k > 0$.

4.3. Substitution tiling spaces. Let S be a tiling substitution on $\mathcal{T}$. Take $D \in \mathcal{T}$ and define a sequence of patches x_k inductively: $x_1 = \{D\}$, and $x_k = S(x_{k-1})$, $k > 1$. Since we assume S satisfies the 2-patch closure property, it follows that $x_k \in \mathcal{T}^*$ for all k.

The patches x_k are used to define a tiling space as follows. First we define a set $\mathcal{F}_S$ of forbidden patches by stipulating that $y \in \mathcal{T}^*$ is forbidden if it is not a sub-patch of x_k for any k. The tiling space $X_S = X_{\mathcal{F}_S} \subseteq X_{\mathcal{T}}$ is called the *substitution tiling space* corresponding to S. The dynamical system (X_S, T) is called a *substitution tiling dynamical system*.

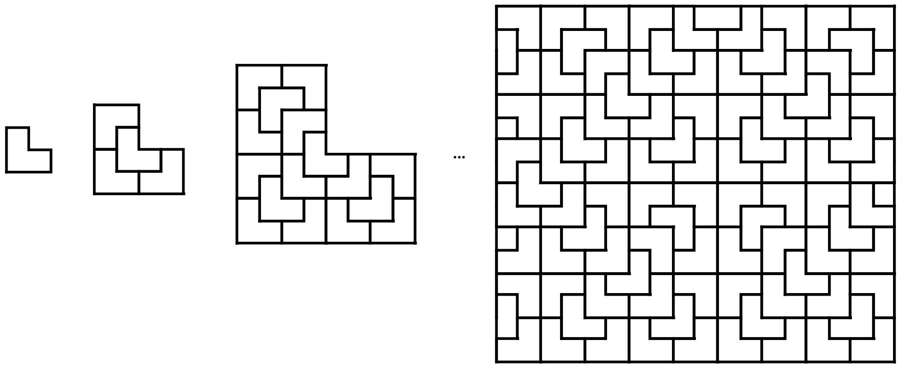

FIGURE 11. The patches x_k, $k = 1, 2, 3$, for the chair tiling substitution (Figure 4(a)), and a patch of *chair tiling*.

LEMMA 4.7. *Let $X_S \subseteq X_{\mathcal{T}}$ be the substitution tiling space corresponding to a primitive tiling substitution S on $\mathcal{T}$. Then $X_S \neq \emptyset$ and X_S is independent of the initial tile $D \in \mathcal{T}$. Moreover $S(X_S) \subseteq X_S$.*

PROOF. First we observe that $x_k \in \mathcal{T}^+$ for all k, since by the definition of $\mathcal{F}_S$, no patch x_k contains any forbidden sub-patches. We also note that since L is expansive, $\operatorname{diam}(\operatorname{supp}(x_k)) \to \infty$. Thus given $r > 0$, there exists $\mathbf{t} \in \mathbb{R}^d$ so that $B_r = \mathbf{t} \subseteq \operatorname{supp}(x_k)$ for k sufficiently large. It follows from the Extension Theorem that $X_S = X_{\mathcal{F}_S} \neq \emptyset$. The independence of X_S from the choice of D follows from primitivity. The S-invariance is clear. □

COROLLARY 4.8. *Let $\mathcal{P}$ be the set of marked rhombic Penrose tiles shown in Figure* 2*, and let $\mathcal{P}_1$ be the marked triangular Penrose tiles shown in Figure* 6*. Then the corresponding finite type tiling spaces $X_{\mathcal{P}}$ and $X_{\mathcal{P}_1}$ are nonempty, i.e.,* Penrose tilings exist.

PROOF. For the two tiling substitutions, S from Figure 8 and the substitution S_1 from Figure 6, the corresponding substitution tiling spaces satisfy $X_{\mathcal{P}} \supseteq X_S \neq \emptyset$ and $X_{\mathcal{P}_1} \supseteq X_{S_1} \neq \emptyset$. □

This is essentially Penrose's argument, although he did not couch it in dynamical terms. We conclude with a result that characterizes exactly which tilings belong to a substitution tiling space.

PROPOSITION 4.9. *Let $X_S \subseteq X_{\mathcal{T}}$ be a substitution tiling space for a primitive tiling substitution S. Let $x \in X_{\mathcal{T}}$. Then $x \in X_S$ if and only if there is an infinite sequence $x_n \in X_S$ so that $S^n x_n = x$ (i.e., x has infinitely many S-preimages).*

EXERCISE 5. Prove Proposition 4.9.

COROLLARY 4.10. *For Penrose tilings, the finite type tiling spaces are the same as the substitution tiling spaces: $X_{\mathcal{P}} = X_S$ and $X_{\mathcal{P}_1} = X_{S_1}$.*

PROOF. By Proposition 4.9 and Corollary 4.8 it suffices to show S^{-1} and S_1^{-1} exist on $X_{\mathcal{P}}$ and $X_{\mathcal{P}_1}$ respectively. This is easy to see by inspection for $X_{\mathcal{P}_1}$. Although it is a little harder to see for $X_{\mathcal{P}}$, it is also true in that case.

Alternatively, this result for $X_{\mathcal{P}}$ also follows from $X_{\mathcal{P}_1}$ using Example 5.20 below. □

5. Applications of topological dynamics

5.1. Repetitivity and minimality. Tilings x constructed from aperiodic sets $\mathcal{Q}$ of prototiles, as well as substitution tilings, tend to have the following curious property.

DEFINITION 5.1. A tiling x is called *repetitive* if for any patch y in x there is an $r > 0$ such that for any $\mathbf{t} \in \mathbb{R}^d$ there is a translation $T^{\mathbf{s}}y$ of y in x such that $\mathrm{supp}(T^{\mathbf{s}}y) \subseteq B_r + \mathbf{t}$.

In other words, a copy of y occurs "nearby" any given location $\mathbf{t}$ in x. Since all periodic tilings are repetitive, we think of repetitivity as a generalization of periodicity.

Let (X, T) be a dynamical system. Let $U \subseteq X$ be open and let $x \in X$. Define the *return set* of x to U to be

$$R(x, U) = \{\mathbf{t} \in \mathbb{R}^d : T^{\mathbf{t}}x \in U\}. \tag{5.1}$$

A set $R \subseteq \mathbb{R}^d$ is called *relatively dense* if there is an $r > 0$ such that every r-ball in $\mathbb{R}^d$ intersects R.

DEFINITION 5.2. A point $x \in X$ is *almost periodic* if $R(x, U)$ is relatively dense for every open $U \subseteq X$ with $R(x, U) \neq \emptyset$.

For a tiling space $X \subseteq X_{\mathcal{T}}$, let $y \in \mathcal{T}^*$ and let $R = \mathrm{supp}(y)$. Define $X(y) = \{x \in X : x[R] = y\}$. For $\epsilon > 0$ define the *cylinder set*

$$U_{y,\epsilon} = T^{B_\epsilon}X(y) = \{T^{\mathbf{t}}x : x \in X(y), \mathbf{t} \in B_\epsilon\}.$$

Clearly $U_{y,\epsilon}$ is open.

Without loss of generality, we can assume by translating that the support of each patch $y \in \mathcal{T}^*$ contains the largest possible ball B_r around the origin. It follows that the cylinder sets $\{U_{y,\epsilon_n} : y \in \mathcal{T}^*, \epsilon_n \to 0\}$ form a basis for the tiling topology on X. In Definition 5.2, it suffices to check only the sets U belonging to this basis. Thus we have the following.

PROPOSITION 5.3. *Let (X, T) be a tiling dynamical system. Then a tiling $x \in X$ is repetitive if and only if it is an almost periodic point*[7].

A dynamical system (X, T) is called *minimal* if there are no proper closed T-invariant subsets of X. For a point $x \in X$ we define its *orbit* by $O(x) = \{T^{\mathbf{t}}x : \mathbf{t} \in \mathbb{R}^d\}$, and its orbit closure $\overline{O(x)} \subseteq X$ to be the closure of $O(x)$ in X.

GOTTSCHALK'S THEOREM. ([**Got44**]) *A dynamical system $(\overline{O(x)}, T)$ is minimal if and only if x is almost periodic.*

It follows from the minimality of the dynamical system $(\overline{O(x)}, T)$ that $\overline{O(y)} = \overline{O(x)}$ for all $y \in \overline{O(x)}$. In this case it follows from Gottschalk's Theorem that y is almost periodic too.

DEFINITION 5.4. Two repetitive tilings x, y are said to be *locally isomorphic* if $\overline{O(x)} = \overline{O(y)}$.

[7] Because of this, the author previously used the term "almost periodic tiling" to mean a repetitive tiling.

Geometrically, two locally isomorphic tilings x and y have exactly the same patches. Dynamically, local isomorphism means x and y belong to the same minimal tiling dynamical system.

A dynamical system (X, T) is *transitive* if there is a single orbit: $O(x) = X$ for all $x \in X$. This is a special case of minimality. A tiling x is periodic if and only if $O(x) = \overline{O(x)}$. In this case $(T, O(x))$ is transitive, and $O(x) = \mathbb{R}^d/\Gamma_x$ is a d-dimensional torus (e.g., $X_{\mathcal{S}} = \mathbb{Z}^d/\mathbb{R}^d$).

We say a tiling is *properly repetitive* if it is repetitive but not periodic. An easy application of Zorn's lemma shows that every dynamical system (X, T) has a minimal T-invariant subset $Y \subseteq X$. It follows that every tiling space contains a repetitive tiling (this argument appears in[**RW92**]). Of course in general this tiling may be periodic, but if we know X is an aperiodic tiling space, then it must contain a properly repetitive tiling.

Exercise 6. Starting with the Penrose tiles $\mathcal{P}$, construct a new example of an aperiodic prototile set $\mathcal{P}'$ such that not every tiling $x \in X_{\mathcal{P}'}$ is repetitive.

A dynamical system (X, T) is called *topologically transitive* if there exists $x \in X$ such that $\overline{O(x)} = X$. Clearly transitive implies minimal which implies topologically transitive. In each case the converse is false. Exercise 7 shows that $(X_{\mathcal{S}_n}, T)$ is topologically transitive but not minimal. Later we will show that the Penrose tiling dynamical system $(X_{\mathcal{P}}, T)$ is minimal but not transitive. A minimal dynamical system which is not transitive is called *properly minimal*.

Exercise 7. Show that the tiling dynamical system $(X_{\mathcal{S}_n}, T)$ is topologically transitive but not minimal. What can you say about $(X_{\mathcal{R}_n}, T)$? (See Examples 2.5 and 2.6.)

Remark 5.5. When x periodic, $\overline{O(x)}$ is a torus, i.e., a connected manifold. It turns out that this is an exceptional situation. One can show that if (X, T) is aperiodic and topologically transitive, then for each patch y, $X(y)$ is homeomorphic to a Cantor set. Since for ϵ sufficiently small, $U_{y,\epsilon} = T^{B_\epsilon}X(y)$ is homeomorphic to $B_\epsilon \times X(y)$, every point $x \in X$ has a neighborhood homeomorphic to a product of $\mathbb{R}^d$ and a Cantor set. Such a space is called a *lamination*. In particular, a tiling space X is almost never connected.

5.2. The repetitivity of substitution tilings.

Proposition 5.6. *Let X_S be the substitution tiling space corresponding to an invertible primitive tiling substitution S. Then any X_S is an aperiodic tiling space .*

Proof. Suppose $T^{\mathbf{t}_0}x = x$ for some $\mathbf{t}_0 \neq 0$. Since C is invertible, (4.2) implies

$$T^{\mathbf{t}_0}C^{-n}x = C^{-n}x \tag{5.2}$$

for all $n \geq 1$. Choose n so large that $T^{\mathbf{t}_0}\mathrm{int}(L^n D) \cap \mathrm{int}(L^n D) \neq \emptyset$ for all $D \in \mathcal{T}$. Since for some $D \in \mathcal{T}$, $L^n D \in x$, this contradicts (5.2). $\square$

Remark 5.7. In Proposition 5.6, invertibility is also necessary [**Sol98**].

The following generalizes a well known result for discrete substitution dynamical systems (see [**Que87**]).

THEOREM 5.8. *Let X_S be a substitution tiling space corresponding to an primitive tiling substitution S. Then any $x \in X_S$ is repetitive. Moreover, any $x, y \in X_S$ are locally isomorphic. In particular, (X_S, T) is minimal.*

COROLLARY 5.9. *The tiling space X_S corresponding to invertible primitive tiling substitution S consists of properly repetitive tilings.*

PROOF. This follows from Gottschalk's Theorem, Theorem 5.6 and Theorem 5.8. □

PROOF OF THEOREM 5.8. Assume without loss of generality $A > 0$ (otherwise replace S with S^k). Let $x \in X$ and let y be a patch in x. Fix $D_0 \in \mathcal{T}$ and choose k so large that y is a patch in $S^{k-1}(\{D_0\})$. Then y is a patch in $S^k(\{D\})$ for all $D \in \mathcal{T}$.

Since $x \in X$ it follows from Proposition 4.9 that there exists $x_k \in X$ so that $S^k x_k = x$. Let $x'_k = L^k x_k \in X_{L^k\mathcal{T}}$. Note that $C^k x'_k = x$.

Let s be the largest diameter of $L^k D \in L^k\mathcal{T}$ and let $r = 2s$. It follows from the triangle inequality that for any $D' \sim L^k D$, if $\mathbf{t} \in D'$ then $D' \subseteq B_r + \mathbf{t}$.

Thus any r-ball $B_r + \mathbf{t}$ in $\mathbb{R}^d$ contains a tile $D' \in x'_k$, and the patch $C^k(\{D'\})$ in x, which has support D', contains a sub-patch that is a copy of y. □

5.3. Self-affine tilings. Let $S = LC$ be a tiling substitution on $\mathcal{T}$. A tiling $x_0 \in X_{\mathcal{T}}$ is called a *self-affine tiling* with *expansion map* L if $Sx_0 = x_0$. When L is a similarity, x_0 is called a *self-similar tiling*. Self-affine tilings play the same role in the theory of substitution tiling dynamical systems that fixed points play in the theory of discrete substitutions. In particular, self-affine tilings provide a alternative definition for substitution tiling spaces.

THEOREM 5.10. *If S is a primitive tiling substitution then there exists $k > 0$ and $x_0 \in X_S$ such that $S^k x_0 = x_0$.*

PROOF. First, we assume without loss of generality that $A > 0$, since otherwise we can replace S with S^k so that $A^k > 0$. Fixing $D_1 \in \mathcal{T}$, there is a translation of $L^{-1}(D_1)$ in $C(D_1)$. By taking additional powers of S, if necessary, we can assure that $L^{-1}(D_1) \subseteq \mathrm{int}(\mathrm{supp}(D))$. Similarly, there is a sequence D_k of tiles equivalent to D_1 so that for all $k > 0$, $L^{-1}(D_k) \in C(D_{k-1})$ and $L^{-1}(D_k) \subseteq \mathrm{int}(\mathrm{supp}(D_{k-1})) \subseteq \mathrm{int}(D_1)$. Since L is expanding, there exists a unique $\mathbf{c} \in \mathrm{int}(D_1)$ satisfying

$$\mathbf{c} \in \bigcup_{k=1}^{\infty} L^{-k} D_k.$$

The point $\mathbf{c}$ is called a *control point.*

Let $D_0 = D_1 - \mathbf{c}$. This is a tile with a control point at the origin. Define the sequence $x_k = S^k(\{D_0\})$ and use this sequence to construct the substitution tiling space X_S. Because of the location of the control point, we have that x_{k-1} is a sub-patch of x_k for all k, and $x_k = Sx_{k-1}$.

Let $x_0 = \cup_{k \geq 1} x_k$, and note that x_0 is a tiling of $\mathbb{R}^d$ by the choice of an *interior* control point. By Proposition 4.9 we have $x_0 \in X_S$, and also $Sx_0 = x_0$. □

It follows from Gottschalk's Theorem and Theorem 5.8 that for any tiling substitution S, $\overline{O(x)} = X_S$ for any $x \in X_S$. By Theorem 5.10 it follows that there exists a self-affine tiling $x_0 \in X_S$, (i.e., $S^k x_0 = x_0$). Thus for any tiling substitution S there exists a self-affine tiling x_0 such that $X_S = \overline{O(x_0)}$. This hints at the alternative definition of X_S mentioned above. However, as the next example shows it is not completely straightforward.

EXAMPLE 5.11. Let S be the *table* substitution and let y_1 be the patch consisting of two rows of two horizontal table tiles, arranged in a 4×2 rectangle. Note that $y_1 \in \mathcal{F}$, the set of forbidden patches for table tilings. Put $y_n = S^{2(n-1)} y_1$, so that $y_n \subseteq y_{n+1}$. Then $y_0 = \cup_{k \geq 1} y_k$ satisfies $S^2 y_0 = y_0$. However $y_0 \notin X_S$ since it contains the patch $y_1 \subseteq y_0$ and $y_1 \in \mathcal{F}$.

One can show that the patch y_1 in the occurs only at the origin in y_0, but nowhere else. In particular, the self-similar tiling y_0 is not repetitive.

The following result shows the correct way to define a substitution tiling space in terms of self-affine tilings.

PROPOSITION 5.12. *Let $S = LC$ be a primitive tiling substitution and let $x_0 \in X_{\mathcal{T}}$ be repetitive and satisfy $S^k x_0 = x_0$ for some $k \geq 1$. Then $X_S = \overline{O(x_0)}$.*

REMARK 5.13. Suppose we start with a tiling substitution S that we do not assume *a priori* satisfies the 2-patch closure property, but that satisfies $S^k x_0 = x_0$ for some $x_0 \in X_{\mathcal{T}}$. It then follows that S does satisfy 2-patch closure. Once this is known, we can use Proposition 5.12 to define X_S provided we can verify that x_0 is repetitive. This may be easier in practice than verifying that S satisfies 2-patch closure.

Next, we consider the question of what linear maps L can occur as the expansion map for a self-affine tiling.

Let $S = \lambda C$ be a primitive tiling substitution for $d = 1$, and let $Sx_0 = x_0$ be a self-similar tiling of $\mathbb{R}$. If A is the structure matrix for S, then it is easy to see that λ must be the *Perron-Frobenius eigenvalue* of A: the unique real eigenvalue of largest modulus (see Section 6.3 below).

EXAMPLE 5.14. A *discrete substitution* σ is a mapping from a finite alphabet $\mathcal{A} = \{1, \ldots, n\}$ to the set of all non-trivial finite words in the alphabet. For a concrete example, see (8.2) in Section 8.6 below (see also [**Que87**]).

We define the *structure matrix* A of σ to be the $n \times n$ matrix such that the entry $A_{i,j}$ is the number of times the letter i occurs in $\sigma(j)$. Assuming A is primitive, we let $\lambda > 0$ and $\mathbf{a} = (a_1, \ldots, a_n) > 0$ be its Perron-Frobenius eigenvalue and eigenvector. Given a primitive non-negative integer matrix A, it is easy to manufacture a substitution σ with structure matrix A (i.e., this amounts to choosing orders for letters in the words corresponding the columns of A).

Now let $\mathcal{T} = \{[0, a_i] : i = 1, \ldots, n\}$ be a set of prototiles in $\mathbb{R}$ and for convenience, identify $\mathcal{T}$ with $\mathcal{A}$ by identifying $[0, a_i]$ with i. Define a tiling substitution S in such a way that $S([0, a_j])$ is the partition of $\lambda \cdot [0, a_j]$ into translates of the intervals corresponding to $i_1, i_2, \ldots, i_{m_j}$, in order, where $\sigma(j) = i_1 i_2 \ldots i_{m_j}$. It follows that S is a tiling substitution for $d = 1$ with expansion λ. Thus we can construct the corresponding substitution tiling dynamical system (X_S, T), and by Theorem 5.10 there exists a self-similar tiling $x_0 \in X_S$ with $S^k x_0 = x_0$ for some k.

Those numbers λ that can be obtained as Perron-Frobenius eigenvalues of a primitive non-negative integer matrix where classified by Doug Lind [**Lin84**], who called them *Perron numbers*. They consist of all positive real algebraic integers λ such that any Galois conjugate λ' of λ satisfies $|\lambda'| < \lambda$. Modulo some technicalities, we have essentially proved the following observation of Thurston.

PROPOSITION 5.15. (Thurston, [**Thu89**]) *A positive real number* λ *is the expansion for a self-similar tiling of* $\mathbb{R}$ (*or equivalently a* 1*-dimensional tiling substitution*) *if and only if it is a Perron number.*

A similar result holds for $d = 2$ in the the self-similar case.

THEOREM 5.16. (Thurston [**Thu89**], Kenyon [**Ken96**]) *Given* $\lambda \in \mathbb{C}$ *there is a primitive* 2*-dimensional self-similar tiling substitution* $S = \lambda C$ *and a self similar-tiling* $x_0 \in X_S$ *with expansion* λ *if and only if* λ *is a* complex Perron number*: an algebraic integer* λ *such that any Galois conjugate* λ' *of* λ*, except possibly the complex conjugate* $\overline{\lambda}$*, satisfies* $|\lambda'| < |\lambda|$.

We call an expansion $L \in Gl(d, \mathbb{R})$ a *Perron expansion* if its eigenvalues $\Lambda = \{\lambda_1, \ldots, \lambda_d\}$, written with multiplicity, satisfy the condition that for every $\lambda \in \Lambda$ with multiplicity k, if λ' is a Galois conjugate of λ with $|\lambda'| > |\lambda|$, then $\lambda' \in \Lambda$ with multiplicity $k' \geq k$. This idea generalizes both real and complex Perron numbers viewed as expansions of $\mathbb{R}$ and $\mathbb{C} \cong \mathbb{R}^2$ respectively.

THEOREM 5.17. (Kenyon, [**Ken90**]) *If a diagonalizable linear map* $L \in Gl(d, \mathbb{R})$ *is the expansion for a primitive self-similar tiling substitution* $S = LC$*, and* $Sx_0 = x_0$ *for some* $x_0 \in X_S$*, then* L *is a Perron expansion.*

Kenyon [**Ken90**] claims that the converse is also true.

5.4. Local mappings. Local mappings play much the same role in tiling dynamical systems that sliding block codes play in symbolic dynamics. The following version of the definition comes from [**PS01**].

DEFINITION 5.18. A continuous mapping between tiling spaces $Q : X \to Y$ is called a *local mapping* if there is an $r > 0$ so that for all $x \in X$, $Q(x)[\{0\}]$ depends only on $x[B_r]$. We say $Q(x)$ is *locally derivable* from x. If Q is invertible, we say x and $Q(x)$ are *mutually locally derivable.*

EXERCISE 8. Show that a local mapping is continuous and T-equivariant (i.e., $T^{\mathbf{t}}Qx = QT^{\mathbf{t}}x$). Moreover, if a local mapping is invertible then its inverse is also a local mapping. Thus a composition mapping is local.

Now consider two dynamical systems (X, T) and (Y, T). A surjective continuous mapping $Q : X \to Y$ so that $T^{\mathbf{t}}Q = QT^{\mathbf{t}}$ for all $\mathbf{t} \in \mathbb{R}^d$ is called a *factor* mapping, and (Y, T) is called a factor of (X, T). An invertible factor mapping Q is called a *topological conjugacy.* In this case the two dynamical systems are said to be *topologically conjugate.*

LEMMA 5.19. *If* $Q : X \to Y$ *is a surjective local mapping between tiling spaces then it is a factor mapping between tiling dynamical systems. If* Q *is invertible then it is a topological conjugacy.*

EXAMPLE 5.20. (Equivalence of different Penrose tilings). Let $X_{\mathcal{P}}$ denote rhombic Penrose tilings and let let $X_{\mathcal{P}_1}$ be the triangular Penrose tilings. Figure 12 shows how these two types of tilings are mutually locally derivable. Similar invertible local mappings connect these examples to several other famous types of Penrose tilings not discussed here (e.g., the "kites and darts" tilings and Penrose's original "pentagon" tilings: see [**GS87**]). It follows that all the various corresponding Penrose tiling dynamical systems are topologically conjugate.

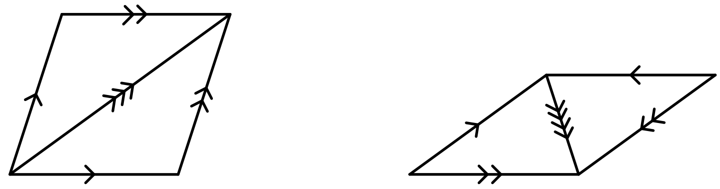

FIGURE 12. The local equivalence between rhombic Penrose tilings and triangular Penrose tilings.

DEFINITION 5.21. A factor mapping $Q : X \to Y$ is *almost* 1:1 if there is a point $y_0 \in Y$ so that $\mathrm{card}Q^{-1}(\{y_0\}) = 1$.

The name of this kind of factor comes from the fact that if (Y, T) is topologically transitive, then $\{y \in Y : \mathrm{card}(Q^{-1}(y)) = 1\}$ is a dense G_δ set.

Now suppose $\mathcal{T}_\#$ is a marked version of a prototile set $\mathcal{T}$ (e.g., in the way $\mathcal{P}$ is a marked version of $\mathcal{R}_5$). Then the local mapping F that erases or "forgets" the markings is an example of a local mapping.

THEOREM 5.22. (Goodman-Strauss [**G-S98**]) *Suppose (X_S, T) is a self-similar substitution tiling dynamical system with $X_S \subseteq X_\mathcal{T}$, for 2-dimensional set $\mathcal{T}$ of prototiles with finite local complexity. Then there exists a marking $\mathcal{T}_\#$ of $\mathcal{T}$, and a local matching rule $\mathcal{T}_\#^{(2)}$ such that the forgetful mapping $F : X_{\mathcal{T}_\#} \to X_\mathcal{T}$ satisfies $F(X_{\mathcal{T}_\#}) = X_S$, and $F : X_{\mathcal{T}_\#} \to X_S$ is almost 1:1.*

In most cases this mapping is *not* invertible (i.e., is not a topological conjugacy). In the terminology of symbolic dynamics, a non finite type factor of a finite type tiling dynamical system should be called *strictly sofic*. It follows that a substitution tiling space is always sofic and usually strictly sofic. An exception is the following. Let $F : \mathcal{P} \to \mathcal{R}_5$ be the mapping that erases the arrows defining the Penrose matching rules. The tilings $X = F(X_\mathcal{P}) \subseteq X_{\mathcal{R}_5}$ are called the *unmarked Penrose tilings*.

THEOREM 5.23. (de Bruijn [**dB81**]) *The marked Penrose tilings are mutually locally derivable with the unmarked Penrose tilings.*

It is interesting to compare these examples to the the situation for 1-dimensional symbolic dynamics. In that case, the intersection between substitution dynamical systems and sofic shifts (or shifts of finite type) consists only of purely periodic examples.

REMARK 5.24. Even though the finite type property is not closed under factorization, it is closed under topological conjugacy. See [**RS01**] for a proof.

To illustrate one further application of local mappings, we return to the idea of an imperfect tiling substitution.

DEFINITION 5.25. An *imperfect tiling substitution* is a mapping $S = LC : X_\mathcal{T} \to X_\mathcal{T}$ where $L \in Gl(d, \mathbb{Z})$ is expansive and $C : X_\mathcal{T} \to X_{L^{-1}\mathcal{T}}$ is a local mapping. For emphasis, the corresponding tiling dynamical system $X_S \subseteq X_\mathcal{T}$ is called an *imperfect substitution* tiling dynamical system .

Note that Examples 4.4 and 4.5 both satisfy this. An imperfect substitution is invertible if C is an invertible local mapping and it is also possible extend the notion of primitivity to this case.

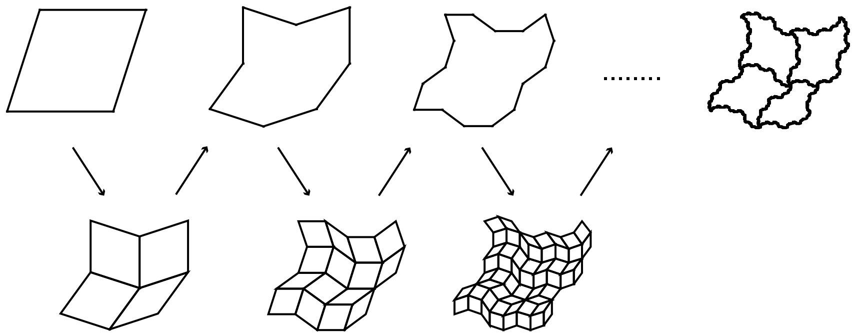

FIGURE 13. Iterating the boundary in the binary tiling decomposition to obtain a perfect decomposition with a fractal boundary.

THEOREM 5.26. (Priebe and Solomyak [**PS01**]) *Suppose* $X_S \subseteq X_{\mathcal{T}}$ *is a imperfect substitution tiling dynamical system with* $d = 2$. *Then there is new prototile set* $\mathcal{T}'$ *and a perfect substitution* S' *on* $\mathcal{T}'$ *so that the substitution tiling dynamical system* $X_{S'} \subseteq X_{\mathcal{T}'}$ *and* X_S *are topologically conjugate via a local mapping (i.e., corresponding tilings are mutually locally derivable).*

This theorem is proved using a construction called "iterating the boundary" which often results in producing fractal tiles. In Figure 13 this idea is illustrated in the case of the binary tiling system of Example 4.5

REMARK 5.27. One way in which tiling dynamical systems differ from discrete symbolic dynamical systems is the following. The Curtis-Lyndon-Hedlund Theorem [**Hed69**] says that any factor mapping between discrete symbolic dynamical systems is implemented by a sliding block code. In tiling dynamical systems the equivalent question is whether every topological conjugacy is implemented by a local mapping. i.e., is the converse to Lemma 5.19 true? A negative answer was provided by Petersen [**Pet99**] and Radin and Sadun [**RS01**].

5.5. Incongruent tilings. In the case that x is a periodic tiling one has $O(x) = \overline{O(x)}$. In other words there is a single orbit. It follows that, up to translation or congruence, there is just a single tiling.

THEOREM 5.28. *If* x *is a properly repetitive tiling then the number of orbits in* $\overline{O(x)}$ *is uncountable. There are uncountably many incongruent tilings in a local isomorphism class.*

This follows directly form the next lemma, which illustrates the power of simple topological ideas in this subject.

PROPOSITION 5.29. *Suppose* (X, T) *is a minimal dynamical system with* T *an action of* $\mathbb{R}^d$. *Let* $\Omega \subseteq X$ *be such that* X *can be expressed as a disjoint union of orbits* $X = \cup_{x\in\Omega} O(x)$. *Then either* $\mathrm{card}(\Omega) = 1$ *or* $\mathrm{card}(\Omega) > \aleph_0$.

PROOF. If Ω is finite then $\mathrm{card}(\Omega) = 1$ since (X, T) is minimal. Thus we suppose Ω is infinite.

For $x \in \Omega$, write x in terms of its tiles $x = \{D_1, D_2, \ldots\}$. Let $V_i(x) = \{T^{\mathbf{t}}x : 0 \in T^{\mathbf{t}}D_i\}$. Then $O(x) = \cup_{i=1}^{\infty} V_i(x)$ is a countable decomposition of $O(x)$ into nowhere dense sets (i.e., they have an empty interior). Thus $X = \bigcup_{x\in\Omega} \bigcup_{i=1}^{\infty} V_i(x)$.

But since X is a compact metric space, it follows from the Baire Category Theorem (see [**Mun75**]) that X is not a countable union of nowhere dense sets. This implies Ω is uncountable. $\square$

COROLLARY 5.30. (Penrose) *In any Penrose tiling space X there are uncountably many incongruent Penrose tilings.*

5.6. Quasicrystallography. Let $M(d)$ denote the set of all rigid motions of $\mathbb{R}^d$ (i.e., the set of congruence transformations). Let $O(d)$ be the subgroup of $M(d)$ fixing the origin. Denote the subgroup of translations in $M(d)$ by $\mathbb{R}^d$.

Suppose x is a periodic tiling. The following ideas are basic to mathematical crystallography (see [**Sen95**]). The *space group* or *symmetry group* of x is defined $G_x = \{M \in M(d) : Mx = x\}$. The translation group Γ_x is a normal subgroup, and the quotient $H_x = G_x/\Gamma_x$, called the *point group* is isomorphic to a finite subgroup of $O(n)$. The *Crystallographic Restriction* is the theorem which says that in any dimension d, there are only finitely many possibilities for H_x. In particular, for $d = 2$ no $M \in H_x$ can have order 5.

Now we sketch the outlines of a theory of *quasicrystallography* (see [**Rob96b**] for more details). For a tiling $x \in X$ define $G_{x,X} = \{M \in M(d) : Mx \in X\}$.

LEMMA 5.31. [**Rob96b**] *If (X, T) is minimal then $G_{x,X} = G_{y,X}$ for all $x, y \in X$.*

EXERCISE 9. Prove lemma 5.31.

In the minimal case, we write G_X. It follows that G_X is a closed subgroup of $M(d)$ containing $\mathbb{R}^d$ as a normal subgroup. We call G_X the *quasisymmetry group* and we call the quotient $H_X = G_X/\mathbb{R}^d$ the *quasicrystallographic* point group. The algebraic situation is simpler than in the case of symmetry groups. One always has that G_X is a semi-direct product of $\mathbb{R}^d$ and H_X. In particular, H_X is isomorphic to a closed subgroup of $O(d)$.

PROPOSITION 5.32. *For the Penrose tiling space $X_{\mathcal{P}}$, $H_{X_{\mathcal{P}}} = D_{10}$ (the dihedral group of order 20). For the Pinwheel tiling space X, $H_X = O(2)$. Thus H_X contains the circle $\mathbb{T}$ as a subgroup.*

In fact, one can construct examples X so that H_X contains any finite order rotation. It follows that there is no Crystallographic Restriction for quasicrystals.

6. Applications of ergodic theory

6.1. Measures. Let (X, T) be a dynamical system. In this section we discuss the set $M(X)$ of Borel probability measures in X.

Without going into a lot of details (see for example [**Wal82**]) we mention that a *measure* $\mu \in M(X)$ is a function that assigns a number $0 < \mu(E) < 1$ to a Borel set $E \subset X$. One way to interpret μ is as a "probability law" in which $\mu(E)$ measures the probability that a randomly chosen point $x \in X$ belongs to E. Borel sets can be complicated, but include all open sets, closed sets, and most of the other sets that typically arise in practice. We will assume all sets mentioned are Borel sets.

The integral of a function with respect to a measure μ is denoted $\int_X f(x)\, d\mu$. If we think of f as a random variable on X, then the integral is its expectation.

Of particular interest to us will be T-invariant measures. These are the measures that satisfy $\mu(T^{\mathbf{t}}E) = \mu(E)$ or $f(T^{\mathbf{t}}x)\, d\mu = \int_X f(x)\, d\mu$ for all $\mathbf{t} \in \mathbb{R}^d$. We denote the set of all invariant measures by $M(X,T)$. One can show that always $M(X,T) \neq \emptyset$.

An important feature of measure theory is that a measure μ is completely determined by its values on a collection of sets smaller than the collection of Borel sets. For a tiling dynamical system (X,T), a measure $\mu \in M(X,T)$ is determined by its values on cylinder sets. One can show that there is a function $\mu_0 : \mathcal{T}^* \to \mathbb{R}$ such that

$$\mu(U_{y,\epsilon}) = \mu_0(y)\mathrm{Vol}(B_1)\epsilon^d, \tag{6.1}$$

for ϵ sufficiently small. This generalizes a similar and well known result that holds for discrete symbolic dynamical systems.

6.2. Unique ergodicity. An invariant measure is said to be *ergodic* if $T^{\mathbf{t}}E = E$ for all $\mathbf{t} \in \mathbb{R}^d$ implies $\mu(E) = 0$ or $\mu(E) = 1$ (note the similarity to the idea of minimality). A dynamical system is called *uniquely ergodic* if $M(X,T) = \{\mu\}$. In this case μ is always an ergodic measure.

THEOREM 6.1. *If S is a primitive tiling substitution then the corresponding tiling dynamical system (X_S,T) is uniquely ergodic.*

A similar result is well known in the case of discrete substitution systems (see [**Que87**]). We will prove Theorem 6.1 below. Later we will also discuss a different kind of uniquely ergodic tiling dynamical system, but first we discuss the consequences of unique ergodicity.

Uniquely ergodic dynamical systems satisfy the following especially strong version of the *Ergodic Theorem.*

THEOREM 6.2. *If a dynamical system (X,T) is uniquely ergodic then for all complex valued continuous functions f on X*

$$\lim_{t\to\infty} \frac{1}{\mathrm{Vol}(B_t)} \int_{B_t} f(T^{\mathbf{t}}x)\,d\mathbf{t} = \int_X f(x)d\mu, \tag{6.2}$$

where the expression on the left, viewed as a function of x, converges uniformly *to the integral the right (a constant). Conversely, if for all continuous f and for all x, the limit in (6.2) exists, then (X,T) is uniquely ergodic.*

Now let us assume (X,T) is a minimal uniquely ergodic tiling dynamical system. Let $x \in X$ be a tiling and let $y \in \mathcal{T}^*$ be a patch that occurs in x. We know that the occurrences of y are relatively dense, but suppose we want a more quantitative description of this repetitivity. For simplicity we assume, as above, that $\mathrm{supp}(y)$ contains a maximal ball around the origin. Let $P(x,y) = \{\mathbf{t} \in \mathbb{R}^d : T^{\mathbf{t}}y \subseteq x\}$. Note that this is a subset of $R(x,U_{y,\epsilon})$. Recall that the *characteristic function* of a set U is

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$$

We have in particular $\int_X \chi_U(x)d\,\mu = \mu(U)$. It turns out that even though characteristic functions of cylinder sets are not continuous, Theorem 6.2 still holds for them.

COROLLARY 6.3. *A tiling dynamical system (X, T) is uniquely ergodic if and only if for any $x \in X$ and any $y \in \mathcal{T}^*$ the following limit exists:*

$$\lim_{t\to\infty} \frac{1}{\mathrm{Vol}(B_t)} \mathrm{card}(B_t \cap P(x, y)). \tag{6.3}$$

If (X, T) is uniquely ergodic, then the value of limit (6.3) is $\mu_0(y)$.

This result explains the combinatorial and geometric meaning of the unique invariant measure in the uniquely ergodic case: it determines the frequency of all the different tiling patches y in all the tilings $x \in X$.

In fact, a similar result holds under the weaker assumption that $\mu \in M(X, T)$ is just ergodic. We say x is *generic* for μ if (6.3) holds. That is, every patch has a well defined frequency. It follows from the Birkhoff Ergodic Theorem (see [**Wal82**]) that μ a.e. $x \in X$ is generic. If (X, T) is not uniquely ergodic, different tilings will be generic for different ergodic measures. This will result in different patch frequencies for different tilings. Moreover, by Corollary 6.3 there will always be some tilings $x \in X$ so that the limit (6.3) diverges. In these tilings, certain patches will not have well defined frequencies.

6.3. Perron Frobenius theory. In this section we present the main part of the proof of Theorem 6.2. We use a well known argument (see [**Que87**]) based on the Perron-Frobenius Theorem to show that in a substitution tiling space, every prototile occurs in every tiling with a well defined density. Unfortunately this is not quite enough to prove Theorem 6.2. In the next section we show how to generalize the idea of a higher block code from symbolic dynamics to tiling spaces, and how to use this idea to finish the argument.

THEOREM 6.4 (Perron-Frobenius Theorem (see [**Rue69**])). *Let $A \geq 0$ be a real square matrix with $A^k > 0$ for some $k \geq 1$. Then there is a simple positive eigenvalue $\omega > 0$ with $\omega > |\omega'|$ for all other eigenvalues ω'. Let $\mathbf{a}$ and $\mathbf{b}$ be the eigenvectors corresponding to ω for A and A^T. Then $\mathbf{a}, \mathbf{b} > 0$ and for any $\mathbf{v} \in \mathbb{R}^d$*

$$\lim_{n\to\infty} \frac{1}{\omega^n} A^n \mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}. \tag{6.4}$$

The eigenvalue $\omega > 0$ and eigenvector $\mathbf{a} > 0$ are called the *Perron-Frobenius eigenvalue* and *Perron-Frobenius eigenvector* of A .

COROLLARY 6.5. *let A be the structure matrix for a primitive tiling substitution $S = LC$ and let ω be the Perron-Frobenius eigenvalue of A. Then $\omega = \det(L)$.*

This is because both ω and $\det(L)$ measure how the substitution S expands volumes.

COROLLARY 6.6. *Let X be a primitive substitution tiling space. For $D_i \in \mathcal{T}$ let $D_i^n = \mathrm{supp}(S^n D_i)$. Then for any $x \in X$ and $D_i, D_j \in \mathcal{T}$ the limit*

$$\lim_{n\to\infty} \frac{1}{\mathrm{Vol}(D_i^n)} \mathrm{card}(D_i^n \cap P(x, \{D_j\}))$$

exists.

PROOF. Since $\omega^n = \det(L^n)$, we have $\mathrm{Vol}(D_i^n) = \omega^n \mathrm{Vol}(D_i)$. Write $d_i = \mathrm{Vol}(D_i)$. Let $A_{i,j}^n$ be the i,jth entry of A^n. Note that $A_{i,j}^n$ is the number of tiles equivalent to D_j that occur in $S^n(\{D_i\})$, so

$$A_{i,j}^n = \mathrm{card}(D_i^n \cap P(x, \{D_j\})) \tag{6.5}$$

Since $A^n_{i,j} = \mathbf{e}_i \cdot (A^n \mathbf{e}_j)$, it follows from (6.4)

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{\mathrm{Vol}(D_i^n)} \mathrm{card}(D_i^n \cap P(x, \{D_j\})) &= d_i^{-1} \lim_{n\to\infty} \frac{1}{\omega^n} A^n_{i,j} \\
&= d_i^{-1} \lim_{n\to\infty} \frac{1}{\omega^n} \mathbf{e}_i \cdot (A^n \mathbf{e}_j) \\
&= d_i^{-1} \mathbf{e}_i \cdot \left(\lim_{n\to\infty} \frac{1}{\omega^n} (A^n \mathbf{e}_j) \right) \\
&= d_i^{-1} (\mathbf{b} \cdot \mathbf{e}_j)(\mathbf{e}_i \cdot \mathbf{a}) \\
&= \frac{a_i b_j}{d_j},
\end{aligned}$$

where $\mathbf{a} = (a_1, \ldots, a_d)$ and $\mathbf{b} = (b_1, \ldots, b_d)$. □

With a bit more care one can prove the following refinement of the above.

COROLLARY 6.7. *Let X be a primitive substitution tiling space. Then for any $x \in X$ and $D \in \mathcal{T}$ the limit*

$$\lim_{r\to\infty} \frac{1}{\mathrm{Vol}(B_r)} \mathrm{card}(B_r \cap P(x, \{D\})) \tag{6.6}$$

exists.

6.4. Higher patch tiles. This section discusses a technical result needed to complete the proof of Theorem 6.1.

Let (X, T) be a tiling dynamical system $X \subseteq X_{\mathcal{T}}$. Fix $r > 0$ and let $y_1, y_2, \ldots, y_m$ be the equivalence classes of patches $x[B_r]$, $x \in X$. Let

$$E_i = \overline{\{\mathbf{t} \in \mathbb{R}^d : x[B_r + \mathbf{t}] \sim y_i\}},$$

where we assume E_i has nonempty interior. Define a new tiling x' by subdividing each tile $D \in x$ into the smaller tiles $D \cap E_i$, $i = 1, \ldots, m$. Up to equivalence there are only finitely prototiles in x' for all $x \in X$ and we denote this new prototile set by $\mathcal{T}_r$. Moreover, the mapping $H_r : X_{\mathcal{T}} \to X_{\mathcal{T}_r}$ is clearly an invertible decomposition mapping. It is called *higher patch mapping*. This is similar to the idea of a *higher block code* in symbolic dynamics (see [**LM95**], [**Que87**]).

PROPOSITION 6.8. *If S is a tiling substitution on $X_{\mathcal{T}}$ then $S_r = H_r^{-1} S H_r$ is a tiling substitution on $X_{\mathcal{T}_r}$. If S is primitive then so is S_r.*

Theorem 6.1 now follows by applying Proposition 6.8 and Corollary 6.7.

7. Mixing properties

7.1. Mixing and eigenvalues: the geometric interpretation. Let (X, T) be a dynamical system and $\mu \in M(X, T)$. A complex function $f \in L^2(X, \mu)$ is called an *eigenfunction* if there exists a corresponding *eigenvalue* $\mathbf{w} \in \mathbb{R}^d$ such that

$$f(T^{\mathbf{t}} x) = e^{2\pi i \langle \mathbf{t}, \mathbf{w} \rangle} f(x), \tag{7.1}$$

for μ a.e. x. Note that this "eigenvalue" is actually a vector! In Physics this might be called a "wave vector."

A constant function f is always an eigenfunction, corresponding to $\mathbf{w} = \mathbf{0}$. Ergodicity is equivalent to $\mathbf{w} = \mathbf{0}$ being a simple eigenvalue. Moreover, in the ergodic case all the eigenvalues are simple, and the set Σ of eigenvalues is a countable subgroup of $\mathbb{R}^d$ (see [**Wal82**]).

If the only eigenfunctions are constants, then T is said to be *weakly mixing.* The opposite situation is called *pure discrete spectrum*; it occurs when the eigenfunctions have a dense span in $L^2(X, \mu)$. An eigenfunction f is continuous if it is equal μ a.e. to a continuous function.

A dynamical system is called *strongly mixing* (or just *mixing*)if for any Borel sets A and B

$$\lim_{||\mathbf{t}|| \to \infty} \mu(T^{\mathbf{t}} A \cap B) = \mu(A)\mu(B). \tag{7.2}$$

A well known theorem (see [**Wal82**]) says that weak mixing is equivalent to (7.2) holding except on a set of $\mathbf{t}$ of density zero (this set depends on A and B).

Now consider a tiling dynamical system (X, T). As we will discuss below, the eigenvalues are related to the the "diffraction" properties of tilings $x \in X$. Heuristically, such diffraction is caused by constructive reinforcement of waves reflecting off atoms, usually thought of as being located at the vertices of a tiling. When a tiling exhibits diffraction it can be interpreted as evidence that the tiling has some sort of long range spatial order in the arrangement of its tiles. Periodic tilings always diffract, but as we will see below, so do some properly repetitive tilings.

Conversely, if a tiling system satisfies a mixing property (i.e., a lack of diffraction) then it indicates that its tilings enjoy some sort of long-range spatial disorder. Consider, for example, (7.2) applied to a pair of cylinder sets U_{ϵ,y_1} and U_{ϵ,y_2} in a mixing tiling dynamical system (X, T). For a randomly chosen $x \in X$ and for $\mathbf{t}$ sufficiently large, the probability of seeing y_1 and $T^{\mathbf{t}} y_2$, (up to an ϵ-translation), is approximately $\epsilon^2 \mathrm{Vol}(B_1)^2 \mu_0(y_1)\mu_0(y_2)$. Thus, the knowledge that y_1 sits at one place in x is approximately statistically independent of the knowledge that a copy of y_2 sits at any particular distant location.

7.2. Weakly mixing tiling spaces. There are two known mechanisms for producing weakly mixing tiling dynamical systems. The first is related to the algebraic properties of the eigenvalues of the expansion. It generalizes ideas from the theory of discrete 1-dimensional substitutions.

Let $D \in \mathcal{T}$ and $x \in X_{\mathcal{T}}$. Define

$$\Xi(x) = \{\mathbf{t} \in \mathbb{R}^d : \exists D_1, D_2 \in x, D_2 = D_1 - \mathbf{t}\}.$$

If (X, T) is a properly minimal tiling dynamical system, then $\Xi(x)$ is the same for all tilings $x \in X$, and we write $\Xi(X)$. In addition, $\Xi(X)$ satisfies

$$\overline{\{\mathbf{t}/||\mathbf{t}|| : \mathbf{t} \in \Xi(X), \mathbf{t} \neq 0\}} = S^{d-1} \subseteq \mathbb{R}^d \tag{7.3}$$

(see [**Sol97**], Proof of Theorem 4.4). Note that this is the case when $X = X_S$ is a primitive substitution tiling space.

THEOREM 7.1. (Solomyak [**Sol97**]) *A number* $\mathbf{w} \in \mathbb{R}^d$ *is an eigenvalue for an invertible primitive self-affine substitution tiling system* (X_S, T) *with* $S = LC$ *if and only if*

$$\lim_{n \to \infty} e^{2\pi i \langle L^n \mathbf{t}, \mathbf{w} \rangle} = 1 \tag{7.4}$$

for all $\mathbf{t} \in \Xi(X_S)$*. Moreover, the eigenfunctions can always be chosen to be continuous.*

This is a combination of Theorems 4.3 and 5.1 in [**Sol97**]. It generalizes a similar result of Host [**Hos86**] for 1-dimensional discrete substitutions.

REMARK 7.2. In certain non-weakly mixing cases Solomyak [**Sol97**] describes some eigenvalues explicitly.

Our interest here is on weak mixing, and for this purpose the beauty of Theorem 7.1 is that it reduces the question to number theory. A real algebraic integer $\lambda > 1$ is called a *Pisot number* if all of its Galois conjugates λ' satisfy $|\lambda'| < 1$. *Pisot's Theorem* says that a positive real algebraic number λ satisfying $e^{2\pi i\omega\lambda^n} \to 1$ for some for some $\omega \in \mathbb{R}$ must be a Pisot number (see [**Sal63**]). A *complex Pisot number* is a complex algebraic integer λ all of whose Galois conjugates λ', except its complex conjugate, satisfy $|\lambda'| < 1$. The following generalization of this idea is due to Mauduit [**Mau89**].

DEFINITION 7.3. A set $\Lambda' = \{\lambda_1, \ldots, \lambda_{d'}\}$ of distinct algebraic integers with $|\lambda_i| \geq 1$ is called a *Pisot family* if λ' is Galois conjugate of some $\lambda \in \Lambda'$ with $\lambda' \notin \Lambda'$, then $|\lambda'| < 1$. Otherwise Λ' is called *non-Pisot.*

A real Pisot number by itself is a Pisot family, as is a complex Pisot number together with its complex conjugate. The next result generalizes Pisot's Theorem.

THEOREM 7.4. (Mauduit, [**Mau89**]) *If $\Lambda' = \{\lambda_1, \ldots, \lambda_{d'}\}$ is set of distinct algebraic numbers such that*

$$\lim_{n\to\infty} e^{2\pi i \sum_{i=1}^{d'} v_i \lambda_i^n} = 1$$

for some[8] $(v_1, \ldots, v_{d'}) \in (\mathbb{C}\backslash\{0\})^{d'}$*, then Λ' is a Pisot family.*

A nonempty set Λ' of distinct algebraic integers can be written as a disjoint union

$$\Lambda' = \Lambda_1 \cup \Lambda_2 \cup \cdots \cup \Lambda_\ell \tag{7.5}$$

where for each i there exists a monic irreducible polynomial $p_i \in \mathbb{Z}[t]$ (the minimal polynomial) such that $p_i(\lambda) = 0$ for all $\lambda \in \Lambda_i$.

DEFINITION 7.5. We say Λ' is *totally non-Pisot* if each Λ_i in the decomposition (7.5) is non-Pisot.

It is clear that a totally non-Pisot family Λ' is non-Pisot, and moreover, any nonempty subset $\Lambda'' \subseteq \Lambda'$ is totally non-Pisot. We call an expansion L *totally non-Pisot* if its set Λ' of eigenvalues, written without multiplicity, is a totally non-Pisot family.

THEOREM 7.6. *Suppose $S = LC$ is a primitive invertible tiling substitution such that the expansion L is diagonalizable and totally non-Pisot. Then the substitution tiling system (X_S, T) is weakly mixing.*

This is essentially due to Solomyak [**Sol97**], although our formulation is different.

[8] Mauduit [**Mau89**] proves this for $(\mathbb{R}\backslash\{0\})^{d'}$, but the proof works in the complex case.

PROOF. Let $\Lambda' = \{\lambda_1, \dots, \lambda_{d'}\}$ be the eigenvalues of L, written without multiplicity. Then Λ' is totally non-Pisot. Given $\mathbf{t} \in \mathbb{R}^d$ express $\mathbf{t} = \sum_{j=1}^{d'} P_j \mathbf{t}$ where P_j is the projection to the eigenspace for λ_j, parallel to the all the other eigenspaces, and $\mathbf{t}_j = P_j\, \mathbf{t} \in \mathbb{C}^d$. Then $L^n \mathbf{t} = \sum_{j=1}^{d'} \lambda_j^n \mathbf{t}_j$.

Now suppose $\mathbf{w} \neq \mathbf{0}$ is an eigenvalue for (X_S, T). Then by Theorem 7.1 we have

$$(7.6) \qquad 1 = \lim_{n\to\infty} e^{2\pi i \langle L^n \mathbf{t}, \mathbf{w}\rangle} = \lim_{n\to\infty} e^{2\pi i \sum_{j=1}^{d'} \lambda_j^n \langle \mathbf{t}_j, \mathbf{w}\rangle} = \lim_{n\to\infty} e^{2\pi i \sum_{j=1}^{d'} \lambda_j^n \langle \mathbf{t}, P_j^* \mathbf{w}\rangle},$$

where $\langle \mathbf{t}, \mathbf{w}\rangle = \mathbf{t} \cdot \overline{\mathbf{w}}$ is the inner product on $\mathbb{C}^d$, and P_j^* is the complex-conjugate transpose of P_j.

Let $\Lambda''(\mathbf{w}) = \{\lambda_j \in \Lambda' : P_j^*(\mathbf{w}) \neq 0\}$, and note that $\Lambda''(\mathbf{w}) \neq \emptyset$ since L is diagonalizable. By (7.3) there exists $\mathbf{t} \in \Xi(X)$ such that $\langle \mathbf{t}, P_j^* \mathbf{w}\rangle \neq 0$ for all j such that $\lambda_j \in \Lambda''(\mathbf{w})$. Applying (7.6), it follows from Theorem 7.4 that $\Lambda''(\mathbf{w})$ is a Pisot family. But since $\Lambda''(\mathbf{w}) \subseteq \Lambda'$, this contradicts the fact that Λ' is totally non-Pisot $\square$

REMARK 7.7.

(1) Suppose $d \geq 1$ and let $S = \lambda C$ be a self-similar tiling substitution with $\lambda \in \mathbb{R}$. If λ is not real Pisot then (X_S, T) is weakly mixing.
(2) Suppose $d = 2$ and $S = \lambda C$, where $\lambda \in \mathbb{C}\backslash\mathbb{R}$. If λ is not complex Pisot, then (X_S, T) is weakly mixing.
(3) In both cases above, for $d = 2$, the converse is also true, [**Sol97**].

COROLLARY 7.8. *The binary tiling dynamical system (X_{S_b}, T) is weakly mixing.*

PROOF. We use the fact that $C_b^2(\mathcal{T}) \subseteq \lambda^{-2}\mathcal{T}^*$, $\lambda \in \mathbb{R}$, so that $S_b^2 = \lambda C_b^2$. The structure matrix for S_b^2 is A^2, where $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$. Thus, the expansion λ for S_b^2 is the Perron-Frobenius eigenvalue λ for A, which is not real Pisot. $\square$

The reader should compare the disordered appearance of binary tilings to the more regular appearance of the Penrose tilings. As we will see below, Penrose tiling dynamical systems have pure discrete spectrum.

REMARK 7.9. Let $S = \lambda C$ be a tiling substitution with $d = 2$ and $\lambda \in \mathbb{R}\backslash\mathbb{C}$. We show here that in order to establish that (X_S, T) is weakly mixing it is *not* sufficient (i) that $|\lambda|$ is not real Pisot, nor (ii) that $|\lambda|^2 = \omega$ (where ω is the Perron-Frobenius eigenvector for A) is not real Pisot.

For (i) consider $p(t) = t^4 + t^2 - 1$. This has complex Pisot root $\lambda = i\sqrt{\tau}$, $\tau = \frac{1+\sqrt{5}}{2}$, but $|\lambda| = \sqrt{\tau}$ is not real Pisot. For (ii) consider $q(t) = t^3 - t^2 + 10t - 5$. This has a complex Pisot root λ, but $\omega = |\lambda|^2$ is a root of $r(t) = t^3 - 10t^2 + 5t - 25$, and so is not real Pisot.

Finally, we observe that λ complex Pisot implies that λ is complex Perron. Thus by Theorem 5.16, in each case above, there exists a tiling substitution S with expansion λ. The fact that the corresponding substitution tiling system (X_S, T) is not weakly mixing follows from Remark 7.7, part 3. My thanks to the referee for providing these two examples (see also [**Sol99**]).

The second known mechanism responsible for producing weakly mixing tiling dynamical systems involves quasisymmetry.

PROPOSITION 7.10. [**Rob96b**] *The group of eigenvalues Σ_X of a tiling dynamical system is invariant under the action of the quasicrystallographic point group H_X.*

COROLLARY 7.11. (Radin, [**Rad94**]) *The pinwheel tiling dynamical system is weakly mixing.*

PROOF. Let X be the pinwheel tiling space. Then $\mathbb{T} \subseteq H_X$. Since Σ_X must be discrete and $\mathbb{T}$-invariant, it follows that $\Sigma_X = \{0\}$. □

REMARK 7.12. Since the almost 1:1 extension in Theorem 5.22 is always a metric isomorphism, we can obtain examples (starting e.g. with the binary tilings) of minimal uniquely ergodic finite type tiling spaces that are weakly mixing. On the other hand, the next result shows that none of these examples can be strongly mixing.

THEOREM 7.13. (Solomyak, [**Sol97**]). *No self-affine substitution tiling dynamical system (X, T) can be strongly mixing.*

7.3. Diffraction. X-ray diffraction experiments provide a powerful method for studying the microscopic structure of solids. In particular, quasicrystals were discovered (see [**SBGC84**]) as a result of the observations of unusual diffraction patterns.

Mathematically we model diffraction as follows. A *Delone set* is a uniformly discrete and relatively dense subset of $\mathbb{R}^d$ (see [**Sen95**]). Starting with a tiling x, we let $z = v(x)$ be the Delone set of all its vertex points. We place an "atom" $\delta_{\mathbf{t}}$, consisting of a unit atomic measure, at each vertex point $\mathbf{t} \in z$. The corresponding "solid" consists of the measure

$$\mu_x = \sum_{\mathbf{t} \in v(x)} \delta_{\mathbf{t}}. \tag{7.7}$$

The physical diffraction pattern corresponds mathematically to the locations Σ'_x of the point masses in the Fourier transform $\hat{\mu_x}$. There are many technicalities involving how this is actually computed. For example, one should really compute the transform of the *autocorrelation measure*, which is supported on the difference set $z - z$ of z. We will not discuss these issues here, but rather refer the reader to [**Hof97**] for a nice overview. The main result in this area is the following.

THEOREM 7.14 ([**Dwo93**], [**Hof97**]). *Let (X, T) be a uniquely ergodic tiling space with eigenvalue set Σ. Then for any $x \in X$, $\Sigma'_x \subseteq \Sigma$.*

Because of this result, physicists have been especially interested in tiling dynamical systems with pure discrete spectrum. On the other hand, one would expect a weakly mixing repetitive tiling (like the binary tiling) to have no spots in its diffraction pattern. In spite of this, it still seems reasonable to regard this example as a "quasicrystal": as we will see in the next section, it has entropy zero.

7.4. Entropy.

DEFINITION 7.15. Let (X, T) be a tiling dynamical system. For $n > 0$ the *complexity* $c(n)$ of (X, T) is the number of different equivalence classes of tilings

$x[B_n]$ for all $x \in X$. The *topological entropy* is defined as the exponential growth rate in complexity:

$$h(X) = \lim_{n\to\infty} \frac{1}{n^d} \mathrm{Vol}(B_1)^{-1} \ln(c(n)). \tag{7.8}$$

Nonzero entropy is, in some sense, the ultimate indication of disorder in a dynamical system. A tiling whose orbit closure is a positive-entropy tiling dynamical system should probably be considered too disordered to be regarded as a quasicrystal. Some of the full tiling shifts, discussed above, do have positive entropy. Since they are topologically transitive, there are tilings among them with a positive entropy orbit closure. However, as expected, most of the tiling dynamical systems that arise in the study of quasicrystals do indeed have zero entropy.

THEOREM 7.16. *Suppose X is either* (a) *a substitution tiling space for a primitive invertible tiling substitution, or* (b) *a finite type tiling space which is uniquely ergodic. Then $h(X) = 0$.*

Part (b) is due to Radin [**Rad91**] in the $\mathbb{Z}^d$ case and generalized to the case of tiling dynamical systems by Shieh [**Sh**]. Part (a) follows from the next theorem.

THEOREM 7.17. [**HR02**] *Let (X_S, T) be a substitution tiling space, where $S = LC$ is primitive and invertible. Suppose L has eigenvalues $\lambda_1, \dots, \lambda_d$ where $|\lambda_d| \le |\lambda_i|$ for all i. Let*

$$c = \frac{\log|\det(L)|}{\log|\lambda_d|} = \frac{\log(|\lambda_1|\cdot\dots\cdot|\lambda_d|)}{\log|\lambda_d|}.$$

Then the complexity satisfies

$$c(n) \le K \cdot n^c.$$

for some $K > 0$. In the self-similar case, $|\lambda_1| = \dots = |\lambda_d|$, so $c = d$.

The proof follows the dissertation of Clifford Hansen ([**Han00**]), who studied the case of discrete multi-dimensional substitutions. It is based on the following consequence of local finiteness.

LEMMA 7.18. *Let $X_{\mathcal{T}}$ be a finite local complexity tiling space. Given $m \ge 1$ there exists a constant $J = J(m) > 0$ so that for all n sufficiently large*

$$\#\{x \in \mathcal{T}^{(m)} : x \subseteq y \text{ for some } y \in \mathcal{T}^{(n)}\} \le J \cdot n. \tag{7.9}$$

PROOF OF THEOREM 7.17. For $D_j \in \mathcal{T}$ and $\mathbf{v} = (1, 1, \dots, 1)^t$ we have $\#(C^p D_j) = (A^p \mathbf{v})_j$, where A is the structure matrix for S. Since S is primitive, the Perron-Frobenius Theorem implies

$$\lim_{p\to\infty} \frac{A^p\mathbf{v}}{\omega^p} = (\mathbf{b}\cdot\mathbf{v})\mathbf{a} \stackrel{\text{def}}{=} \mathbf{r},$$

where $\omega > 0$ and $\mathbf{a}, \mathbf{b} > 0$. Since also $\mathbf{v} > 0$, we have $\mathbf{r} > 0$. Thus there exists N so that for all sufficiently large p,

$$\max_{D\in\mathcal{T}} \#(C^p D) \le N \cdot \omega^p. \tag{7.10}$$

For $p \ge 0$, call a patch $y \in L^p\mathcal{T}^*$ a *p-basic* patch if for some $D \in y$, each $D' \in y$ satisfies $D' \cap D \ne \emptyset$. We denote the p-basic patches, up to equivalence, by $y_1^p, \dots, y_M^p$, where M is independent of p. Then $M' = \max\{\#(y_j^p) : j = 1, \dots, M\}$ is also independent of p.

For a p-basic patch y_j^p, we have $C^p y_j^p \in \mathcal{T}$. Let

$$M_p = \max\{\#(C^p y_j^p) : j = 1, \ldots, M\}. \tag{7.11}$$

Since $\#(C^p y_j^p) \leq M' \cdot \max_{D \in \mathcal{T}} \#(C^p D)$, it follows from (7.10) that

$$M_p \leq M' N \cdot \omega^p. \tag{7.12}$$

Let $\delta = \max\{\operatorname{diam}(D) : D \in \mathcal{T}\}$ and let ϵ be the maximum of all $r > 0$ such that all $D \in \mathcal{T}$ satisfy $B_r \subseteq D - \mathbf{s}$ for some $\mathbf{s} \in \mathbb{R}^d$.

Fix $q \geq (\delta + 1)/\epsilon$. Then for all $n \geq 1$, $\epsilon q n \geq n + \delta$. Define

$$p_n = \frac{\log(qn)}{\log|\lambda_d|}.$$

It follows that

$$\epsilon |\lambda_d|^{p_n} = \epsilon q n \geq n + \delta. \tag{7.13}$$

By the choice of δ, $B_{n+\delta} \supseteq \operatorname{supp}(x[B_n])$ for all $n > 0$ and all $x \in X_S \subseteq X_{\mathcal{T}}$. Since S is invertible, one can define a *super-tiling* $C^{-p}x \in X_{L^p \mathcal{T}}$ for any $p \geq 1$ and $x \in X_S$. It follows from (7.13) that $L^p B_\epsilon \supseteq B_{n+\delta}$ for all $p \geq p_n$. Thus for each $x \in X_S$, the patch $(C^{-p}x)[B_{n+\delta}]$ is a sub-patch of some p-basic patch $T^{\mathbf{t}} y_j^p$ in the super-tiling $C^{-p}x$. Hence,

$$x[B_n] \subseteq x[B_{n+\delta}] = C^p(C^{-p}x[B_{n+\delta}]) \subseteq T^{\mathbf{t}} C^p y_j^p.$$

Now we apply Lemma 7.18, (7.11) and (7.12) to conclude that

$$c(n) \leq J \cdot M_p \leq J M' N \cdot \omega^p \stackrel{\text{def}}{=} K' \cdot \omega^p$$

for all $p \geq p_n$ once n is sufficiently large. In particular, we take n large enough that (7.10) holds for $p = p_n$. It follows that

$$\begin{aligned} c(n) &\leq K' \omega^{p_n} = K' \omega^{\frac{\log(qn)}{\log|\lambda_d|}} \\ &= K' e^{\frac{\log(\omega)\log(qn)}{\log|\lambda_d|}} \\ &= K' q^c \cdot n^c \\ &= K \cdot n^c, \end{aligned}$$

where $K = K' q^c = J M' N q^c$.

□

We conclude with two open problems.

- Can a uniquely ergodic finite type tiling space X be strongly mixing?
- Is there an example of a uniquely ergodic, finite type tiling space that not an almost 1:1 extension of a substitution tiling space (or, more generally, that is not a tiling space having some other type of hierarchical structure)?

8. Quasiperiodic tilings and models

In this section we describe the *projection method*, which is the third general method (after local matching rules and tiling substitutions) for constructing aperiodic tilings. This method also provides an algebraic/geometric model for the tiling spaces it produces. After describing the projection method, we briefly describe some other geometric models.

8.1. Kronecker dynamical systems. Let $\mathbb{G}$ be a compact abelian group, written additively, with normalized Haar measure γ. Suppose there exists an injective continuous homomorphism $\iota : \mathbb{R}^d \to \mathbb{G}$. The continuous γ-preserving $\mathbb{R}^d$ action

$$T^{\mathbf{t}}\mathbf{g} = \iota(\mathbf{t}) + \mathbf{g} \tag{8.1}$$

on $\mathbb{G}$ is called a *Kronecker dynamical system.* Every point in such a dynamical system is almost periodic. Any coset $W = \overline{\iota(\mathbb{R}^d)} + \mathbf{g}$ is T-invariant. It follows that, up to topological conjugacy, (W, T) is a minimal Kronecker system. Any minimal Kronecker system is uniquely ergodic (see [**Wal82**]).

Remark 8.1. By an appropriate choice of ι one can ensure that (W, T) is properly minimal.

Every Kronecker system has pure discrete spectrum. The eigenfunctions are the *characters* $\chi \in \widehat{\mathbb{G}}$ of $\mathbb{G}$. In particular, for $\chi \in \widehat{\mathbb{G}}$ we have $\chi(\iota(\mathbf{t})) \in \widehat{\mathbb{R}^d} = \mathbb{R}^d$, so there exists $\mathbf{w} \in \mathbb{R}^d$ such that $\chi(\iota(\mathbf{t})) = e^{\pi i \langle \mathbf{t}, \mathbf{w} \rangle}$, and the set Σ of eigenvalues of (W, T) is the set of all such $\mathbf{w}$ (see [**Wal82**] for details).

Two dynamical systems (X, T) and (Y, T), preserving measures μ and ν respectively, are said to be *metrically isomorphic* (see [**Wal82**]) if there exist T-invariant subsets $X_0 \subseteq X$ and $Y_0 \subseteq Y$ with $\mu(X_0) = \nu(Y_0) = 1$, and an invertible measure preserving Borel mapping $Q : X_0 \to Y_0$ so that $QT = TQ$. Metric isomorphism is the primary notion of isomorphism studied in ergodic theory. If (X, T) and (Y, S) are uniquely ergodic then topological conjugacy implies metric isomorphism. Note that any two metrically isomorphic ergodic dynamical systems must have the same eigenvalues.

Halmos-von Neumann Theorem. (See [**Wal82**]) *Any dynamical system with pure discrete spectrum is metrically isomorphic to a Kronecker system. Every countable subgroup $\Sigma \subseteq \mathbb{R}^d$ is the eigenvalue group for a Kronecker system with $\mathbb{R}^d$ acting.*

Remark 8.2. Every Kronecker system has entropy zero (see [**Wal82**]).

8.2. The projection method. A landmark in the theory of aperiodic tilings is de Bruijn's algebraic theory of Penrose tilings [**dB81**]. Originally, this theory described Penrose tilings as being dual (in the sense of graph theory) to so-called "grid" tilings. The generalization of this idea is called the *grid* method, and the tilings it produces are called *quasiperiodic tilings*. There are two alternate equivalent constructions of quasiperiodic tilings, mostly developed by physicists: the *projection* method and the *cut* method (see [**ODK88**]). Here we discuss the projection method because it is conceptually the simplest.

Let $E^{\|}$ be a d-dimensional subspace of $\mathbb{R}^n$ and let $\iota : \mathbb{R}^n \to E^{\|}$ be an isometric isomorphism. Let $E^{\perp}$ be the perpendicular subspace, so that $\mathbb{R}^n = E^{\|} \oplus E^{\perp}$. Denote the projections to these two subspaces by $\pi^{\|}$ and $\pi^{\perp}$. Consider the integer lattice $\mathbb{Z}^n \subseteq \mathbb{R}^n$. Let W_0 be the closure of $\iota(\mathbb{R}^d)$ in $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and let $W_0 + \mathbf{g}$, $\mathbf{g} \in E^{\perp}$, be an arbitrary coset. The tiling systems we will construct is closely related to the Kronecker system (W_0, T).

Let $K \subseteq E^{\perp}$ be compact with a Lebesgue measure zero boundary and a nonempty interior. Let $S_K = K + E^{\|}$. For $\mathbf{s} \in \mathbb{R}^d$ let

$$z_{\mathbf{s}} = (\iota^{-1}\pi^{\|})(S_K \cap (\mathbb{Z}^n + \mathbf{s})) \subseteq \mathbb{R}^d.$$

Note that $z_{\mathbf{s}} = z_{\mathbf{s}'}$ if $\mathbf{s} - \mathbf{s}' \in \mathbb{Z}^d$. Thus we can index using $\mathbf{s} \in \mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$, and this makes the mapping $\mathbf{s} \mapsto z_{\mathbf{s}}$ 1:1.

The set $z_{\mathbf{s}}$ is a *Delone set*. It is possible to topologize the collection of all Delone sets (using something similar to a tiling metric) in such a way that the set $\{z_{\mathbf{s}} : \mathbf{s} \in \mathbb{T}^n\}$ is homeomorphic to $\mathbb{T}^n$. We also have $z_{T^{\mathbf{t}}\mathbf{s}} = T^{\mathbf{t}} z_{\mathbf{s}}$, so in fact $\mathbf{s} \mapsto z_{\mathbf{s}}$ is a topological conjugacy.

We call $\mathbf{s} \in \mathbb{T}^n$ *regular* if $\partial(S_K) \cap (\mathbb{Z}^n + \mathbf{s}) = \emptyset$. Let $\mathbb{T}_0^n$ denote the set of all regular points in $\mathbb{T}^n$, and note that $\mathbb{T}_0^n$ has full Lebesgue measure. In many cases it will turn out that the points in $z_{\mathbf{s}}$ are the vertices of a tiling x of $\mathbb{R}^d$. Let us now specialize to such a situation.

Let $d = 2$ and consider the full tiling space $X_{\mathcal{R}_n}$, $n \geq 4$, from Example 2.6. Let $K = \pi^{\perp}(Q)$ where $Q = \{\mathbf{q} \in \mathbb{R}^n : 0 \leq q_i \leq 1\}$ is the unit cube. Let s be as in Example 2.6, and let B be matrix having the vectors $\mathbf{v}_k$ as rows. Define $\iota(\mathbf{t}) = B\mathbf{t}$.

PROPOSITION 8.3 (de Bruijn, [**dB81**]). *For each $\mathbf{s} \in \mathbb{T}_0^n$ there exists a tiling $x \in X_{\mathcal{R}_n}$ with $v(x) = z_{\mathbf{s}}$, i.e., x has the Delone set $z_{\mathbf{s}}$ as its vertex points.*

In addition, de Bruijn [**dB81**] showed that the non-regular points in $\mathbb{T}^n$ correspond to more than one tiling (but always a finite number). Moreover, each of these tilings can be obtained as a limit of the regular cases. This result can be made into a statement about tiling dynamical systems as follows:

THEOREM 8.4. [**Rob96b**] *Let $W = \overline{\iota(\mathbb{R}^2)} + \mathbf{g}$ for $\mathbf{g} \in W^{\perp}$. For $\mathbf{s} \in W \cap \mathbb{T}_0^n$ define $x = H(\mathbf{s}) \in X_{\mathcal{R}_n}$ to be the tiling with vertex set $v(x) = z_{\mathbf{s}}$. Then*

$$X = \overline{H(W \cap \mathbb{T}_0^n)}$$

is a tiling space that is minimal and uniquely ergodic. Moreover, H^{-1} extends to a continuous mapping $P : X \to W$ which is an almost 1:1 factor mapping and a metric isomorphism.

In the case where ι is chosen so that (W, T) is properly minimal, each tiling $x \in X$ will be aperiodic (i.e., properly repetitive). The tilings $x \in X$ are called *quasiperiodic tilings* and (X, T) is called a *quasiperiodic tiling dynamical system.*

COROLLARY 8.5. *Every quasiperiodic tiling dynamical system (X, T) is properly minimal, uniquely ergodic and contains uncountably many incongruent tilings. Moreover, it has an almost* 1:1 *Kronecker system factor (W, T), to which it is metrically isomorphic. It thus has pure discrete spectrum and entropy zero.*

It is possible to find explicitly the set Σ for these quasiperiodic tiling dynamical systems (see [**Rob96a**]). In particular,

$$\Sigma = B^t\mathbb{Z}^n = \{\sum_{j=1}^{n} n_j \mathbf{v}_j : n_j \in \mathbb{N}\}.$$

When $d = 2$ and $n = 2^m$ we have $\overline{\iota(\mathbb{R}^2)} = \mathbb{T}^n$. The case $n = 4$ gives a well known example called the *octagonal* or *Ammann-Beenker tilings.* Like the Penrose tilings, the Amman-Beenker tilings can also be generated by a local matching rule and by a tiling substitution (see [**Sen95**] for details).

In the case that $d = 2$ and $n = p$ is an odd prime, $\overline{\iota(\mathbb{R}^2)} \cong \mathbb{T}^{p-1}$. Here $E^{\perp}$ is the 1-dimensional subspace generated by $(1, 1, \ldots, 1)$, and $\mathbb{T}^n = W \oplus W^{\perp}$ where

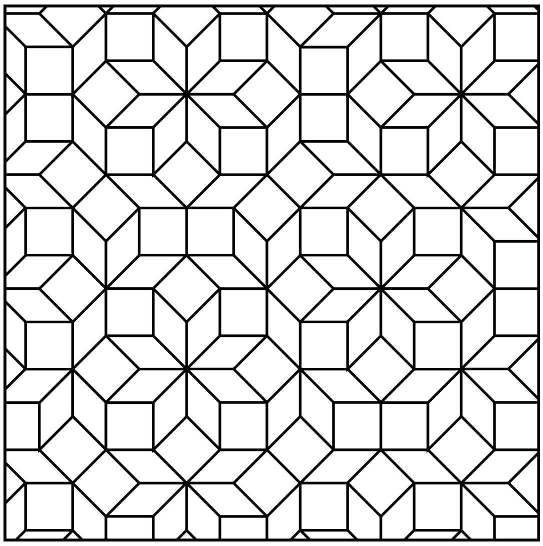

FIGURE 14. A patch of Ammann-Beenker tiling.

$W^\perp = E^\perp/\mathbb{Z}^2 \cong \mathbb{T}$. let us define $\varphi : \mathbb{T}^n \to W^\perp$ by $\varphi(\mathbf{s}) = s_1 + s_2 + \cdots + s_n$ mod 1. For $0 \le t \le 1$, we define

$$X_{n,t} = \overline{H((W + \varphi^{-1}(t)) \cap \mathbb{T}^n_0)} \subseteq X_{\mathcal{R}_n}.$$

We can now state de Bruijn's remarkable algebraic structure theorem for Penrose tilings.

THEOREM 8.6. (de Bruijn [**dB81**]) *$X_{5,0}$ is the precisely the set $F(X_{\mathcal{P}})$ of unmarked Penrose tilings.*

COROLLARY 8.7. [**Rob96a**] *The Penrose tiling dynamical system $(X_{\mathcal{P}}, T)$ has pure discrete spectrum, with an almost* 1:1 *Kronecker factor T on the* 4*-torus $\mathbb{T}^4$. The eigenvalue group Σ_p is the subgroup of $\mathbb{R}^2$ is generated by the "*5*th roots of unity" $\mathbf{v}_0, \dots, \mathbf{v}_4$.*

REMARK 8.8. In a slight abuse of notation we write $\Sigma_p = \mathbb{Z}[e^{2\pi i/5}] \subseteq \mathbb{R}^2$.

The tilings on which the factor map $P : X_{0,5} \to \mathbb{T}^5$ fails to be 1:1 are a well known special class Penrose tilings. In particular, P is 2:1 on the Penrose tilings that contain *infinite worms*, and P is 10:1 on the various *cartwheel tilings*. In particular, $P(x) = 0$ for the cartwheel tilings x centered at the origin. See [**GS87**] for the geometric description of these special Penrose tilings. Similar results hold for all other classes of quasiperiodic tilings.

The cases $X_{5,t}$ for $t \ne 0$, are called *generalized Penrose tilings*. All the generalized Penrose tiling dynamical systems have pure discrete spectrum and they all have the eigenvalue group $\Sigma = \mathbb{Z}[e^{2\pi i/5}]$, which they share with the true Penrose tilings. Thus, by the Halmos von Neumann Theorem, all the corresponding tiling dynamical systems are metrically isomorphic. However, one can also show (see [**Rob96a**], [**Le97**]) that in some cases they are not topologically conjugate. This illustrates the interesting fact that ergodic theory and topological dynamics sometimes provide different invariants when applied to tiling theory.

EXERCISE 10. Show that the point groups satisfy $H_{X_{5,t}} = D_{10}$ only if $t = 0$ or $t = 1/2$, and otherwise $H_{X_{5,t}} = D_5$.

REMARK 8.9. A *Sturmian shift dynamical system* is a symbolic dynamical system obtained by coding an irrational rotation on the circle using a partition into two intervals (see [**Ber00**]). Consider the partition η of a quasiperiodic tiling space X according to which prototile contains the origin. The factor mapping $P : X \to W$ pushes η forward into a partition on W. We can recover the tilings by seeing how this partition tiles the orbits of $\mathbb{R}^2$ in W. In this way, quasiperiodic tiling dynamical systems generalize the idea of Sturmian systems.

8.3. Quasiperiodicity and the finite type property. An interesting general question concerns the relation between quasiperiodic tilings and the finite type property. Such questions were studied primarily by researchers interested in quasicrystals. See [**Le97**] for a good survey with complete references. Here, we discuss mainly the case of generalized Penrose tilings.

THEOREM 8.10. [**Le95**] *Let* $\tau = \frac{1+\sqrt{5}}{2}$.

(1) *The tiling space* $X_{5,t}$ *is a finite type tiling space if and only if* $t \in \mathbb{Z}[\tau]$.
(2) *There exists a finite type tiling space* $X_{\mathcal{T}_\#}$ *by a marked version* $\mathcal{T}_\#$ *of the tiles* $\mathcal{R}_5$ *so that the forgetful mapping* $F : X_{\mathcal{T}_\#} \to X_{5,t}$ *is an almost 1:1 factor if and only if* $t \in \mathbb{Q}(\tau)$. *If* $t \in \mathbb{Q}(\tau)\backslash\mathbb{Z}[\tau]$ *then* F *is not a topological conjugacy; such examples are strictly sofic.*

REMARK 8.11. One should compare this with Theorem 5.22.

REMARK 8.12. In general, a necessary condition for a quasiperiodic tiling system to be finite type or sofic is that the "slope" of $E^{\|}$ must be *algebraic* (in a certain precise sense). Thus sofic examples are very special (see [**Le97**]).

REMARK 8.13. It seems to be still unknown exactly which quasiperiodic tiling systems are substitution tiling systems, and *vice versa*.

8.4. Algebraic and geometric models. The subspace $E^{\perp}$ in the definition of quasiperiodic tiling dynamical systems can be replaced by an arbitrary locally compact abelian group. In this case one obtains more general Delone sets called *model sets*. Such sets can be studied without the benefit of tilings or even dynamical systems theory. However, one can show that quite generally, the corresponding tiling dynamical systems have almost 1:1 Kronecker factors, defined on more general groups than tori. The chair tiling dynamical system can be obtained this way. (See [**BMS98**] for a discussion of this point of view.)

Alternatively, one can ask, given a tiling dynamical system (X, T), whether there is a "classical" dynamical system that is its almost 1:1 factor. We refer to such a factor as an algebraic/geometric *model* for the tiling system. For example the Penrose tilings are modeled by $\mathbb{T}^4$. Here is another example.

Let $\mathbb{G}$ be the locally compact group whose dual $\hat{\mathbb{G}}$ is the group $\mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}]$ of dyadic rational vectors in $\mathbb{R}^2$. By Halmos-von Neumann theory, there is a unique minimal Kronecker $\mathbb{R}^2$ action T on $\mathbb{G}$ with $\Sigma = \mathbb{Z}[\frac{1}{2}] \oplus \mathbb{Z}[\frac{1}{2}]$ (this Kronecker system is called the $\mathbb{R}^2$-*adding machine* in [**Rob99**]).

THEOREM 8.14. [**Rob99**] *The chair tiling dynamical system* (X_{S_c}, T) *has* $(\mathbb{G}, T)$ *as an almost 1:1 factor. Thus it has pure discrete spectrum with* $\Sigma = \hat{\mathbb{G}}$. *The table* *tiling dynamical system* (X_{S_t}, T) *has* $(\mathbb{G}, T)$ *as an almost 4:1 factor. The table system has precisely the same eigenvalues* Σ *as the chair, and thus is not weakly mixing. However, the table tiling system also does not have pure discrete spectrum; it has* mixed spectrum.

Remark 8.15. The spectral properties of the table and chair were first computed in [**Sol97**], where it was observed that the table system is similar to the well known discrete *Morse sequence* substitution dynamical system (see [**Que87**]).

In the chair (or the table), the points where the factor mapping P fails either to be 1:1 (or 4:1) can be completely classified (see [**Rob99**]). In particular, chair tilings can have worms and cartwheels very similar to the ones that occur in Penrose tilings. The proofs in [**Rob99**] are completely general and show that similar results hold for all polyomino substitutions. In particular, there is an easy criterion for pure discrete spectrum. This turns out to be equivalent to the idea of "coincidence" in the theory of substitutions, and to the idea of "synchronizing" or "magic" words in the theory of discrete 1-dimensional finite type shifts (see [**Rob99**]).

Remark 8.16. Solomyak [**Sol97**] defines a more general notion of coincidence, and uses it to prove that some examples (e.g., chair tiling systems) have pure discrete spectrum.

Models based on Kronecker systems won't work for weakly mixing examples because weak mixing implies the absence of nontrivial eigenvalues. Thus we ask, what could a model for a weakly mixing tiling dynamical system possibly look like?

8.5. The dynamics of the substitution map and Markov partitions. Before describing an example of a model for a weakly mixing tiling dynamical system, we consider the dynamical properties of the action of an invertible substitution mapping S acting on a tiling space X_S.

Theorem 8.17. [**Rob96a**] *The Penrose substitution* S *on the set* $X_{\mathcal{P}}$ *of Penrose tilings has the hyperbolic toral automorphism*

$$A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & -1 \\ 1 & 0 & -1 & -1 \end{pmatrix}$$

acting on $\mathbb{T}^4$ *as is an almost 1:1 factor.*

Similarly, one can show that the chair tiling dynamical system has a "hyperbolic" automorphism of $\mathbb{G}$ as an almost 1:1 factor. These examples illustrate that tiling substitutions S tend to be *hyperbolic*.

In [**AP98**] it is shown that the action S of a tiling substitution on X_S (i.e., the dynamical system (X_S, S)) is always a kind of generalized hyperbolic system called a *Smale space*. If the substitution S is based on a perfect decomposition, then then the tilings induce a partition on X_S (according to which prototile occurs at the origin). The perfect decomposition property implies this partition is a *Markov partition* (see [**Bow78**]). Conversely, if a Smale space has a Markov partition whose partition elements are connected, then the partition induces tilings on the stable manifolds, and these tilings are self-affine.

Connectedness is needed because our definition of tilings requires connected tiles. Unfortunately, it is unknown whether every hyperbolic toral automorphism has a Markov partition with connected partition elements. However, some partial results appear in [**FI98**].

REMARK 8.18. It is known that for all Markov partitions for hyperbolic toral automorphisms of $\mathbb{T}^3$ (see [**Bow78**]) and for typical hyperbolic toral automorphisms of $\mathbb{T}^n$, $n > 3$, (see [**Caw91**]) the boundaries of partition elements must be fractal. This implies that self-affine quasiperiodic tilings satisfying a perfect decomposition will almost always have tiles with fractal boundaries.

8.6. A geometric model for a weakly mixing system. The example discussed in this section is based on a kind of hyperbolic dynamical system J called a *pseudo-Anosov diffeomorphism* (see [**FS79**]). In this example, J is defined on a surface M of genus 2. Pseudo-Anosov diffeomorphisms always have Markov partitions, and we obtain self-similar tilings, as in the previous section, by intersecting the Markov elements partition with the unstable manifolds for J. Since for almost every point $m \in M$, the stable manifold through m is homeomorphic to $\mathbb{R}$, this example consists of 1-dimensional tilings. See [**Fit98**] or [**FHR00**] for details.

Consider the discrete substitution σ given by

$$\begin{aligned} 1 &\rightarrow 1424 \\ 2 &\rightarrow 142424 \\ 3 &\rightarrow 14334 \\ 4 &\rightarrow 1434. \end{aligned} \tag{8.2}$$

The structure matrix for this substitution is

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 2 & 3 & 2 & 2 \end{pmatrix},$$

which has the non-Pisot Perron-Frobenius eigenvalue $\lambda = \frac{1}{4}(7+\sqrt{5}+\sqrt{2}\sqrt{19+7\sqrt{5}})$. Following example 5.14, we construct the corresponding 1-dimensional tiling substitution S with expansion λ. It follows from Theorem 7.6 that (X_S, T) is weakly mixing.

THEOREM 8.19. [**FHR00**] *Let X_S be the tiling space corresponding to the tiling substitution S in described above. Then (X_S, S) has an almost* 1:1 *factor that is a pseudo-Anosov diffeomorphism (M, J) on a surface M of genus* 2.

The corresponding tiling dynamical system (X_S, T) is metrically isomorphic a unit speed flow along the unstable manifolds of (M, J). Since not all the stable manifolds are homeomorphic to $\mathbb{R}$, this flow is not defined everywhere on M. However, after removing a "singular set" of measure zero, one can show that this flow is metrically isomorphic to a suspension of a self-inducing interval exchange transformation of four intervals (see [**FHR00**]).

REMARK 8.20. One can obtain a 2-dimensional weakly mixing example by taking the Cartesian square. The geometric model for such an example is a 4-manifold. By Theorem 5.22, this 2-dimensional tiling dynamical system has the property that it is an almost 1:1 factor of a weakly mixing finite type tiling system.

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