Cartan's method of equivalence
and its applications in geometry of differential equations

Oleg I. Morozov
Institute of Mathematics and Statistics
University of Tromsø
Norway

Abstract:

The theory of continuous transformation groups was founded by Sophus Lie and greatly influenced on diverse branches of mathematics and physics. One of Lie's greatest contributions was the discovery of the connection between continuous transformation groups and algebras of their infinitesimal generators. Lie had originally used the theory of transformation groups to provide the most widely applicable technique to find closed form solutions of ordinary differential equations and to recognize many diverse and ad hoc integration methods for solving special cases of ODEs to be manifestations of the infinitesimal transformation method. Applied to partial differential equations, the method leads to group-invariant solutions, conservation laws and other techniques of great significance in studying physically important systems of differential equations in classical mechanics, fluid dynamics, relativity, elasticity, optics and many other areas.

The application of Lie's infinitesimal method to concrete systems of differential equations requires an analysis and integration of over-determined defining systems for symmetry algebras. Even the writing out of symmetry generators for a modest system involves big amount of tedious calculations. Computer algebra systems are useful in their simplification. Many symbolic packages can find the defining equations of the Lie symmetry algebra. The most sophisticated packages can then reduce the determining system into an equivalent but more suitable form, and solve it using heuristics in integration.

An alternative methodology of studying transformation groups was developed by Elie Cartan. His theory's main tool is the possibility to characterize the transformations by means of a set of invariant differential 1-forms. Calculating their exterior differentials and expressing these differentials in terms of the original 1-forms yields the Cartan structure equations of the transformation group. These equations contain all information about the group, so finding the Cartan structure equations is the same problem as to describe the group. An important advantage of Cartan's approach is that it does not require integration and uses only linear algebra operations and differentiation.

The mini-course presents the technique of Cartan's method and is focused on its applications to symmetry groups of differential equations. We will discuss the main constructions of the method and illustrate them on examples of calculation of invariant 1-forms and structure equations for ordinary and partial differential equations.