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Ordinary differential equations and symmetry-based methods of solving them

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Introduction

Let us consider a scalar ordinary differential equation (ODE to abbreviate)

$$\frac{d\,y)}{d\,x} = f\left[\,x, y(x)\right]. \tag{0.0.1}$$

A large amount of mathematical textbooks starts from the setting under what condition the Cauchy problem

$$\frac{d\,y(x)}{d\,x} = f\left[\,x, y(x)\right], \qquad y(x_0) = y_0$$

has a solution, and when does such solution is unique.

The main goal of these lectures is different. We rather assume from the very beginning that the function $f(x,\, y)$ is continuous or even more regular, such that solution does exist. Of interest is thus not the existence of solutions but rather the ways of obtaining them.

Let us accept the following definition.

Definition 0.0.1. *We say that Eq. (0.0.1) is integrable if we are able to deliver a procedure enabling to get the general solution in explicit or implicit form.*

Example 0.0.1. It is evident that implicit solution to the equation

$$\frac{d\,y(x)}{d\,x} = \frac{P(x)}{Q(y)}, \tag{0.0.2}$$

where $P(x)$, $Q(y)$ are continuous functions, is given by the expression

$$\int Q(y)\,d\,y - \int P(x)\,d\,x = C.$$

Remark 0.0.1. Equation (0.0.2) is called a separable differential equation

From the standard course of ODEs you should know that there is a number of equations which can be made separable by the changes of variables. Let us give examples of such equations.

1. A *homogeneous equation*

$$\frac{d\,y}{d\,x} = f\left(\frac{y}{x}\right). \tag{0.0.3}$$

Using the change of variables $z = \frac{y}{x}$ and treating z as a new dependent variables, we get the following separable equation:

$$\frac{d\,z}{d\,x} = \frac{f(z) - z}{x}.$$

2. Equation

$$\frac{d\,y}{d\,x} = x^{k-1}\,f\left(\frac{y}{x^k}\right). \tag{0.0.4}$$

is reduced to the separable differential equation by the ansatz $z = y\,x^{-k}$. Inserting it to the source euqation we get, after the differentiation and some algeabraic manipulation the following equation:

$$x\frac{d\,z}{d\,x} = f(z) - k\,z.$$

3. Equation

$$\frac{d\,y}{d\,x} = f(\alpha\,x + \beta\,y) \tag{0.0.5}$$

is reduced to the separable equation by means of the ansatz $z(x) = \alpha\,x + \beta\,y(x)$. Indeed, differentiating the function z and using the source equation we get:

$$\frac{d\,z}{d\,x} = F(z) = \alpha + f(z).$$

A natural question arises: do these examples have some features distinguishing them from a generic case? The answer is positive, but to describe the features in question we should introduce an extra definition.

Let us consider a one-parameter family of diffeomorphisms:

$$\bar{x} = \varphi(x,\,y;\;a), \qquad \bar{y} = \psi(x,\,y;\;a), \qquad (x,\,y)\,\in\,U\,\in\,\mathbb{R}^2, \quad a\,\in\,(-\varepsilon,\,\varepsilon). \tag{0.0.6}$$

Definition 0.0.2. *We say that the equation (0.0.1) admits the transformations (0.0.6) if it maintains its form in new variables, in other words, if relations*

$$\frac{d\,y}{d\,x} = f\,[x,\,y] \;\;\Leftrightarrow\;\; \frac{d\,\bar{y}}{d\,\bar{x}} = f\,[\bar{x},\,\bar{y}]$$

take place (we also say in this case that (0.0.1) has the symmetry or is invariant w.r.t the transformations (0.0.6)).

Lemma 0.0.1. *1. Eq. (0.0.3) admits the change of variables*

$$\bar{x} = e^a\,x, \qquad \bar{y} = e^a\,y, \qquad a\,\in\,\mathbb{R}^1.$$

2. Eq. (0.0.4) admits the change of variables

$$\bar{x} = e^a\,x, \qquad \bar{y} = e^{k\,a}\,y, \qquad a\,\in\,\mathbb{R}^1.$$

3. Eq. (0.0.5) admits the change of variables

$$\bar{x} = x + a, \qquad \bar{y} = y - a\,\frac{\alpha}{\beta}, \quad a\,\in\,\mathbb{R}^1, \quad \beta \neq 0\,.$$

Now let us consider the second-order equation

$$y'' = F(x,\,y,\,y'). \tag{0.0.7}$$

The usual way of integrating this equation is to reduce it to the first order ODE, using some change of variables. Below we deliver a couple of examples illustrating when it is possible.

1. Equation

$$\frac{d^2\,y}{d\,x^2} = f\,(x,\,y')\,. \tag{0.0.8}$$

is reduced to the first-order ODE

$$\frac{d\,w}{d\,x} = f(x,\,w)$$

by means of the substitution $y' = w$.

2. Equation

$$y'' = F\left(y,\, y'\right) \tag{0.0.9}$$

is reduced to the first order ODE

$$w\,\frac{d\,w}{d\,y} = F(y,\, w)$$

by means of the substitution $y' = w\,[y(x)]$.

In all the second order ODEs, for which the lowering of the order is possible, we notice the existence of some transformations retaining the form of these equations. Below we repeat almost word for word the definition of symmetry formulated for the first order ODE:

Definition 0.0.3. *We say that Eq. (0.0.7) admits the transformations (0.0.6) if, being written in the new variables, it maintain its form, in other words, if the following relation of equivalence takes place:*

$$y'' = F(x,\, y,\, y') \;\; \Leftrightarrow \text{and} \;\; \bar{y}'' = F(\bar{x},\, \bar{y},\, \bar{y}')$$

Now it is easy to show that the following statement holds true.

Lemma 0.0.2. *1. Eq. (0.0.8) admits the change of variables*

$$\bar{x} = x, \qquad \bar{y} = y + a.$$

2. Eq. (0.0.9) admits the change of variables

$$\bar{x} = x + a, \qquad \bar{y} = y.$$

In order to be completely integrable, the equation of the form (0.0.8) should possess an extra symmetry, which imposes restrictions on the function $F(x,\, y')$. For example, if $F(x,\, y') = f(y'/x)$, i.e. the equation takes the form

$$\frac{d^2\,y}{d\,x^2} = f\left(\frac{y'}{x}\right), \tag{0.0.10}$$

then the following statement holds

Lemma. Eq. (0.0.10) admits an extra transformation

$$\bar{y} = e^{2\,a}\,y, \qquad \bar{x} = e^{a}\,x. \tag{0.0.11}$$

We can integrate Eq. (0.0.10) using the following procedure. First we employ the ansatz $z = w/x$, and this enables to obtain the separable equation

$$\frac{d\,z}{d\,x} = \frac{f(z) - z}{x}.$$

Integrating this equation we can get the following result:

$$x = C_1\,\phi(z),$$

where

$$\phi(z) = \exp\left[\int \frac{d\,z}{f(z) - z}\right].$$

Inverting the above formula, we obtain:

$$z = \frac{w}{x} = \phi^{-1}\left[\frac{x}{C_1}\right].$$

Returning to the initial variables, we obtain the separable equation

$$\frac{d\,y}{d\,x} = x\,\phi^{-1}\left[\frac{x}{C_1}\right].$$

So the general solution to the equation (0.0.10) can be expressed in the form of quadrature

$$y = \int x\,\phi^{-1}\left[\frac{x}{C_1}\right]\,d\,x + C_2.$$

The connections of the integrability (and the possibility of lowering order of higher-order ODE) with invariant transformation properties of ODEs to be invariant under some transformations were noticed for the first time by the outstanding Norwegian mathematician Sophus Lie at the end of XIX-th sentury. He was familiar with the Galois' theory giving the answer to the milestone problem on when it is possible to express the roots of the $n-th$ order algebraic equation

$$z^n + a_{n-1}\,z^{n-1} + ... + a_0 = 0,$$

in terms of its coefficients. Basing on the group of permutaions, Galois explained why it is possible to express the roots in therms of the polynomial equations of the order 2, 3 an 4 and is impossible for the polynomial of the higher order. S.Lie wanted to make a classification of all integrable ODEs, and for this purpose he put forward a concept of continuous groups, called afterward the Lie groups. To these special groups an application of calculus is possible, which gives a very powerful tools for investigations of the integrability of ODEs. Maybe the most brilliant idea of S. Lie is a treatment of ODEs as some "surfaces" in the extended space or sa called *jet spaces*. As an example consider the space $(x,\,y,\,y') \in \mathbb{R}^3$. In this space the differential equation

$$x^2 + y^2 + [y']^2 = \rho^2, \quad \rho \in \mathbb{R}$$

can be treated as a sphere. Standing on this position, S. Lie observed that there is a chances to integrate and ODE if the "surface" defined by this equation (or system of equations) possesses *sufficiently large symmetry*, i.e. admits a family of diffeomorphisms of the form (0.0.6), mapping this surface into itself.

Examples of the surfaces possessing the symmetry.

1. The spherical surface can be rotated with respect to any axis going through its center

2. The infinite cylindrical surface can be rotated with respect to the axis of symmetry and shift along this axis

3. The infinite conical surface admits the rotations with respect to to the axis of symmetry.

Sophus Lie not only revealed connections between the methods of solving DE (ones known at that time) and symmetry properties of the DE, but proposed an algorithmic method of solving them, basing on the above concept of symmetry.

The plan of the course is following:

a) We discuss the concept of the continuous (Lie) group of transformations acting on Euclidean space $\mathbb{R}^n$.

b) We introduce the notion of the Lie group of transformations and the concept of symmetry of some geometric and analytical objects (functions and surfaces).

c) Following Lie's main idea, we apply all the concepts from the previous items to ODEs, prolonging the group action defined on the set of dependent and independent variables $(x,\, y)$ onto the set of finite order derivatives $y', y'', ...y^{(n)}$. After that we are able to define the symmetry of ODEs in a very natural way, treating them as surfaces defined on the jet space $\left(x,\, y,\, y', ...y^{(n)}\right)$.

d) Next come the applications. Symmetry of ODEs enable us to obtain the particular solutions to nonlinear ODEs, to disseminate already known solutions, to distinguish, e.g., all completely integrable equations of the second order (a lot of another applications are known but we hardly have time to stop on them).

Exercises.

Exercise 0.0.1. Show that Eq.

$$u\,\frac{d^2\,u}{d\,x^2}-\lambda\,\left[\frac{d\,u}{d\,x}\right]^2=0$$

admits the transformation

$$\bar{x}=e^a\,x,\qquad \bar{u}=e^b\,u,\quad a,\,b\,\in\,\mathbb{R}.$$

Solution.

$$\frac{d\bar{u}}{d\bar{x}}=\frac{d\,e^b\,u}{d\,x}\frac{d\,x}{d\,\bar{x}}=e^{b-a}\frac{d\,u}{d\,x};$$

$$\frac{d^2\bar{u}}{d\bar{x}^2}=\frac{d\,x}{d\,\bar{x}}\,\frac{d}{d\,x}e^{b-a}\frac{d\,u}{d\,\bar{x}}=e^{b-2\,a}\frac{d^2\,u}{d\,x^2}.$$

Hence

$$\bar{u}\,\frac{d^2\,\bar{u}}{d\,\bar{x}^2}-\lambda\,\left[\frac{d\,\bar{u}}{d\,\bar{x}}\right]^2=e^{2(b-\,a)}\left[u\,\frac{d^2\,u}{d\,x^2}-\lambda\,\left[\frac{d\,u}{d\,x}\right]^2\right]=0.$$

Exercise 0.0.2. Show that Eq.

$$x^2\,\frac{d^2\,u}{d\,x^2}-F\left(x\,\frac{d\,u}{d\,x}-u\right)=0$$

admits the transformation

$$\bar{x}=\lambda\,x,\qquad \bar{u}=u+\lambda\,x,\quad \lambda\,\in\,\mathbb{R}.$$

Solution.

$$\frac{d\bar{u}}{d\bar{x}}=\frac{d}{d\,x}\left(\frac{du}{dx}+\lambda\right)\,\frac{d\,x}{d\bar{x}}=\lambda^{-1}\,\left(\frac{du}{dx}+\lambda\right);$$

$$\frac{d^2\bar{u}}{d\bar{x}^2}=\frac{d}{d\,x}\,\lambda^{-1}\,\left(\frac{du}{dx}+\lambda\right)\,\frac{d\,x}{d\bar{x}}=\lambda^{-2}\,\frac{d^2\,u}{d\,x^2}.$$

Hence

$$x^2\,\frac{d^2\,u}{d\,x^2}-F\left(x\,\frac{d\,u}{d\,x}-u\right)=(\lambda\,x)^2\,\frac{d^2\,u}{d\,x^2}\lambda^{-2}-F\left[\lambda\,x\left(\frac{du}{dx}+\lambda\right)\,\lambda^{-1}-u-\lambda\,x\right]=$$
$$=x^2\,\frac{d^2\,u}{d\,x^2}-F\left(x\,\frac{d\,u}{d\,x}-u\right)=0.$$

Exercise 0.0.3. Show that Eq.

$$\frac{d\,y}{d\,x}=\frac{y}{x(y+\log x)}$$

admits transformations

$$\bar{x}=x\,e^{a\,y},\qquad \bar{y}=y.$$

Solution.

$$\frac{d\,\bar{y}}{d\,\bar{x}}=\frac{d\,y}{d\,(x\,e^{a\,y})}=\frac{d\,y}{e^{a\,y}\,d\,x+a\,x\,e^{a\,y}\,\frac{d\,y}{d\,x}\,d\,x}=\frac{\frac{d\,y}{d\,x}}{e^{a\,y}\left(1+a\,x\frac{d\,y}{d\,x}\right)}=$$
$$\frac{\frac{y}{x\,(y+\log(x))}}{e^{a\,y}\left[1+a\,x\,\frac{y}{x\,(y+\log(x))}\right]}=\frac{y}{e^{a\,y}\,[x(y+\log(x))+a\,y]}.$$

On the other hand,

$$\frac{\bar{y}}{\bar{x}\,[\bar{y}+\log(\bar{x})]}=\frac{y}{xe^{a\,y}\,[y+a\,y+\log(x)]}=\frac{y}{e^{a\,y}\,[x(y+\log(x))+a\,y]},$$

which proves the statement.

Chapter 1

Lie groups and Lie Algebras

1.1 A concept of a group

Definition 1.1.1. *A group is a set G, together with a group operation $\circ$, such that:*

a)

$$\forall a, b \in G \quad a \circ b \in G \tag{1.1.1}$$

b) (associativity)

$$\forall a, b, c \in G \quad (a \circ b) \circ c = a \circ (b \circ c) \tag{1.1.2}$$

c) There is a distinguished element $e \in G$, called the unit element, which has the property that:

$$\forall a \in G \quad a \circ e = e \circ a = a \tag{1.1.3}$$

d) (inverse element)

$$\forall a \in G \quad \exists a^{-1} \in G, \quad such \quad that \quad a \circ a^{-1} = a^{-1} \circ a = e \tag{1.1.4}$$

Example 1.1.1. The following structures are groups:

1. $G = \mathbb{R} \backslash \{0\}, \circ$ is multiplication
2. $GL(n, \mathbb{R})$, a set of invertible $n \times n$ matrices with $G_{ij} \in \mathbb{R}$ together with the matrice's multiplication $\circ$
3. $O(n, \mathbb{R}) = \{X \in GL(n, \mathbb{R}) : X^T \circ X = I\}$, a subset of orthogonal matrices

Exercise 1.1.1. Show that $O(n, \mathbb{R})$ is closed with respect to $\circ$.

Proof.

$$(X \circ Y)^T (X \circ Y) = Y^T \circ X^T \circ X \circ Y = Y^T \circ (X^T \circ X) \circ Y = Y^T \circ I \circ Y = Y^T \circ Y = I$$

□

1.2 A concept of a local one-parameter group of transformations

Let $U \subset \mathbb{R}^n$ be an open set and $\{T_a\}_{a\in\Delta\subset\mathbb{R}^1}$ be a one-parameter family of transformations $T_a : U \longmapsto \mathbb{R}^n$, given by the formula

$$(T_a x)^i = f^i(x_1, ..., x_n; a), \quad x = (x_1, ..., x_n), \quad i = 1, ..., n,$$

such that f^i is three times differentiable w.r.t x_k variables and $f \in C^\infty$ w.r.t. the parameter a. We assume in addition that $\{T_a\}$ is locally closed in the following sense:

- There exists an open, non-empty set $\Delta' \subset \Delta$, such that $\forall x \in U, \forall a, b \in \Delta'$, there exists a number $c \in \Delta$, such that

$$T_b \circ T_a(x) \equiv T_b(T_a(x)) = T_c(x) \tag{1.2.1}$$

- and there exists a function $\phi : \Delta' \times \Delta' \longmapsto \Delta$, such that $\phi(a, b) = c$ or

$$T_b(T_a(x)) = T_\phi(a, b)(x). \tag{1.2.2}$$

Definition 1.2.1. $\{T_a\}_{a\in\Delta}$ *is called a local 1-parameter group of transformations if*

1. *there exists a unit element* $e \in \Delta'$*, such that* T_e *is an identity transformation (*$\forall x \in U \quad T_e(x) = x$*), that is*

$$\forall a \in \Delta' \quad \phi(a, e) = \phi(e, a) = a \tag{1.2.3}$$

   or, equivalently,

$$T_a \circ T_e(x) = T_e \circ T_a(x) = T_a(x). \tag{1.2.4}$$

2. $\phi(\cdot, \cdot) \in C^3(\Delta' \times \Delta')$

3. $\forall a \in \Delta'$ *the equation* $\phi(a, b) = e$ *has the unique solution* $b \in \Delta'$ *and so is for* $\phi(b, a) = e$

Remark 1.2.1. If $\phi(a, b) = \phi(b, a)$ we say that $\{T_a\}_{a\in\Delta}$ is commutative (or Abelian) local group of transformations. For some reasons (which will be explained later on) one can assume in this case, that $\phi(a, b) = a + b$ and $e = 0$. We say then, that a is a *canonical parameter*.

Theorem 1.2.1. *There always exists a change of variable* $\bar{a} = f(a)$*, such that* $\bar{a}$ *is a canonical parameter*

Proof. Let the superposition $T_c = T_b \circ T_a$ be defined by the function $c = \phi(a, b)$.
If we take the second argument with the increment Δb, such that $|\Delta b| \ll 1$, then on virtue of smoothness of ϕ, the parameter c will increase by a small increment Δc.

$$c + \Delta c = \phi(a, b + \Delta b) \tag{1.2.5}$$

In terms of transformations we'll have what follows:

$$T_{c+\Delta c} = T_{b+\Delta b} \circ T_a. \tag{1.2.6}$$

Multiplying by $T_a^{-1} \circ T_b^{-1} = T_c^{-1} \equiv T_{c^{-1}}$ from the right, we get:

$$T_{b+\Delta b} \circ (T_a \circ T_a^{-1}) \circ T_b^{-1} = T_{b+\Delta b} \circ T_b^{-1} = T_{c+\Delta c} \circ T_{c^{-1}}.$$

In terms of function ϕ we'll have what follows:

$$\phi(c^{-1}, c+\Delta c) = \phi(b^{-1}, b+\Delta b) \tag{1.2.7}$$

Let us introduce a function

$$V(b) = \frac{\partial \phi(a,b)}{\partial b}\,|_{a=b^{-1}} \tag{1.2.8}$$

From the Taylor's series decomposition, we have:

$$\phi(b^{-1}, b+\Delta b) = \phi(b^{-1}, b) + \frac{\partial \phi(a,b)}{\partial b}\,|_{a=b^{-1}}\,\Delta b + O(\Delta b^2) \tag{1.2.9}$$

Smoothness of ϕ implies that $|\Delta c|$ and $|\Delta b|$ are of the same order. Hence:

$$\phi(c^{-1}, c+\Delta c) = \phi(c^{-1}, c) + \frac{\partial \phi(a,c)}{\partial c}\,|_{a=c^{-1}}\,\Delta c + O(\Delta c^2) \tag{1.2.10}$$

From (1.2.7),(1.2.9) and (1.2.10) we have:

$$V(c)\Delta c = V(b)\Delta b + O(|\Delta b|^2)$$

Dividing this expression by Δb and taking the limit $\Delta b \to 0$, we obtain:

$$V(c)\frac{dc}{db} = V(b) \qquad c = \phi(a,b); \quad \phi(a,b)|_{b=e} = a \tag{1.2.11}$$

Now we introduce a function:

$$\bar{a} = \int_e^a V(s)\mathrm{d}s \tag{1.2.12}$$

Integrating (1.2.11) within the interval (e, b) we get for the l.h.s.:

$$\int_e^b V(\phi(a,s))\frac{\partial \phi(a,s)}{\partial s}\mathrm{d}s = \int_e^b V(\phi(a,s))\mathrm{d}\phi(a,\ s) = \int_a^c V(\sigma)\mathrm{d}\sigma$$

while for the r.h.s we have the integral $\int_e^b V(s)\mathrm{d}s$. Equating these integrals and taking into account the identity

$$\int_e^c () = \int_e^a () + \int_a^c (),$$

we get:

$$\bar{c} = \int_e^c V(s)\mathrm{d}s = \int_e^a V(s)\mathrm{d}s + \int_a^c V(s)\mathrm{d}s = \int_e^a V(s)\mathrm{d}s + \int_e^b V(s)\mathrm{d}s \equiv \bar{a} + \bar{b}.$$

□

Example 1.2.1. **Scaling group**

$\bar{x} = ax = T_a x \Rightarrow \bar{\bar{x}} = T_b(T_a x) = b\bar{x} = bax$, $\phi(a,b) = ab$ and $e = 1$.
A passage to canonical parameter:

$$V(b) = \frac{\partial}{\partial b}\phi(a,b)|_{a=b^{-1}=\frac{1}{b}} = \frac{\partial}{\partial b}ab|_{a=\frac{1}{b}} = \frac{1}{b}$$

$$\bar{a} = \int_1^a \frac{ds}{s} = \ln a - \ln 1 = \ln a \quad \Rightarrow \quad a = e^{\bar{a}}$$

$$\bar{x} = T_{\bar{a}}x = e^{\bar{a}}x \qquad \bar{\bar{x}} = T_{\bar{b}}\bar{x} = e^{\bar{a}+\bar{b}} = T_{\bar{c}}x \quad \Rightarrow \quad \bar{c} = \bar{a} + \bar{b}$$

Exercise 1.2.1. **Rotation group**

$$\begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix} = \begin{pmatrix} x\cos a - y\sin a \\ x\sin a + y\cos a \end{pmatrix}$$

Find out $c = \phi(a, b)$ and show that these transformations form a group.

Example 1.2.2. The group $\{T_a\}: \ (0, \rho) \times \mathbb{R}^1 \to \mathbb{R}^2$, $\rho > 0$, is given by the formula

$$T_a(x, y) = (\bar{x}, \bar{y}) = (\frac{x}{1-ax}, \frac{y}{1-ax}).$$

Correspondingly, the composition $T_b \circ T_a$ acts as follows:

$$\bar{\bar{x}} = \frac{\bar{x}}{1-b\bar{x}} = \frac{\frac{x}{1-ax}}{1-b\frac{x}{1-ax}} = \frac{x}{1-(a+b)x}$$

$$\bar{\bar{y}} = \frac{\bar{y}}{1-b\bar{x}} = \frac{\frac{y}{1-ax}}{1-b\frac{x}{1-ax}} = \frac{y}{1-(a+b)x}.$$

Note that a natural set for the parameters' values is $\Delta = (-\infty, 1/\rho)$ and, of course, a, $b > 0$ should be taken from such set $\Delta' \subset \Delta$ that $a + b$ does not exceed $1/\rho$. So this local group cannot be extended to a global one.

1.3 Infinitesimal generator of a local Lie group. The first fundamental Lie's theorem

Let

$$\bar{x}^k = f^k(x; a) \tag{1.3.1}$$

be a local Lie group and let a be the canonical parameter.

Definition 1.3.1. *A. The function:*

$$\xi^k(x) = \frac{\partial f^k(x; a)}{\partial a}|_{a=0} \qquad k = 1, ...n \tag{1.3.2}$$

is called the k-th coordinate of the infinitesimal generator (the **IFG** *to abbreviate) of the group* (1.3.1)*.*

B. The first order operator

$$\hat{X} = \sum_{k=1}^{n} \xi^k(x) \frac{\partial}{\partial x^k} \tag{1.3.3}$$

*is called the Lie group infinitesimal generator (***IFG***).*

We postpone with the explanation on why the operator $\hat{X}$ is called so. Yet the notion *"infinitesimal"* will be explained at once. In fact, for $|a| << 1$ the equation (1.3.1) can be written down in the following form:

$$\bar{x}^k = x^k + \xi^k(x)a + O(a^2).$$

So the vector $\xi(x) = (\xi^1(x), ..., \xi^n(x))$ does generate infinitesimal (very small) Lie group transformations.

Examples of the IFG.

Example 1.3.1. A. If $\bar{x} = x + a, \qquad \bar{y} = \frac{x\,y}{x+a}$, then, using the definition, we easily get

$$\hat{X} = \frac{\partial}{\partial\, x} - \frac{y}{x}\frac{\partial}{\partial\, y}$$

B. For the rotation group
$\bar{x} = \cos a\, x - \sin a\, y, \quad \bar{y} = \sin a\, x + \cos a\, y$ we get

$$\hat{X} = -y\frac{\partial}{\partial\, x} + x\,\frac{\partial}{\partial\, y}.$$

C. For the projective group $\bar{x} = \frac{x}{1-a\,x}, \qquad \bar{y} = \frac{y}{1-a\,x}$ we obtain

$$\hat{X} = x^2\,\frac{\partial}{\partial\, x} + yx\,\frac{\partial}{\partial\, y}.$$

Theorem 1.3.1. ***(the first fundamental Lie's theorem)***
The functions $\{f^i(x;a)\}_{i=1}^n$ *satisfy the initial value problem*

$$\frac{\partial f^i(x;a)}{\partial a} = \xi^i(f(x;a)) \qquad f^i(x;0) = x^i, \quad i = 1, ...n. \tag{1.3.4}$$

Conversely, for any smooth vector field $\{\xi^i(x)\}_{k=1}^n$ *the initial value problem (1.3.4) defines a local one-parameter group for which* $\{\xi^i(x)\}_{i=1}^n\}$ *is a set of coordinates of infinitesimal generator.*

Proof. ($\Rightarrow$)
Using the fact, that $T_{a+\Delta a} = T_a \circ T_{\Delta a}$ we have:

$$f^i(x; a + \Delta a) = f^i(f(x;a); \Delta a) \tag{1.3.5}$$

Now for a small Δa we have two Taylor decompositions:

$$f^i(x; a + \Delta a) = f^i(x;a) + \frac{\partial f^i(x;a)}{\partial a}\Delta a + O(\Delta a^2) \tag{1.3.6}$$

$$\begin{aligned} f^i(f(x;a); \Delta a) &= f^i(f(x;a); 0) + \frac{\partial f^i(f(x;a);\Delta a)}{\partial \Delta a}|_{\Delta a=0}\Delta a + O(\Delta a^2) = \\ &= f^i(f(x;a)) + \xi^i\,(f(x;a))\,\Delta a + O(\Delta a^2). \end{aligned} \tag{1.3.7}$$

We equalize (1.3.6) and (1.3.7), divide both sides of the equation obtained this way by Δa and then take the limit directing Δ toward zero. Finally, we get:

$$\frac{\partial f^i(x;a)}{\partial a} = \xi^i(f(x;a))$$

with the additional condition $f^i(x;a)|_{a=0} = x^i$ □

Proof. ($\Leftarrow$)
We know[1], that for any set of smooth functions $\xi^k(x)$ the system

$$\frac{\mathrm{d}\bar{x}^k}{\mathrm{d}a} = \xi^k(\bar{x}) \qquad \bar{x}^k(0) = x^k \quad k = 1, ...n \tag{1.3.8}$$

[1]see, e.g., Ph. Hartman, *Ordinary Differential Equations, Ch. II,* John Wiley and Sons, New York, 1964

has a local solution which can be designated as $\bar{x}(a) = f(x;a) = T_a\,x$. So the statement will be proved if we show that $T_b(T_a x) = T_{a+b}x$. Let us introduce the following functions:

$$y^i(b) := f^i(f(x;a);b), \qquad z^i(b) := f^i(x;a+b)$$

Differentiating these functions and using the initial conditions we will have:

$$\frac{\partial y^i}{\partial b} = \frac{\partial f^i(f;b)}{\partial b} = \xi^i(y) \qquad y^i(0) = f^i(f;0) = f^i(x;a)$$

and

$$\frac{\partial z^i}{\partial b} = \frac{\partial f^i(x;a+b)}{\partial b} = \frac{\partial f^i(x;\tau)}{\partial \tau}|_{\tau=a+b} = \xi^i(f^i(x;\tau))|_{\tau=a+b} = \xi^i(z)$$

$$z^i(0) = f^i(f;a+0) = f^i(x;a)$$

So, up to the notation, both of the functions satisfy the same initial value problem and, therefore, on virtue of the uniqueness of the solution to the i.v.p., $T_b \circ T_a = T_{a+b}$. The statement about the coordinates of the **IFG** appears from the fact that the functions $f^i(x;\, a)$ satisfy (1.3.8). ☐

Example 1.3.2. (reconstructing the transformations $\bar{x}$,$\bar{y}$ from the coordinates of infinitesimal generator)

Let the coordinates of the **IFG** for the family of transformations $T_a : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ are as follows:

$$\xi^1 = 1 \qquad \xi^2 = -\frac{y}{x}.$$

Then solving the equation

$$\frac{\mathrm{d}\bar{x}}{\mathrm{d}a} = 1 \qquad \bar{x}(0) = x$$

we obtain that $\bar{x} = C + a$ with $C = x$. so that $\bar{x} = x + a$. Next we solve the equation

$$\frac{\mathrm{d}\bar{y}}{\mathrm{d}a} = -\frac{\bar{y}}{\bar{x}} = -\frac{\bar{y}}{x+a} \qquad \bar{y}(0) = y.$$

Separating variables, next integrating and taking into account the initial conditions, we finally get $\bar{y} = \frac{xy}{x+a}$.

In order to show that this is a group, we consider the superposition of transformations:

$\bar{\bar{x}} = \bar{x} + b = x + (a+b) = x + \phi(a,b)$

$\bar{\bar{x}} = \frac{\bar{x}\bar{y}}{\bar{x}+b} = \frac{(x+a)\frac{xy}{x+a}}{x+a+b} = \frac{xy}{x+\phi(a,b)}$.

Exercise 1.3.1. Find the one-parameter groups of transformations and canonical coordinates corresponding to the IFG

$$\hat{X}_1 = x\frac{\partial}{\partial\, x} + y\frac{\partial}{\partial\, y},$$

$$\hat{X}_2 = x\frac{\partial}{\partial\, x} - y\frac{\partial}{\partial\, y},$$

$$\hat{X}_3 = x^2\frac{\partial}{\partial\, x} + y^2\frac{\partial}{\partial\, y}.$$

1.4 Canonical coordinates

Suppose that a one parameter group T_a

$$\bar{x}^k = f^k(x;a) = (T_a x)^k$$

with IFG

$$\hat{X} = \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i}$$

acts on $\mathbb{R}^n$. Let us consider a one-to-one and continuously differentiable change of coordinates (diffeomorphism) $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$:

$$\begin{aligned} y(x) &= (y^1(x), ..., y^n(x)) = F(x) \\ y^i(x) &= F^i(x^1, ..., x^n) \quad i = 1, ..., n \end{aligned} \tag{1.4.1}$$

First of all, let us note, that T_a defines in the coordinates $(y^1(x), ..., y^n(x))$ a one-parameter set of transformations

$$\bar{y}^i = \Phi^i(y;a) := F^i(f(x;a)) = F^i(f(F^{-1}(y);a)) \tag{1.4.2}$$

Let us show, that (1.4.2) defines a Lie group. Acting on $x \in \mathbb{R}^n$ by two consecutive transformations belonging to this family, we have:

$$\begin{aligned} \bar{\bar{y}}^i = \Phi^i(\bar{y};b) = F^i[f(\bar{x};b)] = F^i[f(x;\, b+a)] = \\ = F^i[f(F^{-1}(y);\, b+a)] = \Phi^i[y;\, a+b]. \end{aligned}$$

Since (1.4.2) is a local Lie group, then it has an infinitesimal generator, which by definition is as follows:

$$\begin{aligned} \eta^i(y) = \frac{\partial \bar{y}^i}{\partial a}|_{a=0} = \frac{\partial}{\partial a} F^i(f(x;a))|_{a=0} = \sum_{k=1}^{n} \frac{\partial F^i}{\partial f^k} \frac{\partial f^k(x;a)}{\partial a}|_{a=0} = \\ = \sum_{k=1}^{n} \xi^k(x) \frac{\partial F^i}{\partial x^k} = \hat{X} F^i(x) = \hat{X} F^i(F^{-1}(y)) \end{aligned} \tag{1.4.3}$$

The above formula is very important, since it tells us **how the coordinates of the IFG transform under the nonsingular change of variables**. One of the effective technical tools purposed at solving ODEs admitting a Lie group is based on the search of so called *canonical coordinates*, in which the IFG coordinates are *as simple as possible.*

Definition 1.4.1. *A change of coordinates* (1.4.1) *defines a set of canonical variables for the one-parameter Lie group* $\{T_a\}$ *(with the generator* $\hat{X} = \sum_{k=1}^{n} \xi^k(x) \frac{\partial}{\partial x^k}$*) if in terms of such coordinates the group acts as follows:*

$$\begin{aligned} \bar{y}^i &= y^i \quad i = 1, ..., n-1 \\ \bar{y}^n &= y^n + a \end{aligned} \tag{1.4.4}$$

Remark 1.4.1. If $(y^1, ..., y^n)$ are canonical, then $\eta^i(y) = \delta^i_n$.

Example 1.4.1. Let us consider the scaling group

$$\bar{x}_1 = e^a\, x, \qquad \bar{x}_2 = e^{2\,a}\, x_2$$

with the generator

$$\hat{X} = x_1 \frac{\partial}{\partial\, x_1} + 2\, x_2 \frac{\partial}{\partial\, x_2}.$$

To obtain the canonical coordinates, we should solve the system

$$\hat{X}\, y_1 = 0, \qquad \hat{X}\, y_2 = 1.$$

The characteristic form, of the first equation is as follows:

$$\frac{d\, x_1}{x_1} = \frac{d\, x_2}{2\, x_2} = \frac{d\, y_1}{0}.$$

Solving this system we conclude that $y_1 = \varphi(x_1^2/x_2)$. Solving then the characteristic system

$$\frac{d\, x_1}{x_1} = \frac{d\, x_2}{2\, x_2} = \frac{d\, y_2}{1}$$

corresponding to the second equation, we get $y_2 = \log x_1 + \psi(x_1^2/x_2)$. Thus, the simplest form of transformation we need is as follows:

$$y_1 = x_1^2/x_2, \qquad y_2 = \log x_1.$$

To make sure that we really obtained the canonical coordinates, it is sufficient to express the operatot $\hat{X}$ in new variables. Let us do this. Thus, we have:

$$\frac{\partial}{\partial\, x_1} = \frac{\partial y_1}{\partial\, x_1}\frac{\partial}{\partial\, y_1} + \frac{\partial y_2}{\partial\, x_1}\frac{\partial}{\partial\, y_2} = \frac{2\, x_1}{x_2}\frac{\partial}{\partial\, y_1} + \frac{1}{x_1}\frac{\partial}{\partial\, y_2};$$

$$\frac{\partial}{\partial\, x_2} = \frac{\partial y_1}{\partial\, x_2}\frac{\partial}{\partial\, y_1} + \frac{\partial y_2}{\partial\, x_2}\frac{\partial}{\partial\, y_2} = -\frac{x_1^2}{x_2^2}\frac{\partial}{\partial\, y_1}.$$

From this we obtain:

$$\hat{X} = x_1 \left[\frac{2\, x_1}{x_2}\frac{\partial}{\partial\, y_1} + \frac{1}{x_1}\frac{\partial}{\partial\, y_2}\right] + 2x_2 \left(-\frac{x_1^2}{x_2^2}\right)\frac{\partial}{\partial\, y_1} = \frac{\partial}{\partial\, y_2}.$$

Exercise 1.4.1. Consider the following groups of transformations:

1. $\bar{x}_1 = e^a\, x_1, \qquad \bar{x}_2 = e^{-a}\, x_2,$
2. $\bar{x}_1 = x_1 + a, \qquad \bar{x}_2 = e^a\, x_2,$
3. $\bar{x}_1 = x_1 + a, \qquad \bar{x}_2 = \frac{x_1\, x_2}{x_1 + a},$
4. $\bar{x}_1 = \frac{x_1}{1 - a\, x_1}, \qquad \bar{x}_2 = \frac{x_2}{1 - a\, x_1}.$

- Find the IFG in each case.
- Find the canonical coordinates.

Exercise 1.4.2. Consider the rotation group

$$\bar{x}_1 = x_1\, \cos a - x_2\, \sin a, \qquad \bar{x}_2 = x_1\, \sin a + x_2\, \cos a.$$

Show that the change of coordinates

$$r = \sqrt{x_1^2 + x_2^2}, \qquad \theta = \arctan\frac{x_2}{x_1}$$

defines a set of canonical variables.

1.5 Invariants of the Lie group

1.5.1 Invariant functions

Suppose that $\{T_a\}|_{a\in\Delta}$ is a Lie group acting on $\mathbb{R}^n$.

Definition 1.5.1. *A function F, mapping $\mathbb{R}^n$ into $\mathbb{R}^1$ is called an invariant function for $\{T_a\}$ if*

$$F[T_a x] = F[x] \quad \forall a \in \Delta \quad \forall x \in \mathbb{R}^n.$$

Remark 1.5.1. In what follows, we use the notation:

$$T_a F[x] := F[\bar{x}] = F[T_a x].$$

Theorem 1.5.1. *A smooth function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^1$ is an invariant function for T_a if and only if*

$$\hat{X}F[x] = 0 \quad \forall x \in \mathbb{R}^n, \tag{1.5.1}$$

where $\hat{X} = \xi^k(x)\frac{\partial}{\partial x^k}$ is the IFG of the group T_a.

Proof. $(\Rightarrow)$
Since $T_a F = F$, then $F[T_a x]$ does not depend on a. Thus, we have:

$$\frac{\partial}{\partial a} T_a F[x]|_{a=0} = \frac{\partial}{\partial a} F[f(x;a)] = \sum_{j=1}^{n} \frac{\partial F}{\partial f_j}\frac{\partial f_j}{\partial a}|_{a=0} = \hat{X}F[x] = 0$$

$(\Leftarrow)$
Assume that $\hat{X}F[x] = 0$. This is true for any point $x \in \mathbb{R}^n$, in particular this is true for $\bar{x} = f(x;a)$ that:

$$0 = \hat{X}[\bar{x}]F[\bar{x}] = \sum_{i=1}^{n} \xi^i(\bar{x})\frac{\partial F}{\partial \bar{x}^i} = \sum_{i=1}^{n} \frac{\partial F}{\partial \bar{x}^i}\frac{\partial \bar{x}^i}{\partial a} = \frac{\partial}{\partial a}F[\bar{x}]$$

But this mean that $F[\bar{x}]$ does not depend on a, so in particular:

$$F[\bar{x}] = F[f(x;a)] = F[f(x;0)] = F[x]$$

□

Example 1.5.1. Let us find out the function $F[x,y]$, which is invariant w.r.t. the rotation group T_a:

$$\bar{x} = x\cos a - y\sin a \qquad \bar{y} = x\sin a + y\cos a$$

We know that $\hat{X} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$. Applying the criterium (1.5.1), we have:

$$-yF_x + xF_y = 0$$

or in the form of the equivalent characteristic system:

$$\frac{\mathrm{d}x}{-y} \underbrace{=}_{1} \frac{\mathrm{d}y}{x} \underbrace{=}_{2} \frac{\mathrm{d}F}{0}$$

Solving the first two equations, we get the invariant $x^2 + y^2 = \Omega$ and hence $F = \varphi\left(x^2 + y^2\right)$.

Remark 1.5.2. Note, that any function $\varphi(x,y) = \Omega$, such that $\hat{X}\varphi(x,y) = 0$ is called the invariant of the Lie group T_a characterized by the IFG $\hat{X}$. So any smooth function $F(\rho)$, where $\rho = \sqrt{x^2 + y^2}$ is the distance from the origin, is invariant with respect to the rotation group.

Example 1.5.2. Let's consider $T_a: \mathbb{R}^3 \to \mathbb{R}^3$:

$$\bar{x} = e^a x \quad \bar{y} = e^{2a} y \quad \bar{z} = e^{-a} z.$$

The IFG of this group is as follows:

$$\hat{X} = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - z\frac{\partial}{\partial z}. \tag{1.5.2}$$

The above remark tells us that a function F(x,y,z) is an invariant of the group T_a if F can be presented as a smooth function of the independent invariants of the operator (1.5.2). We can find out the independent invariants of $\hat{X}$ by solving the characteristic system:

$$\frac{\mathrm{d}x}{x} \underbrace{=}_{1} \frac{\mathrm{d}y}{2y} \underbrace{=}_{2} \frac{\mathrm{d}z}{-z}$$

It has two independent invariants:

$$1 \Rightarrow \Omega_1 = \frac{y}{x^2}$$

$$2 \Rightarrow \Omega_2 = yz^2$$

So any function invariant w.r.t. (1.5.2) possesses the representation $F(x, y, z) = \Phi(\frac{y}{x^2}, yz^2)$

Remark 1.5.3. The set of the functions$\{\Omega_1, \Omega_2\}$ form a complete set of invariants of (1.5.2) in an open set $U \subset \mathbb{R}^3$ if

$$rank\frac{\partial(\Omega_1,\, \Omega_2)}{\partial(x, y, z)}|_U = 2 = const.$$

Let's calculate the Jacobi matrix for our case:

$$J = \frac{\partial(\Omega_1, \Omega_2)}{\partial(x, y, z)} = \begin{pmatrix} -\frac{y}{2x^3} & \frac{1}{x^2} & 0 \\ 0 & z^2 & 2yz \end{pmatrix}$$

The rank of J is 2 in $\mathbb{R}^3 \backslash \{0, 0, 0\}$.

Exercise 1.5.1. For the IFG

$$\hat{X} = x^2\frac{\partial}{\partial x} + x\,y\frac{\partial}{\partial y} - \left(\frac{y^2}{4} + \frac{x}{2}\right) z\frac{\partial}{\partial z}$$

find the set of independent first integrals (invariants). Show the general outlook of a function admitting $\hat{X}_4$.

1.5.2 Invariance of an algebraic manifold

Definition 1.5.2. *We say, that a set*

$$M = \{x \in \mathbb{R}^n | \psi^\nu(x) = 0 \quad \nu = 1, ..., s < n\} \tag{1.5.3}$$

is a regular surface (or an algebraic manifold), if:

(a) ψ^ν *are smooth functions*

(b) $rank\, \frac{\partial(\psi^1,...\psi^s)}{\partial(x_1,...,x_n)}|_M = s = const$

Definition 1.5.3. *A regular surface (algebraic manifold) M, is an invariant surface for a one-parameter Lie group of transformations* $\bar{x}^k = f^k(x;a)$, $a \in \Delta \subset \mathbb{R}$ *if:*

$$\forall a \in \Delta \quad \forall x \in M \qquad f(x;a) \in M$$

Lemma 1.5.1. *There exists a non-singular change of variables (diffeomorphism)* $y^i = \phi^i(x)$, *such that in new variables the algebraic manifold M is defined by the system of equations:*

$$M|_y = \{(y_1, ..., y_n)|y_1 = y_2 = ... = y_s = 0\} \tag{1.5.4}$$

Proof. (by construction)
Since $rank \frac{\partial(\psi^1,...\psi^s)}{\partial(x_1,...,x_n)}|_M = s$, then it is possible to choose a sequence of indices $1 \le j_1 \le j_2 \le ... \le j_s \le n$ and the corresponding set of variables $(x_{j_1}, x_{j_2}, ...x_{j_s})$, such that the vectors:

$$\begin{pmatrix} \frac{\partial \psi^1}{\partial x_{j_1}} \\ \vdots \\ \frac{\partial \psi^s}{\partial x_{j_1}} \end{pmatrix}, ..., \begin{pmatrix} \frac{\partial \psi^1}{\partial x_{j_s}} \\ \vdots \\ \frac{\partial \psi^s}{\partial x_{j_s}} \end{pmatrix}$$

are linearly independent on M.

Let us perform the following change of variables:

$$x_1' = x_{j_1}, \, x_2' = x_{j_2}..., x_s' = x_{j_s}, \, x_{s+1}' = x_{k_1}, ..., x_n' = x_{k_{n-s}},$$

where $1 \le k_1 \le k_2 \le ... \le k_{n-s} \le n$ and $(x_{k_1}, x_{k_2}, ...x_{k_{n-s}})$ is a set of variables complimentary for $\{x_{j_i}\}_{i=1}^s$, i.e. a set $(x_1, ..., x_n) \backslash (x_{j_1}, ..., x_{j_s})$.

Now let us introduce the change of variables:

$$\begin{cases} y_1 = \tilde{\psi}^1(x_1', ..., x_s', x_{s+1}', ..., x_n') = \psi^1(x_1, ..., x_n) \\ \vdots \\ y_s = \tilde{\psi}^s(x_1', ..., x_s', x_{s+1}', ..., x_n') = \psi^s(x_1, ..., x_n) \\ y_{s+r} = x_{s+r}' \quad 1 \le r \le n-s \end{cases}$$

In new variables $M = \{y \in \mathbb{R}^n \quad | \quad y_1 = ... = y_s = 0\}$.

Let us show that the map $x \longrightarrow x' \longrightarrow y$ is a diffeomorphism. It is evident that

$$\det \frac{\partial(y_1, ...y_n)}{\partial(x_1, ...x_n)} = \det \frac{\partial(y_1, ...y_n)}{\partial(x_1', ...x_n')} \cdot \det \frac{\partial(x_1', ...x_n')}{\partial(x_1, ...x_n)}.$$

But $\det \frac{\partial(x_1', ...x_n')}{\partial(x_1, ...x_n)}$ is the determinant of the matrix of permutation

$$\pi \begin{pmatrix} 1, & 2, & ..., & s, & s+1, & ..., & n \\ j_1, & j_2, & ..., & j_s, & k_1, & ..., & k_{n-s} \end{pmatrix},$$

which is equivalent to either $+1$ ora -1. Concerning the Jacobi matrix of the map $x' \longrightarrow y$, we have: $\det \left(\frac{\partial(y_1, ..., y_n)}{\partial(x_1', ..., x_n')} \right) |_M =$

$$= \det \left(\begin{array}{ccc|cccc} \frac{\partial\tilde{\psi}^1}{\partial x'_1} & \dots & \frac{\partial\tilde{\psi}^1}{\partial x'_s} & \frac{\partial\tilde{\psi}^1}{\partial x'_{s+1}} & \cdots & & \frac{\partial\tilde{\psi}^1}{\partial x'_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & & \vdots \\ \frac{\partial\tilde{\psi}^s}{\partial x'_1} & \dots & \frac{\partial\tilde{\psi}^s}{\partial x'_s} & \frac{\partial\tilde{\psi}^s}{\partial x'_{s+1}} & \cdots & & \frac{\partial\tilde{\psi}^s}{\partial x'_n} \\ \hline 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 \end{array} \right) \Bigg|_M =$$

$$= \det \left(\frac{\partial(\tilde{\psi}^1, ..., \tilde{\psi}^s)}{\partial(x'_1, ..., x'_s)} \right) |_M \neq 0 \qquad \square$$

Theorem 1.5.2. *An algebraic manifold M, defined by* (1.5.3)*, is invariant w.r.t.* $\{T_a\}_{a\in\Delta}$ *with IFG* $\hat{X} = \xi^i \frac{\partial}{\partial x_j}$*, if and only if*

$$\hat{X}\psi^\nu|_M = 0 \quad \nu = 1, ..., s \tag{1.5.5}$$

Proof. ($\Rightarrow$)
If $\forall a \in \Delta \quad \psi^\nu[T_a x] = 0$, then

$$0 = \frac{\partial}{\partial a}\psi^\nu[T_a x]|_{a=0} = \frac{\partial\psi^\nu}{\partial z_j}|_{z=f(x;0)=x}\frac{\partial z_j}{\partial a}|_{a=0} = \frac{\partial\psi^\nu}{\partial x_j}\xi_j(x) = \hat{X}\psi^\nu|_M.$$

($\Leftarrow$)
On virtue of the lemma 1.5.1, we can assume that M is given by the set of equations

$$x^\nu = 0, \qquad \nu = 1, ..., s.$$

If $\hat{X} = \xi^k(x)\frac{\partial}{\partial x^k}$, then the equality $0 = \hat{X}\psi^\nu|_{x^\mu=0,\mu=1,...,s}$, takes the form:

$$\sum_{j=1}^n \xi^j(x)\frac{\partial x^\nu}{\partial x^j}|_M = \xi^j(x)\delta^\nu_j|_M = \xi^\nu(\underbrace{0, ..., 0}_{s}, x^{s+1}, ..., x^n) = 0. \tag{1.5.6}$$

Now let us consider the Lie's equations:

$$\begin{cases} \frac{\partial\bar{x}^\nu}{\partial a} = \xi^\nu(\bar{x}^1, ..., \bar{x}^n) & \bar{x}^\nu(0) = 0 \quad \nu = 1, ..., s \\ \frac{\partial\bar{x}^{s+\mu}}{\partial a} = \xi^{s+\mu}(\bar{x}^1, ..., \bar{x}^n) & \bar{x}^{s+\mu}(0) = x^{s+\mu} \quad \mu = 1, ..., n-s \end{cases} \tag{1.5.7}$$

The first s equations satisfy zero initial conditions and, besides,

$$\xi^\nu|_{a=0} = \xi^\nu(0, 0, ..., x^{s+1}, ...x^n) = 0, \qquad \nu = 1, ...s. \tag{1.5.8}$$

It is shown in any standard course of lectures on ordinary differential equations [2] that the solution to the initial value problem (1.5.7) can be obtained as a limit $x^j(a) = \lim_{k\to\infty} x^j_{k+1}(a)$, where

$$x^j_{k+1}(a) = x^j_0 + \int_0^a \xi^j\left(x_k(\tau)\right) d\,\tau, \qquad x^j_0 = \bar{x}^j(0) = x^j.$$

[2] see e. g. K. Maurin, *Analysis, Vol. 1, Ch. IX,* Reidel, Boston 1976 (translation from Polish).

For $j = 1, ...s$ we have on virtue of (1.5.8):

$$x_1^j(a) = x_0^j + \int_0^a \xi^j\,(x_0)\; d\,\tau = \int_0^a \xi^j\,(\underbrace{0, 0..., 0}_{s},\; x^{s+1}, ...x^n)\, d\,\tau = 0;$$

$$x_2^j(a) = x_0^j + \int_0^a \xi^j\,(x_1)\; d\,\tau = \int_0^a \xi^j\,(\underbrace{0, 0..., 0}_{s},\; x_1^{s+1}, ...x_1^n)\, d\,\tau = 0;$$

....

$$x_k^j(a) = x_0^j + \int_0^a \xi^j\,(x_{k-1})\; d\,\tau = \int_0^a \xi^j\,(\underbrace{0, 0..., 0}_{s},\; x_{k-1}^{s+1}, ...x_{k-1}^n)\, d\,\tau = 0.$$

Therefore the solutions of the first s equations will nullify:

$$\bar{x}^\nu = f^\nu(x,\, a) = 0, \quad \forall x \,\in\, M, \quad \forall a \,\in\, \Delta, \quad \nu \,\in\, [1, ..., s],$$

which means that $\bar{x} = (0,\, 0, ..., 0,\, \bar{x}^{s+1}, ...\bar{x}^n) \,\in\, M.$ □

The geometric interpretation of the invariance of the function $F(x_1, ..., x_n)$ and the surface $F(x_1, ..., x_n) = 0$:
Let us consider the level surfaces $F(x_1, ..., x_n) = C, \quad C \,\in\, (a,\, b)$, corresponding to the function $F(\cdot)$ invariant w.r.t. T_a with $\hat{X} = \xi^j \frac{\partial}{\partial x^j}$. The invariance condition

$$\hat{X}F = \xi^j \frac{\partial F}{\partial x^j} = 0$$

means that the vector field $\{\xi^1(x), ..., \xi^n(x)\}$ is perpendicular to the gradient $gr\vec{a}dF$, which means, that it is tangent to **any** level surface.
In the case of the algebraic manifold defined by the equation $F(x_1, ..., x_n) = 0$, the condition $\hat{X}F|_{F=0} = 0$ means that $\vec{\xi}$ is tangent to **the only** level surface $F = 0$.
Example 1.5.3. We know that any smooth function $F(\sqrt{x^2 + y^2})$ is invariant under $\hat{X}_1 = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.
$M: \quad x^2 + y^2 = 1$ is invariant w.r.t. $\hat{X}_1$ and, more generally, w.r.t:

$$\hat{X_\infty} = -y\frac{\partial}{\partial x} + xf\left(\sqrt{x^2 + y^2}\right)\frac{\partial}{\partial y}$$

where f is any smooth function such that f(1)=1.

1.6 Exponential Map

Formal definition of the operator-valued function $e^{t\hat{X}}$, where $\hat{X} = \sum_{k=1}^n \xi^k(x)\frac{\partial}{\partial\, x_k}$ is an IFG of a one-parameter Lie group $\{T_a\}$ acting in R^n is as follows:

Definition 1.6.1. *By* $e^{a\hat{X}}$ *we mean the operator-valued function* $\mathrm{R}^n \mapsto \mathrm{R}^n$

$$\exp\,[a\,\hat{X}]\,x = \,\mathbf{f}(x;\, a) = \{f^k(x;\, a)\}_{k=1}^n$$

$\forall\, x \,\in\, \mathrm{R}^n, \quad \forall a \in \Delta.$

Properties of $e^{a\hat{X}}$:

1. $\exp\,[a\,\hat{X}]\,\exp\,[b\,\hat{X}]\,x = \exp\,[(a+b)\,\hat{X}]]\,x = \exp\,[b\,\hat{X}]\,\exp\,[a\,\hat{X}]\,x.$

Proof. $\exp[a\,\hat{X}]\circ\exp[b\,\hat{X}]\,x = \exp(a\hat{X})f(x;b) = f(f(x;b);a) =$
$= f(x;a+b) = \exp[(a+b)\hat{X}]x = f(f(x;a);b) = \exp[b\hat{X}]\circ\exp[a\hat{X}]x.$ ☐

2. $\exp[0\,\hat{X}]\,x = x$ (evident).

3. $\frac{d}{d\,a}(\exp[a\,\hat{X}]\,x) = \hat{X}\left[\exp(a\,\hat{X})\,x\right].$

   Proof.

$$\frac{d}{d\,a}(\exp[a\,\hat{X}]\,x)^k = \frac{d\,\bar{x}^k}{d\,a} = \xi^k(\bar{x}) = \xi^j(\bar{x})\frac{\partial}{\partial\,\bar{x}^j}\,\bar{x}^k = \hat{X}\left[\exp\left(a\,\hat{X}\right)\,x\right]^k$$

   for $k = 1, 2, ...n$. ☐

4. $\exp(-a\hat{X})x = (\exp a\hat{X})^{-1}x.$

   Proof.

$$e^{-a\hat{X}}\left[e^{-a\hat{X}}x\right]^k = e^{-a\hat{X}}f^k\,(x,\,a) = f^k\,(f(x,\,a);\,-a) = f^k\,(x, a-a) = x^k.$$

   So $e^{-a\hat{X}}\circ e^{a\hat{X}} = I$ and hence $e^{-a\hat{X}} = \left[e^{a\hat{X}}\right]^{-1}$. ☐

Definition 1.6.2. ***(formal definition)*** *An operator-valued function* $\exp a\hat{X}$ *is the only solution of the equation*

$$\frac{d}{d\,a}x = \hat{X}\cdot x, \qquad x(0) = I,$$

where $x = \exp a\hat{X}$ *is an element of a Banach space of linear operators* $L(\mathrm{R}^n, \mathrm{R}^n)$ *which can be formally defined as:*

$$\exp a\,\hat{X} = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{a^k}{k!}\hat{X}^k\right) \tag{1.6.1}$$

where $\hat{X}^k x = X^{\hat{k-1}}(\hat{X}x)$

Example.
$\hat{X} = \frac{\partial}{\partial\,x}$.

$e^{a\,\partial_x}\,x = x + a,$

where $\partial_x = \frac{\partial}{\partial\,x}$.

Exercise 1.6.1. Calculate

$$\exp[a\,\hat{X}]\begin{pmatrix} x \\ y \end{pmatrix},$$

where $\hat{X} = x^2\frac{\partial}{\partial\,x} + y^2\frac{\partial}{\partial\,y}$.

Exercise 1.6.2. Calculate

$$\exp[a\,\hat{X}]\left(\begin{array}{c} x \\ y \end{array}\right),$$

where $\hat{X} = x^2\frac{\partial}{\partial x} + x\,y\frac{\partial}{\partial y}$.

Let's consider the operator

$$X = \left(\sum_{j=1}^{n} A_{i\,j}\,x^j\right)\frac{\partial}{\partial\,x^i}, \tag{1.6.2}$$

acting on R^n, where $A = (A_{i,\,j})$ is a matrix with constant coefficients.

Statement 1.6.1. *The following formula holds true:*

$$\exp\left(a\,\hat{X}\right)\,x = e^{a\,A}x,$$

where

$$e^{a\,A} = I + a\,A + \frac{a^2}{2!}\,A^2 + ...$$

Proof.

$$\hat{X}\,x^k = A_{i\,j}\,x^j\,\frac{\partial x^k}{\partial\,x^i} = A_{i\,j}\,x^j\,\delta_i^k = A_{k\,j}\,x^j = [A\,x]^k\,, \qquad k = 1,\,2,...n;$$
$$\hat{X}^2\,x^k = A_{m\,n}x^n\partial_{x^m}A_{k\,j}\,x^j = A_{m\,n}x^n\,A_{k\,j}\delta_m^j = A_{k\,m}\,A_{m\,n}x^n = \left(A^2x\right)^k.$$

Further, we use the method of induction. Assuming that $\hat{X}^r\,x^k = (A^r\,x)^k$ for $k = 1,\,2,...,\,n$, let us calculate the action of $\hat{X}^{r+1}$:

$$\left[\hat{X}^{r+1}x\right]^k = A_{i\,j}x^j\partial_{x^i}\,[A^r]_{k\,s}\,x^s = A_{i\,j}x^j\,\delta_i^j\,\,[A^r]_{k\,s} =$$
$$= [A^r]_{k\,s}\,A_{s\,j}x^j = \left[A^{r+1}\right]_{k\,j}\,x^j = \left[A^{r+1}\,x\right]^k.$$

Hence

$$\exp\left(a\,\hat{X}\right)\,x^k = \left[\left(I + a\,A + \frac{a^2}{2!}\,A^2 + ...\right)\,x\right]^k = \left[e^{a\,A}\,x\right]^k.$$

□

Example 1.6.1. Let us calculate

$$e^{a\hat{X}}\left(\begin{array}{c} x_1 \\ x_2 \end{array}\right),$$

where

$$\hat{X} = -x_2\frac{\partial}{\partial\,x_1} + x_1\frac{\partial}{\partial\,x_2}.$$

The operator $\hat{X}$ can be written down in the form (1.6.2) with

$$A = \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array}\right).$$

It is easy to get convinced that

$$A^{2n+1} = (-1)^n A, \qquad A^{2n} = (-1)^n I.$$

Hence

$$e^{aA} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left(a - \frac{a^3}{3!} + ... + (-1)^n \frac{a^{2n+1}}{(2n+1)!} + ... \right) +$$

$$+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \left(a^0 - \frac{a^2}{2!} + ... + (-1)^n \frac{a^{2n}}{(2n)!} + ... \right) =$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \sin a + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos a = \begin{pmatrix} \cos a & -\sin a \\ \sin a & \cos a \end{pmatrix}.$$

So we have reconstructed the action of the rotation group in R^2.

Exercise 1.6.3. Calculate the one-parameter Lie group generated by the operator

$$\hat{X} = A_{i\,j} x^j \frac{\partial}{\partial x^i}, \quad i,\, j = 1, 2, \quad A_{1\,2} = A_{2\,1} = 1, \quad A_{ii} = 0.$$

1.7 Lie algebra of the infinitesimal generators

Suppose that some object (a function or an algebraic manifold) is invariant w.r.t. more than one Lie group. Then we have two or more IFGs:

$$\{\hat{X}_1, ...\hat{X}_k\}$$

each corresponding to a local group T_a^i, $i = 1, ..., k$. A superposition of the one-parameter transformations $T_{a_1} \circ ...T_{a_k}$ is a Lie group called the **multi-parameter Lie group of transformations**. The set $\{\hat{X}_1, ...\hat{X}_k\}$ occurs to be closed w.r.t. some algebraic operation called the Lie bracket (or commutator).

Let

$$\hat{X} = \sum_{k=1}^{n} \xi^k(x) \frac{\partial}{\partial x^k}$$

and

$$\hat{Y} = \sum_{k=1}^{n} \eta^k(x) \frac{\partial}{\partial x^k}$$

be two different generators of one-parameter Lie groups $\{T'_a\}_{a \in \Delta'}$, $\{T''_b\}_{b \in \Delta''}$, acting on R^n.

Definition 1.7.1. *The commutator (or the Lie bracket) of $\hat{X}$ and $\hat{Y}$ is the operator $\left[\hat{X},\, \hat{Y}\right]$ acting on any smooth function $f : \mathrm{R}^n \mapsto \mathrm{R}$ as follows:*

$$[\hat{X}, \hat{Y}] f(x) = \hat{X} \left[\hat{Y} f(x) \right] - \hat{Y} \left[\hat{X} f(x) \right].$$

Lemma 1.7.1. *The commutator $\left[\hat{X},\, \hat{Y}\right]$ is the first-order differential operator defined by the following formula:*

$$[\hat{X}, \hat{Y}] = \sum_{j=1}^{n} [\hat{X}(\eta^j) - \hat{Y}(\xi^j)] \frac{\partial}{\partial x_j} \qquad (1.7.1)$$

Proof

- $$\hat{X}[\hat{Y}(f)] = \sum_{i=1}^{n} \xi^i(x) \frac{\partial}{\partial x^i} \sum_{j=1}^{n} \eta^j(x) \frac{\partial f}{\partial x^j} =$$
$$= \sum_{i=1}^{n} \xi^i(x) \sum_{j=1}^{n} \left[\frac{\partial \eta^j(x)}{\partial x^i} \right] \frac{\partial f}{\partial x^j} + \sum_{i=1}^{n} \sum_{j=1}^{n} \xi^i(x)\, \eta^j(x)\, \frac{\partial^2 f}{\partial x^i \partial x^j}.$$

- $$\hat{Y}[\hat{X}(f)] = \sum_{i=1}^{n} \eta^i(x) \frac{\partial}{\partial x^i} \sum_{j=1}^{n} \xi^j(x) \frac{\partial f}{\partial x^j} =$$
$$= \sum_{i=1}^{n} \eta^i(x) \sum_{j=1}^{n} \left[\frac{\partial \xi^j(x)}{\partial x^i} \right] \frac{\partial f}{\partial x^j} + \sum_{i=1}^{n} \sum_{j=1}^{n} \xi^j(x)\, \eta^i(x)\, \frac{\partial^2 f}{\partial x^j \partial x^i}.$$

Substracting the left and right sides of the second equation from the first one, making the replacement of the summation indices and taking into account the equality of the mixed derivatives, we obtain as a result that

$$[\hat{X}, \hat{Y}]f = \sum_j [\hat{X}(\eta^j) - \hat{Y}(\xi^j)] \frac{\partial f}{\partial x_j}.$$

Defined above operation has the following properties, mostly arising directly from the definition.

1. Skew symmetry:
$$[\hat{X},\, \hat{Y}] = -[\hat{Y},\, \hat{X}].$$
2. Bilinearity:
$$[\alpha_1 \hat{X}_1 + \alpha_2 \hat{X}_2,\, \hat{Y}] = \alpha_1\, [\hat{X}_1,\, \hat{Y}] + \alpha_2\, [\hat{X}_2,\, \hat{Y}].$$
3. Jacobi identity:
$$[\hat{Z}, [\hat{X},\, \hat{Y}]] + [\hat{X}, [\hat{Y},\, \hat{Z}]] + [\hat{Y}, [\hat{Z},\, \hat{X}]] = 0.$$

If the set of linearly independent first-order IFGs $\aleph = \{\hat{X}_1,\, \hat{X}_2,\, ...\hat{X}_m, \}$ is closed with respect to the operation $[\cdot\,, \cdot]$, i.e. for each pair $\hat{X}_i,\, \hat{X}_j$ the following decomposition holds

$$[\hat{X}_i,\, \hat{X}_j] = \sum_{k=1}^{n} c_{i\,j}^{k}\, \hat{X}_k$$

for some numbers $\{c_{i\,j}^{k}\}_{i,\,j,\,k=1,...,n}$, then the set $\aleph$ is called an n-dimensional Lie algebra with the basis (or basic elements) $\aleph$, while the constants $c_{i\,j}^{k}$ are called the structure constants.

Exercise 1.7.1. Show that the following properties of the structure constants are true:

$$c_{i\,j}^{k} = -c_{j\,i}^{k}; \tag{1.7.2}$$

$$c_{i\,i}^{k} = 0. \tag{1.7.3}$$

The following statement holds true:

Theorem 1.7.1. *Let the operators $\hat{X}$ and $\hat{Y}$ be the generators of one-parameter groups admitted by the algebraic manifold M (or function $f : \mathrm{R}^n \mapsto \mathrm{R}$). Then the first order operator $[\hat{X},\, \hat{Y}]$ is the generator of a one-parameter group admitted by this manifold (or function).*

Proof Without the loss of generality, we perform the proof assuming that $\hat{X}$, $\hat{Y}$ are symmetry generators for manifold. We define a one parameter family of transformations mapping M into itself by means of the formula

$$\Psi(x;\, a) = e^{\sqrt{a}\,\hat{X}}\, e^{\sqrt{a}\,\hat{Y}}\, e^{-\sqrt{a}\,\hat{X}}\, e^{-\sqrt{a}\,\hat{Y}}[x]. \tag{1.7.4}$$

We are going to show that for small a this transformation can be presented in the form

$$\Psi(x;\, a)^k = x^k + a\,\eta^k(x) + O(|a|^{3/2}), \quad k = 1, ...n,$$

where the vector field $\{\eta^k(x)\}_{k=1}^n$ is tangent to the manifold M. Basing on the first fundamental Lie theorem, we can, thus, conclude that the operator

$$\sum_{k=1}^{n} \eta^k(x) \frac{\partial}{\partial\, x^k}$$

is the generator of a one-parameter Lie group admitted by M.

We get the operator $\sum_{k=1}^{n} \eta^k(x) \frac{\partial}{\partial\, x^k}$ by decomposing the operators appearing in the right-hand side of (1.7.4) in the Taylor series and grouping relevant terms:

$$\begin{aligned}
&\left(e^{\sqrt{a}\,\hat{X}}\, e^{\sqrt{a}\,\hat{Y}}\, e^{-\sqrt{a}\,\hat{X}}\, e^{-\sqrt{a}\,\hat{Y}}\right)[x] = \\
&= \left[\left(1 + \sqrt{a}\,\hat{X} + \tfrac{a}{2}\hat{X}^2\right)\,\left(1 + \sqrt{a}\,\hat{Y} + \tfrac{a}{2}\hat{Y}^2\right)\right] \cdot \left[\left(1 - \sqrt{a}\,\hat{X} + \tfrac{a}{2}\hat{X}^2\right)\,\left(1 - \sqrt{a}\,\hat{Y} + \tfrac{a}{2}\hat{Y}^2\right)\right][x] \\
&= \left(1 + \sqrt{a}\,\hat{X} + \tfrac{a}{2}\hat{X}^2 + \sqrt{a}\,\hat{Y} + a\,\hat{X}\,\hat{Y} + \tfrac{a}{2}\hat{Y}^2 + O(|a|^{3/2})\right) \cdot \\
&\left(1 - \sqrt{a}\,\hat{X} + \tfrac{a}{2}\hat{X}^2 - \sqrt{a}\,\hat{Y} + a\,\hat{X}\,\hat{Y} + \tfrac{a}{2}\hat{Y}^2 + O(|a|^{3/2})\right)[x] = \\
&\left[1 + \sqrt{a}\left(\hat{X} + \hat{Y}\right) + \tfrac{a}{2}\left(\hat{X}^2 + 2\,\hat{X}\,\hat{Y} + \hat{Y}^2\right)\right] \cdot \\
&\left[1 - \sqrt{a}\left(\hat{X} + \hat{Y}\right) + \tfrac{a}{2}\left(\hat{X}^2 + 2\,\hat{X}\,\hat{Y} + \hat{Y}^2\right)\right][x] + O\left(|a|^{3/2}\right) = \\
&\left[1 - a\left(\hat{X} + \hat{Y}\right)\left(\hat{X} + \hat{Y}\right) + a\left(\hat{X}^2 + 2\,\hat{X}\,\hat{Y} + \hat{Y}^2\right)\right][x] + O\left(|a|^{3/2}\right) = \\
&\left[1 - a\left(\hat{X}^2 + \hat{Y}^2 + \hat{X}\,\hat{Y} + \hat{Y}\,\hat{X}\right) + a\left(\hat{X}^2 + 2\,\hat{X}\,\hat{Y} + \hat{Y}^2\right)\right][x] + O\left(|a|^{3/2}\right) = \\
&\left\{1 + a\left(\hat{X}\,\hat{Y} - \hat{Y}\,\hat{X}\right)\right\}[x] + O\left(|a|^{3/2}\right) = x + a\,\eta(x) + O\left(|a|^{3/2}\right).
\end{aligned}$$

Tangency of the vector field $\{\eta^k(x)\}_{k=1}^n$ appear from the fact that the r.h.s of the formula (1.7.4) defines the superposition of transformations mapping the manifold M into itself.

1.8 Examples of Lie algebras

It is convenient to present a finite-dimensional Lie algebra $\left\{\hat{X}_j\right\}_{j=1}^n$ in the form of *commutator table,* whose $(i,\, j)-$th entry expresses the comutator $[\hat{X}_i,\, \hat{X}_j]$.

Remark 1.8.1. It is evident that the commutator table is skew-symmetric, since $[\hat{X}_i,\, \hat{X}_j] = -[\hat{X}_j,\, \hat{X}_i]$.

Example 1.8.1. For the set of operators $\{X_j\}_{j=1}^3$

$$X_1 = -y\partial_x + x\partial_y,$$
$$X_2 = \partial_x,$$
$$X_3 = \partial_y$$

the commutator table is as follows:

	X_1	X_2	X_3
X_1		$-X_3$	X_2
X_2			0
X_3			

Example 1.8.2. For the set of operators $\{X_j\}_{j=1}^4$

$$X_1 = -y\partial_x + x\partial_y,$$
$$X_2 = \partial_x,$$
$$X_3 = \partial_y,$$
$$X_4 = x\partial_x + y\partial_y$$

the commutator table is as follows:

	X_1	X_2	X_3	X_4
X_1		$-X_3$	X_2	0
X_2			0	X_2
X_3				X_3
X_4				

Example 1.8.3. For the set of operators $\{X_j\}_{j=1}^3$

$$X_1 = \partial_x,$$
$$X_2 = x\,\partial_x,$$
$$X_3 = x^2\,\partial_x$$

the commutator table is as follows:

	X_1	X_2	X_3
X_1		X_1	$2\,X_2$
X_2			X_3
X_3			

Example 1.8.4. Let us verify if the following set of operators

$$X_1 = x\,\partial_x,$$
$$X_2 = y\,\partial_x,$$
$$X_3 = \partial_y,$$

is closed w.r.t. the Lie brackets and fill in this set if necessary. The commutators are as follows:

1.

$$[X_1,\, X_2] = [x\,\partial_x,\, y\,\partial_x] = -y(\partial_x\, x)\partial_x = -X_2$$

2.

$$[X_1,\, X_3] = [x\,\partial_x,\, \partial_y] = 0,$$

3.

$$[X_2,\, X_3] = [y\,\partial_x,\, \partial_y] = -(\partial_y\, y)\partial_x = -\partial_x := -X_4.$$

So in order that the above set be closed, it should contain an extra operator $X_4 = \partial_x$. The set of operators X_1, ...X_4 proves to be closed as it is is shown below:

$$[X_1,\, X_4] = [x\,\partial_x,\, \partial_x] = -(\partial_x\, x)\partial_x = -X_4,$$

$$[X_2, X_4] = [y\,\partial_x,\, \partial_x] = 0,$$

$$[X_3,\, X_4] = [\partial_y,\, \partial_x] = 0.$$

The commutator table for the operators $X_1, ...X_4$ is the following:

	X_1	X_2	X_3	X_4
X_1		$-X_2$	0	$-X_4$
X_2			$-X_4$	0
X_3				0
X_4				

Exercise 1.8.1. Make the commutator table for the following set of operators:

$$X_1 = x_2\,\partial_{x_3} - x_3\,\partial_{x_2},$$
$$X_2 = x_3\,\partial_{x_1} - x_1\,\partial_{x_3},$$
$$X_3 = x_1\,\partial_{x_2} - x_2\,\partial_{x_1}.$$

Exercise 1.8.2. [(*a*.)] Verify if the following operators

$$X_1 = x\,\partial_y + y\,\partial_x,$$
$$X_2 = -y\,\partial_x + x\,\partial_y$$

form a closed set with respect to the Lie bracket.

[(*b*.)] If the answer to [(*a*)] is negative, supplement the set X_1, X_2 with the missing operator(s) and make the commutator table for the whole set of operators.

Chapter 2

Groups admitted by the differential equations

2.1 Introductory remarks

We'll discuss here definition of symmetry which is applied to both PDEs and ODEs, because the the formalism is identical in both of these cases (yet the applications are essentially different).

Let the group G_a acts on the space $\left(x^1, x^2, ...x^n; u^1, ...u^m\right) \in R^{n+m}$, where x^i are independent variable, while u^α are functions of x^i. The group G_a action is defined as follows:

$$\bar{x}^k = f^k(x, u; a) = x^k + a\,\xi^k(x, u) + O(a^2), \;\; k = 1, ...n, \tag{2.1.1}$$

$$\bar{u}^\alpha = g^\alpha(x, u; a) = u^\alpha + a\,\eta^\alpha(x, u) + O(a^2), \;\; \alpha = 1, ...m, \tag{2.1.2}$$

where f^k, g^α are three times differentiable with respect to x and u and analytic functions with respect to the group parameter $a \in \Delta \subset R^1$,

$$\xi^k(x, u) = \left(\partial f^k / \partial\, a\right)|_{a=0}, \qquad \eta^\alpha(x, u) = (\partial g^\alpha / \partial\, a)\,|_{a=0}.$$

Remark 2.1.1. Note that temporarily we do not distinguish between the independent and dependent variables, which enter the formulae (2.1.1), (2.1.2) on an equal footing.

Let us consider the manifold M defined in the extended space (the space of **jets** of $r-$th order)

$$(x, ; u, \partial\, u, \partial^2\, u, ...\partial^r\, u)$$

by the system of equations

$$R^\sigma(x, u, \partial\, u, ...\partial^r\, u) = 0, \qquad \sigma = 1, 2, ...s, \tag{2.1.3}$$

where $\partial^k\, u$ denotes the set of all k-th order partial derivatives of the functions u^α (we identify $\partial\, u$ with $\partial^1\, u$ in this context).

Definition 2.1.1. *We say that a one-parameter group $\{G_a\}_{a \in \Delta}$, defined on the set of dependent and independent variables by means of formulae (2.1.1)–(2.1.2), is admitted by (2.1.3) if it maps each solution of (2.1.3) into some other solution of this system.*

Let us try to understand what the definition 2.1.1 does mean. We can identify any given solution $u = \varphi(x)$ with its graph

$$\Gamma_\varphi = \left\{(x, \varphi(x)), \quad x \in \Omega \in R^n\right\},$$

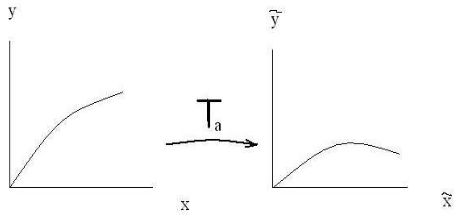

Figure 2.1: Action of the operator T_a on the graph Γ_φ

where $R^n \supset \Omega$ is an open set contained in the natural domain of the function φ. Acting with $T_a \in G_{\{a\}}$ we get the graph

$$T_a \circ \Gamma_\varphi = (\bar{x},\, \bar{u}) = [f(x,\, u;\, a), g(x,\, u;\, a)]|_{(x,\, u=\varphi(x))\,\in\,\Gamma_\varphi}\,.$$

Remark 2.1.2. Since $T_0 = Id$, then $T_a \circ \Gamma_\varphi$ is also the graph of a function at least for sufficiently small a.

Attempting to understand the above definition, let us address the example formulated as the following statement:

Statement 2.1.1. *Transformation*

$$\begin{pmatrix} \bar{x} \\ \bar{u} \end{pmatrix} = \begin{pmatrix} x\,\cos a & -u\,\sin a \\ x\,\sin a & u\,\cos a \end{pmatrix} \tag{2.1.4}$$

maps the solutions of the equation

$$u_{xx} = 0. \tag{2.1.5}$$

into itself.

Proof The general solution to the equation (2.1.5) takes the form

$$u(x) = A\,x + B,$$

where A, B are the arbitrary constants. From the geometric viewpoint invariance means that the rotation group maps the graph $(x,\, A\,x + B)$ into some other graph of this sort:

$$(x,\, A\,x + B) \to (\bar{x},\, \bar{A}\,\bar{x} + \bar{B}).$$

Let us show this analytically.

Applying the rotation group (2.1.4) to the pair $(x, u) \in R^2$, where $u(x) = \varphi(x) = A\,x + B$, we get:

$$(\bar{x},\, \bar{u}) = (x\cos\, a - (A\,x + B)\sin\, a,\; x\,\sin\, a + (A\,x + B)\,\cos\, a)$$

Using the equality $\bar{x} = x\cos\, a - (A\,x + B)\sin\, a$ we can express the old variable in terms of the new one:

$$x = \frac{\bar{x} + B\,\sin\, a}{\cos\, a - A\,\sin\, a}.$$

Hence

$$\bar u = \tilde\varphi(\bar x) = \frac{\bar x + B\sin a}{\cos a - A\sin a}(\sin a + A\cos a) + B\cos a,$$

or

$$\tilde\varphi(\bar x) = \bar x\,\frac{\sin a + A\cos a}{\cos a - A\sin a} + B\left(\sin a\,\frac{\sin a + A\cos a}{\cos a - A\sin a} + \cos a\right) = \bar A\,\bar x + \bar B\,.$$

So the transformed function is linear in new variables and hence satisfies the equation

$$\bar u_{\bar x\,\bar x} = 0.$$

Another definition of the invariance of DE is following. Let's have the transformation group (2.1.1), (2.1.2). We show in the following section that this transformation group defines automatically the corresponding transformations

$$\frac{\partial^r \bar u^\alpha}{\partial\,\bar x_1^{r_1}...\partial\,\bar x_n^{r_n}} = \theta^\alpha_{r_1,...r_n}(x,\,u,\,\partial u,..\partial^r u;a), \quad 1 \le r_1 + ... + r_n \le r, \tag{2.1.6}$$

$r_k \ge 0$, which, together with (2.1.1), (2.1.2) form *the* $r-th$ *extension* $G_a^{(r)}$ of the Lie group G_a, acting on the space of **jets** of $r-th$ order.

Definition 2.1.2. *We say that the set of transformations* $G_a^{(r)}$ *is admitted by the system*

$$R^\sigma(x,\,u,...\partial^r u) = 0, \qquad \sigma = 1,...s. \tag{2.1.7}$$

if for sufficiently small a (2.1.7) implies

$$R^\sigma(\bar x,\,\bar u,...\partial^r \bar u) = 0, \qquad \sigma = 1,...s.$$

Statement 2.1.2. *The Galilei group*

$$\bar x = x + 2at, \quad \bar t = t, \quad \bar u = e^{-ax-a^2 t}\,u \tag{2.1.8}$$

is admitted by the transport equation

$$u_t = u_{xx}. \tag{2.1.9}$$

Remark 2.1.3. The symbol u_ν here and henceforth stands for the partial derivative w.r.t. the corresponding variable

<u>Proof</u> Assume that the function $u = \varphi(t,\,x)$ satisfies (2.1.9). Let us express $\bar u$ in new variables. From (2.1.8) we can express the old independent variables by the new ones as follows:

$$x = \bar x - 2\,a\,\bar t, \qquad t = \bar t.$$

Thus, we have

$$\bar u = e^{-a(\bar x - 2\,a\,\bar t) - a^2\,\bar t}\,\varphi(\bar t,\,\bar x - 2\,a\,\bar t).$$

Calculating the corresponding partial derivatives, we get

$$\frac{\partial \bar u}{\partial \bar t} = e^{-a(\bar x - 2\,a\,\bar t) - a^2\,\bar t}\left\{a^2\varphi + \varphi_1 - 2\,a\,\varphi_2\right\},$$

$$\frac{\partial \bar u}{\partial \bar x} = e^{-a(\bar x - 2\,a\,\bar t) - a^2\,\bar t}\left\{-a\,\varphi + \varphi_2\right\},$$

$$\frac{\partial^2 \bar u}{\partial \bar x^2} = e^{-a(\bar x - 2\,a\,\bar t) - a^2\,\bar t}\left\{-a\,(\varphi_2 - a\varphi) + \varphi_{2\,2} - a\,\varphi_2\right\} =$$

$$= e^{-a(\bar x - 2\,a\,\bar t) - a^2\,\bar t}\left\{\varphi_{2\,2} - 2\,a\,\varphi_2 + a^2\varphi\right\},$$

where

$$\varphi_1=\frac{\partial}{\partial z_1}\varphi(z_1,\,z_2)|_{z_1=\bar t,\;z_2=\bar x-2\,a\,\bar t},\quad \varphi_2=\frac{\partial}{\partial z_2}\varphi(z_1,\,z_2)|_{z_1=\bar t,\;z_2=\bar x-2\,a\,\bar t},$$

$$\varphi_{2\,2}=\frac{\partial^2}{\partial z_2^2}\varphi(z_1,\,z_2)|_{z_1=\bar t,\;z_2=\bar x-2\,a\,\bar t}.$$

So

$$\begin{aligned}\frac{\partial\bar u}{\partial\bar t}-\frac{\partial^2\bar u}{\partial\bar x^2}&=e^{-a(\bar x-2\,a\,\bar t)-a^2\,\bar t}\Big\{a^2\varphi+\varphi_1-2\,a\,\varphi_2-\varphi_{2\,2}+2\,a\,\varphi_2-a^2\,\varphi)\Big\}=\\&=e^{-a(\bar x-2\,a\,\bar t)-a^2\,\bar t}\left\{\varphi_1-\varphi_{2\,2}\right\}|_{z_1=\bar t,\;z_2=\bar x-2\,a\,\bar t}=\\&=e^{-a(\bar x-2\,a\,\bar t)-a^2\,\bar t}\left\{\varphi_t-\varphi_{x\,x}\right\}=0.\end{aligned}$$

2.2 Theory of prolongations

Let us remind that S.Lie proposed to consider DEs as manifolds in a *jet space*. But the local Lie group acts on the $n+m-$ dimensional space $(x_1,\,...x_n;\,u^1...u^m)\in R^{n+m}$ consisting of dependent and independent variables. However, if a local 1-parameter Lie group is defined by the formulae

$$x^k=f^k(x,\,u;\,a)=x^k+a\,\xi^k(x,\,u)+...,\quad k=1,2,...n, \tag{2.2.1}$$

$$u^\alpha=g^\alpha(x,\,u;\,a)=u^\alpha+a\,\eta^\alpha(x,\,u)+...,\quad \alpha=1,2,....m \tag{2.2.2}$$

then the equations (2.2.1)–(2.2.2) induce the transformations of the partial derivatives:

$$\frac{\partial\bar u^\alpha}{\partial\,\bar x^k}=\theta^\alpha_k(x,\,u,\,\partial\,u\,;a)=\frac{\partial u^\alpha}{\partial\,x^k}+a\,\zeta^\alpha_k(x,\,u,\,\partial\,u)+..., \tag{2.2.3}$$

$$\frac{\partial^2\bar u^\alpha}{\partial\,\bar x^k\partial\,\bar x^j}=\theta^\alpha_{k,j}(x,\,u,\,\partial\,u,\,\partial^2\,u;a)=\frac{\partial^2u^\alpha}{\partial\,x^k\,\partial\,x^j}+a\,\zeta^\alpha_{k,j}(x,\,u,\,\partial\,u,\partial^2\,u)+... \tag{2.2.4}$$

....

Definition 2.2.1. *The operator*

$$\hat X^{(r)}=\sum_{k=1}^n\xi^k(x,\,u)\frac{\partial}{\partial\,x^k}+\sum_{\alpha=1}^m\eta^\alpha(x,\,u)\frac{\partial}{\partial\,u^\alpha}+\sum_{1\le\,|J|\le\,r}\zeta^\alpha_J(x,\,u,\partial\,u,...\partial^{|J|}\,u)\frac{\partial}{\partial\,u^\alpha_J},$$

$J=(j_1,\,j_2,\,...j_n),\ j_1\le\ j_2...\le\ j_n,\ |J|=j_1+...+j_n$, $u^\alpha_J=\frac{\partial^{|\,J|}}{\partial\,x^{j_1}\,\partial\,x^{j_2}...\partial\,x^{j_n}}\,u^\alpha$ *is called the* $r-th$ *prolongation of the generator*

$$\hat X=\sum_{k=1}^n\xi^k(x,\,u)\frac{\partial}{\partial\,x^k}+\sum_{\alpha=1}^m\eta^\alpha(x,\,u)\frac{\partial}{\partial\,u^\alpha}.$$

We are interested on how it is possible to find out $\zeta^\alpha_{i_1,\,i_2,\,,,,i_k}$. The answer to this question gives the statement formulated below.

Theorem 2.2.1. *The following formula takes place*

$$\zeta^\alpha_{j,\,i_1,...i_r}=D_j\,\zeta^\alpha_{i_1,...i_r}-u^\alpha_{k,\,i_1,...i_r}D_j\,\xi^k, \tag{2.2.5}$$

where

$$D_j=\frac{\partial}{\partial\,x^j}+u^\alpha_j\frac{\partial}{\partial\,u^\alpha}+u^\alpha_{j,i_1}\frac{\partial}{\partial\,u^\alpha_{i_1}}+....+u^\alpha_{j,i_1,...i_k}\frac{\partial}{\partial\,u^\alpha_{i_1,\,i_2,...i_k}}+....$$

Proof. Let's begin with ζ^α_k:

$$\frac{\partial \bar u^\alpha}{\partial \bar x^k} = \frac{\partial}{\partial \bar x^k}\,[u^\alpha + a\,\eta^\alpha(x,\,u)] + O(a^2) = \frac{\partial}{\partial x^m}\,[u^\alpha + a\,\eta^\alpha(x,\,u)]\;\frac{\partial x^m}{\partial \bar x^k} + O(a^2).$$

In order to calculate $\frac{\partial x^m}{\partial \bar x^k}$, we take advantage of the identity

$$x \to \bar x \to x,$$

giving rise to the equation

$$\frac{\partial x^m}{\partial \bar x^r}\frac{\partial \bar x^r}{\partial x^s} = \delta^m_s,$$

and the formula for the infinitesimal inverse transformation

$$x^m = \bar x^m - a\,\xi^m(\bar x,\,\bar u) + O(a^2).$$

But

$$\frac{\partial \bar x^r}{\partial x^s} = \frac{\partial}{\partial x^s}\,[x^r + a\,\xi^r(x,\,u)] + O(a^2) = \delta^r_s + a\,D_{x^s}\,\xi^r(x,\,u) + O(a^2),$$

while

$$\frac{\partial x^m}{\partial \bar x^r} = \frac{\partial}{\partial \bar x^r}\,[\bar x^m - a\,\xi^m(\bar x,\,\bar u)] + O(a^2) = \delta^m_r - a\,D_{\bar x^r}\,\xi^m(\bar x,\,\bar u) + O(a^2).$$

We claim that the last formula up to $O(a^2)$ is equivalent to

$$\delta^m_r - a\,D_{x^r}\,\xi^m(x,\,u).$$

Indeed, multiplying $\frac{\partial x^m}{\partial \bar x^r}$ by $\frac{\partial \bar x^r}{\partial x^s}$ and summing up over r, we get:

$$\frac{\partial x^m}{\partial \bar x^r}\,\frac{\partial \bar x^r}{\partial x^s} = [\delta^m_r - a\,D_{x^r}\,\xi^m(x,\,u)]\cdot[\delta^r_s + a\,D_{x^s}\,\xi^r(x,\,u)] + O(a^2) =$$

$$= \delta^m_s + O(a^2),$$

which proves our statement. Hence

$$\frac{\partial \bar u^\alpha}{\partial \bar x^k} = \frac{\partial}{\partial x^m}\,[u^\alpha + a\,\eta^\alpha(x,\,u)]\;[\delta^m_k - a\,D_{x^k}\,\xi^m(x,\,u)] + O(a^2) =$$

$$= \frac{\partial u^\alpha}{\partial x^k} + a\,[D_k\,\eta^\alpha - u^\alpha_m D_k\,\xi^m] + O(a^2)$$

and the formulae of the first prolongation of IGF hold true.

Now we use the induction. Assume that (2.2.5) holds for all partial derivatives of the order $p \geq 2$. Then we have the following chain of equalities

$$\frac{\partial}{\partial \bar x^j}\frac{\partial^p \bar u^\alpha}{\partial \bar x^{i_1}\,\partial \bar x^{i_2}...\partial \bar x^{i_p}} = \frac{\partial}{\partial \bar x^j}\left[\frac{\partial^p u^\alpha}{\partial x^{i_1},\partial x^{i_2}...\partial x^{i_p}} + a\,\zeta^\alpha_{i_1,\,i_2,...i_p}\right] + O(a^2) =$$

$$= \frac{\partial}{\partial x^k}\left[\frac{\partial^p u^\alpha}{\partial x^{i_1}\,\partial x_{i_2}...\partial x^{i_p}} + a\,\zeta^\alpha_{i_1,\,i_2,...i_p}\right]\,\frac{\partial x^k}{\partial \bar x^j} + O(a^2) =$$

$$= \left[\frac{\partial^{p+1} u^\alpha}{\partial x^k,\,\partial x^{i_1},\,\partial x^{i_2}...\partial x^{i_p}} + a\,D_k\,\zeta^\alpha_{i_1,\,i_2,...i_p}\right]\;\left[\delta^k_j - a\,D_{x^j}\,\xi^k(x,\,u)\right] + O(a^2) =$$

$$= \frac{\partial^{p+1} u^\alpha}{\partial x_j,\, \partial x_{i_1}\, \partial x_{i_2} ... \partial x_{i_n}} + a \left[D_j\, \zeta^\alpha_{i_1,\, i_2, ... i_n} - u^\alpha_{k,\, i_1,\, ... i_n}\, D_j\, \xi^k \right] + O(a^2) =$$

$$= \frac{\partial^{p+1} u^\alpha}{\partial x^j\, \partial x^{i_1}\, \partial x^{i_2} ... \partial x^{i_p}} + a \zeta^\alpha_{j,\, i_1,\, i_2, ... i_p} + O(a^2).$$

Hence

$$\zeta^\alpha_{j,\, i_1,\, i_2, ... i_p} = D_j\, \zeta^\alpha_{i_1,\, i_2, ... i_p} - u^\alpha_{k,\, i_1,\, ... i_p}\, D_j\, \xi^k.$$

Exercise 2.2.1. Show that the equation

$$\frac{\partial u}{\partial x_1} + u \left(\frac{\partial u}{\partial x_2} \right)^3 = 0,$$

$u = u(x_1,\, x_2)$, admits the operators

$$X_1 = u\, \frac{\partial}{\partial u} - 3\, x_1 \frac{\partial}{\partial x_1},$$

$$X_2 = \frac{1}{\sqrt{u}}\, \frac{\partial}{\partial u},$$

$$X_3 = 3\, x_1\, \frac{\partial}{\partial x_1} + x_2\, \frac{\partial}{\partial x_2}.$$

2.3 Criterium of invariance. Splitting of defining equations

Let

$$F^\sigma(x,\, u, \partial\, u,\, ...\partial^r u) = 0, \qquad \sigma = 1,\, 2,\, ...s. \tag{2.3.1}$$

be a system of PDEs (or ODEs), such that:

$$rank \frac{\partial \left(F^1,\, F^2,\, ...F^s\right)}{\partial \left(x,\, u,\, \partial\, u, ...\partial^{\,r}\, u\right)} \Bigg|_{F^\sigma = 0} = s = const,$$

and let

$$\begin{aligned} &\bar{x}^k = f^k(x, u;\, a) = x^k + \, a\, \xi^k(x,\, u) + O(a^2), \;\; k = 1, ...n, \\ &\bar{u}^\alpha = g^\alpha(x,\, u;\, a) = u^\alpha + a\, \eta^\alpha(x,\, u) + O(a^2), \;\; \alpha = 1, ...m, \\ &\vdots \\ &\tfrac{\partial^r \bar{u}^\alpha}{\partial \bar{x_1}^{r_1} \partial \bar{x_n}^{r_n}} = u^\alpha_{r_1 ... r_n} + a \zeta^\alpha_{r_1 ... r_n} + O(a^2), \quad r = f_1 + ... + r_n, \end{aligned}$$

be r times prolonged 1-parameter group with the following IFG:

$$\hat{X} = \xi^k(x,\, u) \frac{\partial}{\partial\, x^k} + \eta^\alpha(x,\, u) \frac{\partial}{\partial\, u^\alpha} + \sum_\alpha \sum_{1 \le |J| \le r} \zeta^\alpha_J \frac{\partial}{\partial u^\alpha_J}$$

where $J = (j_1, ..., j_n),\;\; |J| = j_1 + ... j_n$.

Definition 2.3.1. *We say, that 2.3.1 admits this group if*

$$\hat{X}^{(r)} F^\sigma |_{F^\sigma = 0}.$$

According to the theorem, which gives us criterium of invariance of manifold, this condition is necessary and sufficient.

Example 2.3.1. Find out symmetry of the following equation:

$$u_1 + uu_2^3 = 0, \tag{2.3.2}$$

where $u = u(x_1,\, x_2)$, $u_k = \partial\, u/\partial\, x^k,\ k = 1, 2.$

a) We are looking for the IFG generator

$$\hat{X} = \xi^1(x_1,\, x_2,\, u)\frac{\partial}{\partial x_1} + \xi^2(x_1,\, x_2,\, u)\frac{\partial}{\partial x_2} + \eta(x_1,\, x_2,\, u)\frac{\partial}{\partial u}.$$

The first prolongation of the generator is as follows:

$$\hat{X}^{(1)} = \hat{X} + \zeta_1\,\frac{\partial}{\partial\, u_1} + \zeta_2\,\frac{\partial}{\partial\, u_2},$$

where

$$\zeta_1 = \eta_1 + \eta_u\, u_1 - u_1\left(\xi_1^1 + \xi_u^1\, u_1\right) - u_2\left(\xi_1^2 + \xi_u^2\, u_1\right),$$
$$\zeta_2 = \eta_2 + \eta_u\, u_2 - u_1\left(\xi_2^1 + \xi_u^1\, u_2\right) - u_2\left(\xi_2^2 + \xi_u^2\, u_2\right).$$

b) Applying the criterium of invariance, we obtain:

$$\hat{X}^{(1)}\left\{u_1 + u\, u_2^3\right\} = \zeta_1 + \eta\, u_2^3 + 3\,\zeta_2\, u\, u_2^2 = \eta_1 + \eta_u\, u_1 - u_1\left(\xi_1^1 + \xi_u^1\, u_1\right) - u_2\left(\xi_1^2 + \xi_u^2\, u_1\right) + \eta\, u_2^3 +$$
$$+ 3\, u\, u_2^2\left[\eta_2 + \eta_u\, u_2 - u_1\left(\xi_2^1 + \xi_u^1\, u_2\right) - u_2\left(\xi_2^2 + \xi_u^2\, u_2\right)\right].$$

c) Changing u_1 with $-u\, u_2^3$, we project the above expression on the manifold $M:\ \{u_1 = -uu_2^3\}$, after which the expression can be equated to zero:

$$\eta_1 - u\, u_2^3\, \eta_u + u\, u_2^3\left(\xi_1^1 - \xi_u^1\, u\, u_2^3\right) - u_2\left(\xi_1^2 - u\, u_2^3\, \xi_u^2\right) + \eta\, u_2^3 +$$
$$+ 3\, u\, u_2^2\left[\eta_2 + \eta_u\, u_2 + u\, u_2^3\left(\xi_2^1 + \xi_u^1\, u_2\right) - u_2\left(\xi_2^2 + \xi_u^2\, u_2\right)\right]. \tag{2.3.3}$$

Now, let us note that Eq. (2.3.3) contains the terms u_2^k, while the unknown functions $\xi_1,\ \xi_2,\ \eta$ depend only on the variables $x_1,\ x_2,\ u$. So the equation (2.3.3) can be treated as the polynimial in the variable u_2. But, like any other polynomial equation, (2.3.3) is equal to zero, if all the coefficients of different powers of u_2 are zero. Equating the coefficients of the corresponding powers $u_2^k,\ k = 0, 1, ...6$ we get the following system of linear PDEs:

u_2^6 :

$$-u^2\xi_u^1 + 3u(u\xi_u^1) = 0 \tag{2.3.4}$$

u_2^5 :

$$3u(u\xi_2^1) = 0 \tag{2.3.5}$$

u_2^4 :

$$u\xi_u^2 + 3u(-\xi_u^2) = 0 \tag{2.3.6}$$

u_2^3 :

$$-u\eta_u + u\xi_1^1 + \eta + 3u(\eta_u - \xi_2^2) = 0 \tag{2.3.7}$$

u_2^2 :

$$3u(\eta_2) = 0 \tag{2.3.8}$$

u_2^1 :

$$-\xi_1^2 = 0 \tag{2.3.9}$$

u_2^0 :

$$\eta_1 = 0 \tag{2.3.10}$$

Thus we've performed *the procedure of splitting* of the defining equation which, as a role, leads to the overdetermined system of PDEs. Now, employing all of the above equation but (2.3.7), we easily conclude that

$$\xi_1 = \xi_1(x_1), \quad \xi_2 = \xi_2(x_2), \quad \eta = \eta(u).$$

Inserting this into the equation (2.3.7) we obtain:

$$\eta(u) + 2u\dot{\eta}(u) = 3u\dot{\xi}^2(x_2) - u\dot{\xi}^1(x_1)$$

This implies:

$$\frac{\eta(u) + 2u\dot{\eta}(u)}{u} = C_1 = 3\,\dot{\xi}^2(x_2) - \dot{\xi}^1(x_1)$$

Solving the homogenous equation

$$\frac{d\,\tilde{\eta}(u)\,d\,u}{\tilde{\eta}(u)} = -\frac{1}{2u}$$

we obtain that $\tilde{\eta}(u) = C/\sqrt{u}$. Then inserting the ansatz $\eta = C(u)/\sqrt{u}$ in accordance with the *method of variation of constance* into the inhomogeneous equation

$$\frac{\eta(u) + 2u\dot{\eta}(u)}{u} = C_1$$

we obtain the following solution:

$$\eta = \frac{C_2}{\sqrt{u}} + \frac{1}{3}C_1 u \tag{2.3.11}$$

Sepatating variables in the equation $C_1 = 3\,\dot{\xi}^2(x_2) - \dot{\xi}^1(x_1)$, we get:

$$C_1 + \dot{\xi}^1(x_1) = 3\dot{\xi}^2(x_2) = 3C_3$$

Equating the second term to the third one we obtain, after one integration:

$$\xi^2(x_2) = C_4 + C_3 x_2. \tag{2.3.12}$$

Equating the first term to the third one, and integrating w.r.t x_1, we get:

$$\xi^1(x_1) = (3C_3 - C_1)x_1 + C_5. \tag{2.3.13}$$

Using the independence of the constants, and putting in (2.3.11)–(2.3.13) each time only one of them not equal to zero, we obtain the coordinates of five IFG admitted by (2.3.2). This way we get the following independent generators:

- $\hat{X}_1 = u\frac{\partial}{\partial u} - 3x_1\frac{\partial}{\partial x_1}$,
- $\hat{X}_2 = \frac{1}{\sqrt{u}}\frac{\partial}{\partial u}$,
- $\hat{X}_3 = 3x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}$,
- $\hat{X}_4 = \frac{\partial}{\partial x_2}$,
- $\hat{X}_5 = \frac{\partial}{\partial x_1}$.

Remark 2.3.1. The operators $\hat{X}_1, ..., \hat{X}_5$ form a closed Lie algebra with the following commutator table:

	X_1	X_2	X_3	X_4	X_5
X_1		$-3X_2/2$	0	0	0
X_2			0	0	0
X_3				$-X_4$	$-3X_5$
X_4					0
X_5					

Exercise 2.3.1.

1. Show that the operators $\hat{X}_4 = \frac{\partial}{\partial\, x_1}$, $\hat{X}_5 = \frac{\partial}{\partial\, x_2}$ are non-prolongable in the sense that $\hat{X}_k^{(1)} = \hat{X}_k^{(2)} = ... = \hat{X}_k^{(s)} = .. = \hat{X}_k$, $k = 4,\, 5$.
2. Find the first prolongation of the operators $\hat{X}_1, ...\hat{X}_3$ and show that Eq. (2.3.2) admits the corresponding transformation groups in the sense of definition 2.3.1.

Example 2.3.2. Let us find the symmetry of the second order ODE

$$u_{xx} = u^2, \tag{2.3.14}$$

where $u = u(x),\ x \in$ R. We are looking for the IFG generator

$$\hat{X} = \xi(x_1,\, u)\frac{\partial}{\partial x} + \eta(x,\, u)\frac{\partial}{\partial u}.$$

Since (2.3.14) is the second-order ODE, we should make the second prolongation of the generator $\hat{X}$, which is as follows:

$$\hat{X}^{(1)} = \hat{X} + \zeta_x\,\frac{\partial}{\partial\, u_x} + \zeta_{xx}\,\frac{\partial}{\partial\, u_{xx}},$$

where

$$\zeta_x = \eta_x + \eta_u\, u_x - u_x\left(\xi_x + \xi_u\, u_x\right),$$

$$\zeta_{xx} = D_x\left(\zeta_x\right) - u_{xx} D_x\left(\xi_x\right) = \eta_{xx} + \eta_{xu}\, u_x + u_{xx}\,\eta_u + u_x\left(\eta_{xu} + \eta_{uu}\, u_x\right) -$$
$$-u_{xx}\left(\xi_x + \xi_u\, u_x\right) - u_x\left[\xi_{xx} + \xi_{xu} u_x + u_{xx}\xi_u + u_x\left(\xi_{xu} + \xi_{uu}\, u_x\right)\right] - u_{xx}\left(\xi_x + \xi_u\, u_x\right).$$

The defining equation is then as follows:

$$\zeta_{xx} - 2\,u\eta|_{(2.3.14)} =$$
$$= \eta_{xx} + \eta_{xu}\, u_x + u^2\,\eta_u + u_x\left(\eta_{xu} + \eta_{uu}\, u_x\right) -$$
$$-u^2\left(\xi_x + \xi_u\, u_x\right) - u_x\left[\xi_{xx} + \xi_{xu} u_x + u^2\xi_u + u_x\left(\xi_{xu} + \xi_{uu}\, u_x\right)\right] - u^2\left(\xi_x + \xi_u\, u_x\right) - 2\,u\,\eta = 0.$$

The above equation contains the terms $u_x^k,\ k = 0,\, 1,\, 2,\, 3$ while the unknown functions $\xi,\ \eta$ depend only on the variables $x,\ u$. So, in order to satisfy the equation, we should equalize to zero the coefficients of the powers of u_x. as a result of applying the "splitting" procedure, we get the following system of defining equations:

u_x^3 :

$$-\xi_{uu} = 0, \tag{2.3.15}$$

u_x^2 :

$$\eta_{uu} - 2\,\xi_{xu} = 0, \tag{2.3.16}$$

u_x^1 :

$$2\,\eta_{xu} - \left[\xi_{xx} + 3\;u^2\,\xi_u\right] = 0, \tag{2.3.17}$$

u_x^0 :

$$\eta_{xx} + u^2\,\eta_u - 2\,u^2\,\xi_x - 2\,u\,\eta = 0. \tag{2.3.18}$$

Solving equation (2.3.15), we get

$$\xi = A(x) + u\,B(x).$$

Inserting ξ into the equation (2.3.16) and then integrating twice, we get

$$\eta = u^2\,\dot{B}(x) + u\,f(x) + g(x).$$

Inserting expression for η and ξ into (2.3.17) we obtain the equation polynomial in u:

$$2\left\{2\,u\,\ddot{B} + \dot{f}\right\} = 3\,u^2\,B + \ddot{A} + u\,\ddot{B}.$$

Equating to zero the coefficients of the different powers of u and solving differential equations appeared this way, we conclude that $B = 0$, while $f = \dot{A}/2 + C_1$, Thus, we obtain the formulae

$$\xi = A(x), \qquad \eta = u\left[C_1 + \frac{1}{2}\dot{A}\right] + g(x).$$

Inserting them into (2.3.18), we obtain the second order polynomial in u variable. Equating to zero the coefficients of different powers of u and solving the system obtained this way, we finally get the following expression for the IFG coordinates:

$$\xi = C_2 - \tfrac{2}{5}\,C_1\,x, \eta = \tfrac{4}{5}\,C_1\,u. \tag{2.3.19}$$

The independent generators corresponding to this solution are as follows:

$$\hat{X}_1 = 2\,u\tfrac{\partial}{\partial\,u} - x\,\tfrac{\partial}{\partial\,x},$$
$$\hat{X}_2 = \tfrac{\partial}{\partial\,x}.$$

Since the operator $\hat{X}_2$ is non-prolongable, it is evident that the equation (2.3.14) which is independent on x admits this generator. Now let us show that $\hat{X}_1$ is admitted by (2.3.14) as well. Acting on this equation with two times prolonged operator $\hat{X}_1^{(2)}$, we get:

$$\left\{4\,u_{xx}\tfrac{\partial}{\partial\,u_{xx}} + 3\,u_x\tfrac{\partial}{\partial\,u_x} + 2\,u\tfrac{\partial}{\partial\,u} - x\,\tfrac{\partial}{\partial\,x}\right\}\left(u_{xx} - u^2\right)|_{(2.3.14)} =$$
$$4\,\left(u_{xx} - u^2\right)|_{(2.3.14)} = 0.$$

Chapter 3

Applications to Ordinary Differential Equations

3.1 Symmetries and integrability of the first order ODEs

As it was mentioned in the introduction, ordinary differential equations and the problem of their integrability in quadratures served as a source of inspirations for the creator of the Lie groups theory. The use of the theory of the Lie groups of transformations in the case of ordinary differential equations is the most fruitful. For example, existence of a one-parameter symmetry group admitted by the $n-th$ order scalar ODEs implies the possibility of lowering the order of the equation by one. If the scalar equation of the n-th order admits an n-parameter symmetry group, then under certain algebraic conditions posed on the generators of that group, this equation is completely integrable in quadratures.

3.1.1 Application of the Lie groups in integrating scalar first-order ODEs

We begin the discussion on the relation between the symmetry and integrability, starting with the scalar first order ODEs. The algorithm of finding the generators of one-parameter groups admitted by the scalar first-order ODE does not differ from the general situation. Yet the family scalar first-order equations is *atypical*, since the use of the standard Lie algorithm described in the previous section is not effective in this case.

Let us demonstrate this. We'll consider the general first-order ODE in the form

$$u_x - F(x,\, u) = 0. \tag{3.1.1}$$

The symmetry group's generator will be sought in the following form:

$$\hat{X} = \xi(x,\, u)\frac{\partial}{\partial\, x} + \eta(x,\, u)\frac{\partial}{\partial\, u}.$$

Extending it one time, we obtain the operator

$$\hat{X}^{(1)} = \hat{X} + \zeta_x\,\frac{\partial}{\partial\, u_x} \qquad \zeta_x = \eta_x + u_x\,(\eta_u - \xi_x) - (u_x)^2\;\xi_u.$$

Acting then with this operator on the equation (3.1.1) and using the criterium of invariance, we get:

$$\hat{X}^{(1)}\,[u_x - F(x,\, u)]\;|_{(3.1.1)} = \eta_x + u_x\,[\eta_u - \xi_x - \xi_u\, u_x] - \xi\, F_x - \eta\, F_u\,|_{(3.1.2)} =$$

$$= \eta_x + F(x,\, u)\,[\eta_u - \xi_x - \xi_u\, F] - \xi\, F_x - \eta\, F_u = 0. \tag{3.1.2}$$

We see that the above equation does not include the powers of derivative u_x, since they have been removed during the procedure of projecting onto the manifold (3.1.1). So we face the situation when the integration of the "unsplitted" defining equation (3.1.2) is not easier than solving the source equation (3.1.1) *Therefore, in the case of first order ODEs the Lie theory is used in somewhat reversing order, namely, starting from the knowledge of symmetry rather than of its search.*

Putting aside the question of finding symmetry, let's concentrate on how can we use a known symmetry group for the purpose of obtaining solution. Two methods of integrating scalar first-order ODEs using symmetries are most often used.

The first method. Straightening the vector field. If the map $(x, u) \to (t, s)$ is a diffeomorphism, then the operator $\hat{X} = \xi(x,\, u)\frac{\partial}{\partial x} + \eta(x,\, u)\frac{\partial}{\partial u}$ in new variables will have the form

$$\hat{X}|_{(t,s)} = \hat{X}[t]\frac{\partial}{\partial t} + \hat{X}[s]\frac{\partial}{\partial s}.$$

Straightening of the vector field consists in posing the conditions

$$X[t] = 0, \qquad X[s] = 1. \tag{3.1.3}$$

If these conditions are fulfilled, then $\hat{X}|_{(t,s)} = \frac{\partial}{\partial s}$. Since the translation operator is non-prolongable, this means that a passage to new variables leads to the equation whose r.h.s is independent of s:

$$\frac{d\,s}{d\,t} = \tilde{F}(t). \tag{3.1.4}$$

Indeed, if the general scalar first-order ODE

$$s_t - F(t,\, s) = 0$$

admits $X = \frac{\partial}{\partial s} = X^{(1)}$, then

$$\partial_s \{s_t - F(t,\, s)\} = -F_s(t,\, s) = 0$$

hence $F = F(t)$.

The general solution to the equation (3.1.4) is obtained by one integration:

$$s = \int \tilde{F}(t)\, d\,t + C.$$

Example 3.1.1. Equation of the form

$$\frac{d\,u}{d\,x} = F\left(\frac{u}{x}\right)$$

admits the scaling group $\bar{x} = e^a\, x, \quad \bar{u} = e^a\, u$ with the IFG

$$\hat{X} = x\frac{\partial}{\partial\, x} + u\frac{\partial}{\partial\, u}.$$

Solving the equation (3.1.3) we get:

$$t = \frac{u}{x}, \qquad s = \log x.$$

Under such a change of variables the equation transforms into

$$\frac{d\,s}{d\,t} = \frac{1}{F(t) - t}.$$

Integrating this equation, we obtain the solution in the form of quadrature

$$s = C + \int \frac{d\,t}{F(t) - t}.$$

Let us assume that $F[z] = z + z^2$. Then the solution can be expressed explicitely as

$$s = C - \frac{1}{t}$$

In the initial variables it takes the form

$$u = \frac{x}{C - \log x}. \tag{3.1.5}$$

Method of integrating factor. The equation (3.1.1) can be presented (in many ways) in the form of *total differential equation*

$$P(x,\, u)\, d\,x + Q(x,\, u)\, d\,u = 0 \tag{3.1.6}$$

(called also *the Pfaffian form*) so that $F(x,\, y) = -P/Q$. The equation is called *exact* if

$$\frac{\partial\,P}{\partial\,u} = \frac{\partial\,Q}{\partial\,x}. \tag{3.1.7}$$

Fulfillment of this condition in some simply connected domain Ω implies that the equation (3.1.6) is a differential of some function $R(x,\, y)$. In this case the solution can be obtained in implicit form $R(x,\, y) = C$ by solving the equation

$$\frac{\partial\,R}{\partial\,x} = P, \qquad \frac{\partial\,R}{\partial\,u} = Q.$$

or taking of the both sides of the equation (3.1.6) the line integral *which is independent of the path of integration.*

If (3.1.6) is not exact, we can try to find out an *integrating factor* $\mu(x,\, u)$ such that when multiplied by μ the equation becomes exact.

Theorem 3.1.1. *Suppose that the equation*

$$\frac{d\,u}{d\,x} + \frac{P(x,\, u)}{Q(x,\, u)} = 0 \tag{3.1.8}$$

admits a one-parameter symmetry group with IFG $\hat{X} = \xi \frac{\partial}{\partial\,x} + \eta \frac{\partial}{\partial\,u}$. *Then the function*

$$\mu = \frac{1}{\xi\,P + \eta\,Q}$$

is an integrating factor for the (equivalent) Pfaffian form (3.1.6).

The proof of the theorem is left to the reader as an exercise. **Hint:** Apply the criterium of invariance to the equation (3.1.8) and then compare equation obtained to the equality

$$\frac{\partial}{\partial\,u}\left[\frac{P}{\xi\,P + \eta\,Q}\right] = \frac{\partial}{\partial\,x}\left[\frac{Q}{\xi\,P + \eta\,Q}\right].$$

Example 3.1.2. The equation

$$\frac{d\,u}{d\,x}=\left(\frac{u}{x}\right)^2+\left(\frac{u}{x}\right)$$

from the previous example can be presented in equivalent Pfaffian form

$$u(x+u)\,d\,x-x^2\,d\,u=0.$$

In accordance with the theorem 3.1.1, the function

$$\mu=\frac{1}{\xi P+\eta\,Q}=\frac{1}{x\,u^2}$$

is the integrating factor. Therefore the equation

$$\frac{1}{x\,u^2}\left[u(x+u)\,d\,x-x^2\,d\,u\right]=\frac{x+u}{x\,u}\,d\,x-\frac{x}{u^2}\,d\,u=0$$

is the total differential equation and therefore the line integral of the l.h.s **does not depend on the path of integration**. We shall reconstruct the function $R(x,\,u)$ by integration the l.h.s along the segment connecting the point $(1,\,1)$ with $(x,\,1)$ and next along the segment connecting the point $(x,\,1)$ with the point $(x,\,y)$. Thus, we obtain:

$$R(x,\,u)=\int_1^x\frac{s+1}{s}\,d\,s-\int_1^y\frac{x}{\tau^2}\,d\,\tau=\frac{x}{u}+\log x=C.$$

The result obtained coincides with the previous one (cf. with the formula (3.1.5)).

3.1.2 Inverse problem of the group analysis applied to scalar first order differential equations

In view of the fact that determining symmetry of a scalar first order ODE, generally speaking, is impossible, we turn around the problem and ask the complementary question: What is the most general type of scalar first order ODE which admits a given group as a group of symmetry?

Let us remind that a general scalar first order ODE admits a symmetry group G_a if and only if this equation treated as an expression defining the algebraic manifold in the jet space $(x,\,u,u_x)\in\mathrm{R}^3$, is invariant under the first prolongation $G_a^{(1)}$ of this group. We reformulate the definition of invariance into more appropriate terms

Definition 3.1.1. *Let $\varphi^1(x),\,\varphi^2(x),\,....,\varphi^k(x)$ be a smooth functions defined on a manifold M. Then*

(a) $\varphi^1(x),\,\varphi^2(x),\,....,\varphi^k(x)$ are called functionally dependent if for eqach $x\,\in\,M$ there is a neighborhood U of x and a smooth real-valued function $F(z_1,\,z_2,\dots,z_k)$ not identically zero on an open subset of R^k such that $F\left(\varphi^1(x),\,\varphi^2(x),\,....,\varphi^k(x)\right)=0$ for all $x\,\in\,U$.

(b) $\varphi^1(x),\,\varphi^2(x),\,....,\varphi^k(x)$ are called functionally independent if they are not functionally dependent.

Example 3.1.3. $\varphi^1(x,\,y)=\frac{x}{y}$ and $\varphi^2(x,\,y)=\frac{x\,y}{x^2+y^2}$ are functionally dependent on $\left\{(x,\,y)\,\in\mathrm{R}^2\,|_{y\neq 0}\right\}$ since

$$\frac{x\,y}{x^2+y^2}=\frac{\frac{x}{y}}{1+\left(\frac{x}{y}\right)^2}=\Phi\left(\frac{x}{y}\right).$$

Let us remind, that an algebraic manifold defined by the equation $F(x) = 0$, where $F(x)$ is the function acting from R^n to R is invariant under the action of a one-parameter group G_a with IFG $\hat{X} = \xi_k\,\partial_k$ if and only if $\hat{X}F(x)\,|_{F=0} = 0$. Before turning to the application of inverse problem of the group analysis to the scalar first-order ODEs, we need to formulate some facts concerning the invariants of differential operators.

Definition 3.1.2. *An arbitrary first integral $\varphi(x) = c$ of the system*

$$\frac{d\,x_1}{\xi_1(x)} = \frac{d\,x_2}{\xi_2(x)} = ... = \frac{d\,x_n}{\xi_n(x)} = d\,t, \tag{3.1.9}$$

associated with the differential operator $\hat{X} = \sum_{i=1}^{n} \xi_i(x)\,\frac{\partial}{\partial\,x_i}$, is called the invariant (or characteristics) of the operator $\hat{X}$.

Lemma 3.1.1. *If $\varphi(x) = c$ is an invariant of $\hat{X}$, then $\hat{X}\,\varphi(x) = 0$.*

Proof Differentiating the expression $\varphi(x) = c$ and taking advantage of (3.1.9), we obtain:

$$d\,\varphi(x) = \frac{\partial\,\varphi}{\partial\,x_i}\,d\,x_i = \frac{\partial\,\varphi}{\partial\,x_i}\,\xi_i\,d\,t = \hat{X}\,\varphi(x)\,d\,t = 0.$$

But this is possible if and only if $\hat{X}\,\varphi(x) = 0$.

Theorem 3.1.2. *([6], Ch. 5). Suppose that $\varphi^1(x) = c_1$, $\varphi^1(x) = c_1, ..., \varphi^{n-1}(x) = c_{n-1}$ is the (complete) set of independent invariants of the operator $\hat{X}$. Then the general solution of the equation*

$$\hat{X}\,z(x) = \sum_{i=1}^{n} \xi_i\,\frac{\partial}{\partial\,x_i}\,z(x) = 0$$

takes the form $z = \Phi\left(\varphi^1(x), \varphi^2(x), ..., \varphi^{n-1}(x)\right)$, where Φ is an arbitrary smooth function.

Corollary 3.1.1. Suppose that the functions $\varphi^1(x,\,u,\,u_x)$, $\varphi^2(x,\,u,\,u_x)$ are two independent solutions of the equation

$$\hat{X}^{(1)}\varphi(x,\,u,\,u_x) = \xi(x,\,u)\frac{\partial\,\varphi}{\partial\,x} + \eta(x,\,u)\frac{\partial\,\varphi}{\partial\,u} + \zeta_1(x,\,u,\,u_x)\frac{\partial\,\varphi}{\partial\,u_x} = 0,$$

where $\hat{X}^{(1)}$ is the first prolongation of the generator $\hat{X} = \xi(x,\,u)\frac{\partial}{\partial\,x} + \eta(x,\,u)\frac{\partial}{\partial\,u}$ of the one-parameter group of transformations G_a acting on R^2. Then any scalar first order ODE

$$\Delta(x,\,u,\,u_x) = 0$$

admits G_a if and only if there is an equivalent equation

$$\tilde{\Delta}(\varphi^1,\,\varphi^2) = 0$$

including only the independent invariants of the operator $\hat{X}^{(1)}$.

Example 3.1.4. Let us find the general form of the scalar first order ODE admitting G_a with the following IFG:

$$\hat{X} = \frac{\partial}{\partial\,x} + \frac{u}{x}\frac{\partial}{\partial\,u}. \tag{3.1.10}$$

The first prolongation of the operator $\hat{X}$ takes the form

$$\hat{X}^{(1)} = \hat{X} + \frac{x\,u_x - u}{x^2}\frac{\partial}{\partial\,u_x}.$$

Thus, in order to obtain the general outlook of equation admitting $\hat{X}^{(1)}$, we should solve the system

$$\frac{d\,x}{1} = \frac{d\,u}{\frac{u}{x}} = \frac{d\,u_x}{\frac{x\,u_x - u}{x^2}}.$$

Equating the first term to the second one, and solving the corresponding equation, we obtain the first (zero-order) invariant $\varphi_1 = u/x$. Equating the first term to the third term, using the zero-order invariant, and then solving the corresponding equation, we obtain the second independent invariant

$$\varphi_2 = \frac{u_x - \frac{u}{x}}{x}.$$

These invariants cannot be functionally depepndend since only one of them depends on u_x. So the general scalar first order ODE admitting the operator (3.1.10) can be expressed in the form

$$u_x = \frac{u}{x} + x\,\Phi\left(\frac{u}{x}\right).$$

Example 3.1.5. Let us find the general form of the scalar first order ODE admitting G_a with the following IFG:

$$\hat{X}_1 = u\frac{\partial}{\partial\,x}. \tag{3.1.11}$$

The first prolongation of the operator $\hat{X}_1$ takes the form

$$\hat{X}_1^{(1)} = \hat{X}_1 - u_x^2\,\frac{\partial}{\partial\,u_x}.$$

In order to obtain the general outlook of equation admitting $\hat{X}_1^{(1)}$, we should solve the system

$$\frac{d\,x}{u} = \frac{d\,u}{0} = \frac{d\,u_x}{-u_x^2}.$$

Equating the first term to the second term, and solving the corresponding equation, we obtain the first (zero-order) invariant $\varphi_1 = u$. Equating the first term to the third term, using the zero-order invariant, and then solving the corresponding equation, we obtain the second independent invariant

$$\varphi_2 = \frac{x}{u} - \frac{1}{u_x}.$$

The general scalar first order ODE admitting the operator (3.1.11) can be presented in the form

$$u_x = \frac{u}{x + \Phi(u)}.$$

Exercise 3.1.1. Find the general scalar first order ODE admitting G_a with the following IFG:

(a) $\hat{X} = x^2\frac{\partial}{\partial\,x} + x\,y\,\frac{\partial}{\partial\,y}$;

(b) $\hat{X} = x\,\frac{\partial}{\partial\,y}$;

(c) $\hat{X} = x\,y\,\frac{\partial}{\partial\,x}$.

3.2 Higher-order ODEs: lowering order using symmetry

Let us consider the scalar $n-th$ order ODE

$$F\left(x,\, u,\, u_x,\, ...,\, u^{(n)}\right) = 0. \tag{3.2.1}$$

The basic observation concerning Eq. (3.2.1) is following: If we know a one-parameter Lie group admitted by (3.2.1), then we can reduce the order by one. It can be done, e.g., by **straightening vector field** defined by the symmetry group generator

$$\hat{X} = \xi(x,\, u)\frac{\partial}{\partial\, x} + \eta(x,\, u)\frac{\partial}{\partial\, u}.$$

This can be accomplished by passing to such variables $(t,\, w)$ in which the operator takes on the form

$$\hat{X}|_{[t,\, w]} = \frac{\partial}{\partial\, w}.$$

The change of variables needed can be gained by solving the system

$$\hat{X}(t) = 0, \qquad \hat{X}(w) = 1.$$

After finding new variables, we express through them the old variables x, $u(x)$, $u'(x), ...\ u^{(n)}(x)$. As a result, we obtain the ODE of the same order. We assume that it can be represented in the form of solvable with respect to higher derivative:

$$\frac{d^n\, w}{d\, t^n} = G\left(t,\, w,\, \frac{d\, w}{d\, t},\, ...\frac{d^{n-1}\, w}{d\, t^{n-1}}\right). \tag{3.2.2}$$

But in new variables $\hat{X} = \frac{\partial}{\partial\, w}$ and, like any other operator with constant coefficients, it is non-prolongable, i.e. $\hat{X}^{(n)} = \hat{X}$ for all n. Using the main criterium of invariance we get:

$$\hat{X}^{(n)}\left[w^{(n)} - G\left(t,\, w,\, \tfrac{d\, w}{d\, t},\, ...\tfrac{d^{n-1}\, w}{d\, t^{n-1}}\right)\right]|_{(3.2.2)} =$$
$$= \tfrac{\partial}{\partial\, w}\left[w^{(n)} - G\left(t,\, w,\, \tfrac{d\, w}{d\, t},\, ...\tfrac{d^{n-1}\, w}{d\, t^{n-1}}\right)\right]|_{(3.2.2)} = -\tfrac{\partial\, G}{\partial\, w} = 0.$$

Hence G does not depend on w and, making change of variables $z = w'$, we obtain the equation of order $n-1$:

$$\frac{d^{n-1}\, z}{d\, t^{n-1}} = G\left(t,\, z\, \frac{d\, z}{d\, t},\, ...\frac{d^{n-2}\, z}{d\, t^{n-2}}\right).$$

Exercise 3.2.1. Consider equation

$$u_{xx} + p(x)\, u_x + q(x) u(x) = 0. \tag{3.2.3}$$

As a linear homogeneous equation, (3.2.3) admits the scaling group

$$\bar{x} = x, \qquad \bar{u} = e^a\, u$$

(show this). Pass to the corrdinates $(t,\, w)$ in which the generator of the scaling group takes the form $\hat{X}\,|_{(t,\, w)} = \partial\,/\partial\, w$ and show then that the equation can be reduced by further change of dependent variable $z = w_t$ to the Riccati-type first order ODE.

Another way of employing symmetry in case of higher-order ODE is to take advantage of the **differential invariants.**

Suppose we have a one-parameter group G_a

$$\bar{x} = f(x,\, u;\, a), \qquad \bar{u} = g(x,\, u;\, a)$$

with the IFG

$$\hat{X} = \xi(x,\, u)\,\frac{\partial}{\partial\, x} + \eta(x,\, u)\,\frac{\partial}{\partial\, u}.$$

Definition 3.2.1. *$n-th$ order invariant of G_a is any smooth function $\Phi(x,\, u,\, u',\, ...,\, u^{(n)})$ such that*

$$\hat{X}^{(n)}\Phi = 0,$$

where $\hat{X}^{(n)}$ is the $n-th$ prolongation of the generator $\hat{X}$.

Theorem 3.2.1. *([2],Ch.2). If $t = f(x,\, u',\, ...u^{(n)})$, $w = g(x,\, u',\, ...u^{(n)})$ are $n-th$ order independent invariants of G_a, then*

$$\frac{d\,w}{d\,t} = \frac{D_x\,w}{D_x\,t} \tag{3.2.4}$$

is an invariant of $n+1-th$ order.

Remark 3.2.1. If $t = f(x,\, u)$, $w = g(x,\, u,\, u')$ are independent zero-order and first-order invariants, correspondingly, then

$$\frac{d\,w}{d\,t} = \frac{D_x\,w}{D_x\,t}$$

is the second-order (independent) invariant,

$$\frac{d^2\,w}{d\,t^2} = \frac{D_x\,\frac{d\,w}{d\,x}}{D_x\,t}$$

is the third-order (independent) invariant, etc. Each subsequent invariant is independent, because it contains the derivative higher than any previous invariant.

Now we formulate the assertion which is essential in employing the method of differential invariants for lowering order of ODE.

Statement 3.2.1. *If G_a is a one-parameter group of transformations, then any $n-th$ order scalar ODE having G_a as a symmetry group is equivalent to a $(n-1)-th$ order equation*

$$\tilde{\Delta}\left(t,\, w,\, \frac{d\,w}{d\,t},\, ...\frac{d^{n-1}w}{d\,t^{n-1}}\right)$$

involving the invariants $t(x,\, u)$, $w(x,\, u,\, u')$ of the first prolongation $G_a^{(1)}$ and their derivatives of the order not higher than $n-1$.

Example 3.2.1. Let us consider equation

$$x^2\,u_{xx} + x\,(u_x)^2 - u\,u_x = 0. \tag{3.2.5}$$

Let us verify if the equation (3.2.5) admits the scaling transformations

$$\bar{x} = e^a\,x, \qquad \bar{u} = e^b\,u.$$

Inserting these transformations into the equation (3.2.5) written in new variables, we get:

$$e^b u_x + e^{2\,b-a} x\,(u_x)^2 - e^{2b-2a} u\, u_x = 0.$$

The above equation will be equivalent to the initial one if $b = a$, so (3.2.5) admits the scaling group with the generator

$$X = x\,\frac{\partial}{\partial\, x} + u\,\frac{\partial}{\partial\, u}.$$

Now we are going to lowering order of Eq. (3.2.5) using the method of differential invariants. It is easily see, that $X^{(1)} = X$. Indeed, $X^{(1)} = X + \zeta_x\,\frac{\partial}{\partial\, u_x,}$ but

$$\zeta_x = D_x(u) - u_x\, D_x(x) = 0.$$

So in order to obtain two independent first order invariants, it is necessary to solve the characteristic system

$$\frac{d\,x}{x} = \frac{d\,u}{u} = \frac{d\,u_x}{0},$$

having the independent invariants $t = u/x$, $w = u_x$. On virtue of the theorem 3.2.1 and the following remark, expression

$$\frac{d\,w}{d\,t} = \frac{D_x\,w}{D_x\,t} = \frac{x^2\,u_{xx}}{x\,u_x - u}$$

defines the independent second-order invariant of the symmetry group. Expressing u_{xx} from the above forula, we obtain:

$$u_{xx} = \frac{w-t}{x}\,\frac{d\,w}{d\,t}.$$

Now we return to the equation (3.2.5) and try to express it in new variables. Thus, we have:

$$\begin{aligned} 0 &= x^2\,u_{xx} + x\,(u_x)^2 - u\,u_x = x^2\,\tfrac{w-t}{x}\,\tfrac{d\,w}{d\,t} + xw^2 - x\,t\,w = \\ &= x\,\left\{x\tfrac{1}{x}(w-t)\tfrac{d\,w}{d\,t} + w^2 - t\,w\right\} = x\,(w-t)\,\left[\tfrac{d\,w}{d\,t} + w\right]. \end{aligned}$$

Assuming that $x \neq 0$ we get the equation expressed in terms of invariants of the scaling group operator:

$$(w-t)\,\left[\frac{d\,w}{d\,t} + w\right] = 0.$$

This equation has two families of solutions: [(a)] the singular solution $w = t$ and [(b)] the solution $w = C_1\,e^{-t}$. Returning to the initial variables, we obtain in the case [(a)] the solution $u = k\,x$.

In the case [(b)] we are to solve the equation

$$u_x = C_1\,e^{-\frac{u}{x}},$$

which, when the standard substitution $z = u/x$ is used, turns into the separable equation

$$x\,\frac{d\,z}{d\,x} = C_1\,e^{-z} - z,$$

which has the implicit solution

$$\log x + C_2 = \int \frac{d\,z}{C_1\,e^{-z} - z}, \qquad z = \frac{u}{x}.$$

Example 3.2.2. Let us consider the equation

$$u'' + u' - \frac{u}{x} = 0.$$

We are going to show that this equation admits the operator $X = x\frac{\partial}{\partial u}$, lower its order using the method of differential invariants, and find its general solution.

Solving the equation

$$X^{(1)}\Phi \equiv x\,\frac{\partial \Phi}{\partial u} + \frac{\partial \Phi}{\partial u'} = 0,$$

we can find out two independent first-order invariants

$$t = x, \qquad u' - \frac{u}{x} = w.$$

Using the theorem 3.2.1 and the remark 3.2.1, we calculate the second-order invariant

$$\frac{dw}{dt} = u'' - \frac{w}{t}$$

and get the separate variables equation

$$\frac{dw}{dt} + \left(\frac{1}{t} + 1\right) w = 0.$$

Integrating we obtain

$$w = \frac{C_1}{x}\, e^{-x}.$$

Passing to the initial variables we obtain the first-order linear ODE

$$u' - \frac{1}{x}\, u = \frac{C_1}{x}\, e^{-x},$$

which is solved by means of the *method of variation of constant*:

$$u = C_2\, x + C_1\, x \int \frac{dx}{x^2}\, e^{-x}. \tag{3.2.6}$$

Exercise 3.2.2. Show that Eq.

$$u_{xx} = (x - u)\,(u_x)^3$$

- admits G_a generated by $X = \frac{\partial}{\partial x} + \frac{\partial}{\partial u}$;
- lower the order of this equation using the method of differential invariants.

Exercise 3.2.3. Show that Eq.

$$u_{xx} - u - x^2 = 0$$

- admits G_a with the generator $X = e^x\,\frac{\partial}{\partial u}$;
- lower its order by passing to the coordinates $(t,\, s)$ in which the symmetry generator takes the form $X|_{t,s} = \frac{\partial}{\partial s}$ and solve the equation.

3.3 Integrating the second order scalar ODEs

3.3.1 One instructive example

Naive assumption: if one symmetry is needed to reduce a second-order ODE to the first-order ODE, then any second-order ODE possessing two symmetries is *completely* integrable. But the point is that we don't know if the reduced equation maintains the other symmetry operator as the local Lie group generator! To confirm our doubts or get rid of them, let us consider the equation (cf. with Example 3.2.2):

$$u_{xx} + u_x - \frac{u}{x} = 0. \tag{3.3.1}$$

Lemma 3.3.1. *The equation (3.3.1) admits*

(a) one-parameter Lie group

$$\bar{x} = x, \quad \bar{u} = u + a\,x$$

with the generator

$$\hat{X}_1 = x\,\frac{\partial}{\partial\,u} \tag{3.3.2}$$

(b) one-parameter Lie group

$$\bar{x} = x, \quad \bar{u} = e^b\,u$$

with the generator

$$\hat{X}_2 = u\,\frac{\partial}{\partial\,u} \tag{3.3.3}$$

We leave the proof of this lemma to the reader as an exercise.

Te first way of reduction: we use the symmetry operator (3.3.2) and construct with its help the change of variables

$$t = x, \qquad w = \frac{u}{x}$$

straightening the vector field. Indeed,

$$\hat{X}_1\,|_{t,\,w} = \hat{X}_1[t]\frac{\partial}{\partial\,t} + \hat{X}_1[w]\frac{\partial}{\partial\,w} = \frac{\partial}{\partial\,w}.$$

Let us calculate $u_x,\ u_{xx}$ in terms of t, w and the derivatives of w:

$$u_x = \tfrac{d}{d\,x}x\,w = w + x\,\tfrac{d\,w}{d\,t}\,\tfrac{d\,t}{d\,x} = w + t\,w_t;$$

$$u_{xx} = \tfrac{d}{d\,t}(w + t\,w_t)\,\tfrac{d\,t}{d\,x} = 2w_t + t\,w_{tt}.$$

Thus, we have:

$$0 = u_{xx} + u_x - \frac{u}{x} = 2w_t + t\,w_{tt} + = w + t\,w_t - w = t\,w_{tt} + (2 + t)\,w_t.$$

Making the substitution $z = w_t$, we obtain the separable equation which after the integration gives another separable equation

$$z = \frac{d\,w}{d\,t} = C_1\,\frac{e^{-t}}{t^2}.$$

Performing the second integration, we finally obtain the general solution in the form of quadrature:

$$w = C_2 + C_1 \int \frac{e^{-t}}{t^2}\,d\,t.$$

After the passage to the initial variables, we obtain the solution (cf. with (3.2.6)):

$$u = C_2\,x + C_1\,x \int \frac{e^{-x}}{x^2}\,d\,x.$$

Now let us calculate $\hat{X}_2$ in new variables:

$$\hat{X}_2 = u\,\frac{\partial}{\partial\,u} = \hat{X}_2\,(x)\,\frac{\partial}{\partial\,t} + \hat{X}_2\left(\frac{u}{x}\right)\frac{\partial}{\partial\,w} = w\,\frac{\partial}{\partial\,w}.$$

So the second symmetry generator will survive as a Lie group generator after the change of variables and we can use it to completely integrate the problem (yet it was not necessary in this particular case).

Te second way of reduction: Let us take the operator $\hat{X}_2 = u\,\frac{\partial}{\partial\,u}$ and use the method of differential invariants. For this purpose we consider the characteristic equation

$$\frac{d\,x}{0} = \frac{d\,u}{u} = \frac{d\,u_x}{u_x},$$

corresponding to the first prolongation of the operator $\hat{X}_2$:

$$\hat{X}_2^{(1)} = \hat{X}_2 + u_x\,\frac{\partial}{\partial\,u_x}.$$

Equating the firs two terms, we obtain the invariant $t = x$. Equation the second term to third term, we obtain the first order differential invariant $w = u_x/u$. As the third independent invariant we use the derivative

$$\frac{d\,w}{d\,t} = \frac{D_x\,w}{D_x\,t} = \frac{u\,u_{xx} - u_x^2}{u^2} \tag{3.3.4}$$

Now let us return to the equation (3.3.1). Dividing it by u, we obtain:

$$\frac{u_{xx}}{u} + \frac{u_x}{u} - \frac{1}{x} = \frac{d\,w}{d\,t} + w(1+w) - \frac{1}{t} = 0.$$

which is not separable equation. The question is: can we use the operator $\hat{X}_1$ in order to solve the above equation? The passage $(x,\,u) \to (t = x,\,w = u_x/x)$, leading to invariant variables involves the derivative u_x, so we should prolong X_1 one time to be able to calculate it in new variables:

$$\hat{X}_1^{(1)} = \hat{X}_1 + [D_x(x) - u_x\,D_x(0)]\,\frac{\partial}{\partial\,u_x} = x\,\frac{\partial}{\partial\,u} + \frac{\partial}{\partial\,u_x}.$$

Thus,

$$X_1^{(1)}|_{t,\,w} = \left(x\,\frac{\partial}{\partial\,u} + \frac{\partial}{\partial\,u_x}\right)[x]\,\frac{\partial}{\partial\,t} + \left(x\,\frac{\partial}{\partial\,u} + \frac{\partial}{\partial\,u_x}\right)\left[\frac{u_x}{x}\right]\frac{\partial}{\partial\,w} = \frac{1}{u}(1 - t\,w)\,\frac{\partial}{\partial\,w}.$$

Inverting the formula $w = \partial \log u/\partial x$, and taking into account that $t = x$, we can express the above operator in the form

$$X_1^{(1)}|_{t,\,w} = C\,e^{-\int w(t)\,d\,t}(1 - t\,w)\,\frac{\partial}{\partial\,w}.$$

This operator does not have the form of the local Lie group generator! **So not only the number of symmetry generators admitted is important.** In addition, one should pay attention to the structure of the Lie algebra used.

3.3.2 Remarks concerning Lie algebras

Definition 3.3.1. *Let AG_r be $n-$dimensional Lie algebra, and let N be a linear subspace of AG_n.*

[(a)] N is called a subalgebra if

$$[X,\,Y] \in N \;\forall\; X,\; Y \in N.$$

[(b)] N is called an ideal if

$$[X,\,Y] \in N$$

for any $X \in N$ and $\forall\; Y \in AG_n$.

if N is an idela then the equivalence relation can be introduced in AG_n, namely X is in relation with Y if $X - Y \in N$. The set of all operators equivalent to a given operator X is called the *coset* represented by X. Every element of this coset has the form $Y = X + Z$, where $Z \in N$.

The coset forms a Lie algebra called the **quotient algebra** *of AG_n with respect to the ideal N.* The quotient algebra is denoted by AG_n/N.

Theorem 3.3.1. *Suppose that a $m - th$-order equation*

$$\Delta(x,\,u,\,u',...,u^{(m)}) = 0 \tag{3.3.5}$$

admits the algebra AG_n of the symmetry generators and $N \subset AG_n$ (with $\dim(N) < \dim(AG_n)$) is an ideal. If we reduce the order of (3.3.5) using the infinitesimal symmetry generator belonging to N, then the lower-order ODE obtained this way admits the quotient-algebra AG_n/N, and the further symmetry reduction is possible

3.3.3 Classification of the second order scalar ODEs admitting two-dimensional Lie algebra

Any second-order ODE admitting a pair $X_1,\; X_2$ of symmetry groups' generators occurs to be completely integrable. To begin with, let us formulate the following statement:

Statement 3.3.1. *Suppose we have two-dimensional Lie algebra AG_2 with the generators $X_1,\; X_2$. Then it is possible, using the linear transformation*

$$\tilde{X}_1 = \alpha_{11}\,X_1 + \alpha_{12}\,X_2,$$
$$\tilde{X}_2 = \alpha_{21}\,X_1 + \alpha_{22}\,X_2,$$

to choose the basis in such a way that one of the following commutation relations holds: either

$$[\tilde{X}_1, \tilde{X}_2] = 0, \tag{3.3.6}$$

or

$$[\tilde{X}_1, \tilde{X}_2] = \tilde{X}_1. \tag{3.3.7}$$

Proof If the two dimension algebra is commutative, then (3.3.6) takes place regardless of a choice of the basic elements. So only non-commutative case is nontrivial. Let us assume that

$$[X_1,\, X_2] = \alpha_1\, X_1 + \alpha_2\, X_2.$$

and $\alpha_1 \neq 0$ (if $\alpha_1 = 0$, then $\alpha_2 \neq 0$, and our assumption will be fulfilled after renaming the operators $X_1 \to X_2$ $X_2 \to X_1$ and constants $\alpha_1 \to \alpha_2$, $\alpha_2 \to \alpha_1$). Under such a supposition, we introduce the new basis as follows:

$$\begin{cases} \tilde{X}_1 = \alpha_1\, X_1 + \alpha_2\, X_2, \\ \tilde{X}_2 = \frac{1}{\alpha_1}\, X_2. \end{cases}$$

Calculating the commutator, we get:

$$[\tilde{X}_1,\, \tilde{X}_2] = [\alpha_1\, X_1 + \alpha_2\, X_2,\, \tfrac{1}{\alpha_1}\, X_2] = \alpha_1\, X_1 + \alpha_2\, X_2 = \tilde{X}_1.$$

Thus, any two-dimensional Lie algebra AG_2 is either commutative or contains the one-dimensional ideal. In view of this remark, it becomes clear why in the example from the section 3.3.1 the reduction ends with success when we first use the operator $X_1 = x\,\frac{\partial}{\partial u}$ rather than $X_2 = u\,\frac{\partial}{\partial u}$. Indeed,

$$[X_1,\, X_2] = X_1$$

and therefore it is the operator X_1 that forms the ideal in the algebra $span\{X_1,\; X_2.\}$

Another essential characteristic of AG_2, along with the commutation relations, is the *pseudoscalar product.*

Definition 3.3.2. *For the generators*

$$X_1 = \xi_1 \frac{\partial}{\partial x} + \eta_1 \frac{\partial}{\partial u},$$

$$X_2 = \xi_2 \frac{\partial}{\partial x} + \eta_2 \frac{\partial}{\partial u}$$

the pseudoscalar product is the map given by the following formula:

$$X_1 \,\vee\, X_1 = \xi_1\, \eta_2 - \xi_2\, \eta_1 = \det \begin{bmatrix} \xi_1, & \eta_1 \\ \xi_2, & \eta_2 \end{bmatrix}. \tag{3.3.8}$$

Lemma 3.3.2. *Consider a non-degenerate change of variables*

$$t = t(x,\, u), \qquad w = w(x,\, u). \tag{3.3.9}$$

1. *The pseudoscalar product (3.3.8) transforms under the change of variables (3.3.9) as follows:*

$$X_1 \,\vee\, X_2\,|_{(t,\,w)} = \frac{\partial(t,\, w)}{\partial(x,\, u)} \cdot X_1 \,\vee\, X_2\,|_{(x,\,u)};$$

2. *the commutation relations remain the same.*

Proof Let us begin with the proof of the first statement: Suppose that

$$\hat{X}_k = \xi_k(x,\, u)\frac{\partial}{\partial\, x} + \eta_k(x,\, u)\frac{\partial}{\partial\, u}, \qquad k = 1,\, 2,$$

so that $X_1 \vee X_2|_{x,\,u} = \xi_1(x,\, u)\,\eta_2(x,\, u) - \xi_2(x,\, u)\,\eta_1(x,\, u)$ (we denote it for brevity as $\xi_1\,\eta_2 - \xi_2\,\eta_1$). In accordance with (1.4.3), the same operators in the new coordinates can be written as follows:

$$X_k\,|_{(t,\,u)} = X_k[t]\frac{\partial}{\partial\, t} + X_k[w]\frac{\partial}{\partial\, w} := \tilde{\xi}_k\frac{\partial}{\partial\, t} + \tilde{\eta}_k\frac{\partial}{\partial\, w} \qquad k = 1,\, 2.$$

So in new coordinates

$$X_1 \vee X_2\,|_{(t,w)} = \tilde{\xi}_1\,\tilde{\eta}_2 - \tilde{\xi}_2\,\tilde{\eta}_1 = X_1[t]\,X_2[w] - X_2[t]\,X_1[w] =$$

$$= \left(\xi_1\,\frac{\partial\, t}{\partial\, x} + \eta_1\,\frac{\partial\, t}{\partial\, u}\right)\left(\xi_2\,\frac{\partial\, w}{\partial\, x} + \eta_2\,\frac{\partial\, w}{\partial\, u}\right) - \left(\xi_2\,\frac{\partial\, t}{\partial\, x} + \eta_2\,\frac{\partial\, t}{\partial\, u}\right)\left(\xi_1\,\frac{\partial\, w}{\partial\, x} + \eta_1\,\frac{\partial\, w}{\partial\, u}\right) =$$

$$= \xi_1\,\eta_2\left(\frac{\partial\, t}{\partial\, x}\frac{\partial\, w}{\partial\, u} - \frac{\partial\, t}{\partial\, u}\frac{\partial\, w}{\partial\, x}\right) + \xi_2\,\eta_1\left(\frac{\partial\, t}{\partial\, u}\frac{\partial\, w}{\partial\, x} - \frac{\partial\, t}{\partial\, x}\frac{\partial\, w}{\partial\, u}\right) +$$

$$+\xi_1\,\xi_2\left(\frac{\partial\, t}{\partial\, x}\frac{\partial\, w}{\partial\, x} - \frac{\partial\, t}{\partial\, x}\frac{\partial\, w}{\partial\, x}\right) + \eta_1\,\eta_2\left(\frac{\partial\, t}{\partial\, u}\frac{\partial\, w}{\partial\, u} - \frac{\partial\, t}{\partial\, u}\frac{\partial\, w}{\partial\, u}\right) =$$

$$= (\xi_1\,\eta_2 - \xi_2\,\eta_1)\,\frac{\partial\,(t,w)}{\partial\,(x,u)} = \frac{\partial(t,w)}{\partial(x,u)}\cdot X_1\,\vee\,X_2\,|_{(x,\,u)}.$$

Now let us address the second statement. We give the proof for more general situation when we have a set of operators

$$\left\{X_k = \sum_{j=1}^{n}\xi_k^j\,\frac{\partial}{\partial\, x^j}\right\}_{k=1}^{s}$$

acting in n-dimensional space and forming the Lie algebra with the commutation relations

$$[X_k,\, X_m] = C_{k\,m}^r X_r. \tag{3.3.10}$$

From the definition of the Lie bracket, we can write down the LHS of the formula (3.3.10) in explicit form as follows:

$$[X_k,\, X_m] = \sum_{i=1}^{n}\left\{X_k\left[\xi_m^i\right] - X_m\left[\xi_k^i\right]\right\}\,\frac{\partial}{\partial\, x^i}.$$

Now suppose that $y = F(x)$ is a diffeomorphic function mapping an open set $U\,\subset\,R^n$ into R^n, and we wish to calculate the commutation relations in new coordinates. We again take advantage of (1.4.3) and obtain:

$$\textstyle\sum_{i,j=1}^{n}\left\{X_k\left[\xi_m^i\right] - X_m\left[\xi_k^i\right]\right\}\,\frac{\partial\, y^j}{\partial\, x^i}\,\frac{\partial}{\partial\, y^j} = [X_k,\, X_m]\,|_{(y)} =$$

$$= \textstyle\sum_{r=1}^{s}\,\sum_{q,j=1}^{n}\,C_{k\,m}^r\,\xi_r^q\,\frac{\partial\, y^j}{\partial\, x^q}\,\frac{\partial}{\partial\, y^j} = \sum_{r=1}^{s}\,\sum_{j=1}^{n}\,C_{k\,m}^r\,X_r\left[y^j\right]\,\frac{\partial}{\partial\, y^j} = \sum_{r=1}^{s}\,C_{k\,m}^r X_r\,|_{(y)}.$$

There remains to check if $X_k\,|_y\,X_m\,|_y - X_m\,|_y\,X_k\,|_y$ coincides with the already obtained formula for $[X_k,\, X_m]|_y$. Thus we have:

$$X_k\,|_y\,X_m\,|_y = \xi_k^i\frac{\partial\, y^j}{\partial\, x^i}\,\frac{\partial}{\partial\, y^j}\xi_m^r\,\frac{\partial\, y^p}{\partial\, x^r}\,\frac{\partial}{\partial\, y^p} =$$

$$= \xi_k^i\left[\frac{\partial\, y^j}{\partial\, x^i}\left(\frac{\partial}{\partial\, y^j}\xi_m^r\right)\frac{\partial\, y^p}{\partial\, x^r} + \frac{\partial\, y^j}{\partial\, x^i}\,\xi_m^r\,\frac{\partial}{\partial\, y^j}\,\frac{\partial\, y^p}{\partial\, x^r}\right]\,\frac{\partial}{\partial\, y^p} =$$

$$= \xi_k^i\left[\left(\frac{\partial}{\partial\, x^i}\,\xi_m^r\right)\,\frac{\partial\, y^p}{\partial\, x^r} + \xi_m^r\,\frac{\partial^2\, y^p}{\partial\, x^i\partial\, x^r} + \xi_m^r\,\frac{\partial\, y^j}{\partial\, x^i}\,\frac{\partial\, y^p}{\partial\, x^r}\,\frac{\partial}{\partial\, y^j}\right]\,\frac{\partial}{\partial\, y^p} =$$

$$= (X_k\,\xi_m^r)\,\frac{\partial\, y^p}{\partial\, x^r}\,\frac{\partial}{\partial\, y^p} + \xi_k^i\,\xi_m^r\left[\frac{\partial^2\, y^p}{\partial\, x^i\partial\, x^r} + \frac{\partial\, y^j}{\partial\, x^i}\,\frac{\partial\, y^p}{\partial\, x^r}\,\frac{\partial}{\partial\, y^j}\right]\,\frac{\partial}{\partial\, y^p}.$$

In a completely analogous way we obtain the formula for the inverse sequence of the operators:

$$X_m\,|_y\, X_k\,|_y = (X_m\, \xi_k^r)\, \frac{\partial\, y^p}{\partial\, x^r}\, \frac{\partial}{\partial\, y^p} + \xi_m^i\, \xi_k^r \left[\frac{\partial^2\, y^p}{\partial\, x^i \partial\, x^r} + \frac{\partial\, y^j}{\partial\, x^i}\, \frac{\partial\, y^p}{\partial\, x^r}\, \frac{\partial}{\partial\, y^j} \right] \frac{\partial}{\partial\, y^p}.$$

Subtracting the second equality from the first one, changing the summation indices in the second and third terms and taking into account the equality of the mixed derivatives, we obtain

$$X_k\,|_y\, X_m\,|_y - X_m\,|_y\, X_k\,|_y = [(X_k\, \xi_m^r) - (X_m\, \xi_k^r)]\, \tfrac{\partial\, y^p}{\partial\, x^r}\, \tfrac{\partial}{\partial\, y^p} = [X_k,\, X_m]\,|_y.$$

Exercise 3.3.1. Consider the operators

$$X_1 = \frac{\partial}{\partial\, x^1}, \qquad X_2 = x^1\, \frac{\partial}{\partial\, x^1} + x^2\, \frac{\partial}{\partial\, x^2}.$$

- Calculate the commutator of these operators and make sure that they form a Lie algebra.
- Write down the operators in the coordinates $z^1 = (x^1)^3, \quad z^2 = x^1\, x^2$ and calculate their commutator in new representation.
- Compare the results obtained and verify the second statement of the lemma 3.3.2.

In order to perform classification, we need and extra statements. Let us return to the change of basic elements. Suppose that we've made a change of basis

$$\begin{cases} X_1' = \alpha_1\, X_1 + \alpha_2\, X_2, \\ X_2' = \beta_1\, X_1 + \beta_2\, X_2. \end{cases}$$

Lemma 3.3.3. *In ordr that $X_1',\ X_2'$ be linearly independent, it is necessary and sufficient that*

$$\Delta = \alpha_1\, \beta_2 - \alpha_2\, \beta_1\, \neq\, 0. \tag{3.3.11}$$

The proof is based on the general statements of the linear algebra.

Lemma 3.3.4. *The following formulae take place*

$$[X_1',\, X_2'] = \Delta\, [X_1,\, X_2]. \tag{3.3.12}$$

$$X_1'\, \vee\, X_2' = \Delta\, \left(X_1\, \vee\, X_2\right). \tag{3.3.13}$$

Proof The proof is left it to the reader as an easy homework exercise

Corollary 3.3.1. The relations $X_1 \vee X_2 = 0$ $(X_1 \vee X_2\, \neq\, 0)$ remain the same both under the change of the basic elements and the invertible change of variables $(x,\, u) \rightleftharpoons (t,\, w)$.

Following statement summarizes the above results:

Theorem 3.3.2. *Any two-dimensional Lie algebra AG_2 acting on R^2 can be reduced by choosing appropriate basis $X_1,\ X_2$ to one of the four distinct types:*

1.

$$[X_1,\, X_2] = 0, \qquad X_1\, \vee\, X_2\, \neq\, 0; \tag{3.3.14}$$

2.

$$[X_1,\,X_2]=0,\qquad X_1\,\vee\,X_2\,=0; \tag{3.3.15}$$

3.

$$[X_1,\,X_2]=X_1,\qquad X_1\,\vee\,X_2\,=0; \tag{3.3.16}$$

4.

$$[X_1,\,X_2]=X_1,\qquad X_1\,\vee\,X_2\,\neq\,0. \tag{3.3.17}$$

The structure relations (1)-(4) are invariant under the change of variables (3.3.9).

The above theorem serves as the basis for classifying all two-dimensional Lie algebras, as is summarized below:

Theorem 3.3.3. *(canonical representations of the Lie algebras AG_2)*

1. *If* $[X_1,\,X_2]=0$ *and* $X_1\vee X_2\neq 0$ *then there exists a change of variables* $(x,\,u)\to(t,w)$ *such that*

$$X_1=\frac{\partial}{\partial\,w},\quad X_2=\frac{\partial}{\partial\,t}. \tag{3.3.18}$$

2. *If* $[X_1,\,X_2]=0$ *and* $X_1\vee X_2\,=\,0$ *then there exists a change of variables* $(x,\,u)\to(t,w)$ *such that*

$$X_1=\frac{\partial}{\partial\,w},\quad X_2=t\,\frac{\partial}{\partial\,w}. \tag{3.3.19}$$

3. *If* $[X_1,\,X_2]=X_1$ *and* $X_1\,\vee\,X_2\,=\,0$ *then there exists a change of variables* $(x,\,u)\to(t,w)$ *such that*

$$X_1=\frac{\partial}{\partial\,w},\quad X_2=w\,\frac{\partial}{\partial\,w}. \tag{3.3.20}$$

4. *If* $[X_1,\,X_2]=X_1$ *and* $X_1\,\vee\,X_2\,\neq\,0$ *then there exists a change of variables* $(x,\,u)\to(t,w)$ *such that*

$$X_1=\frac{\partial}{\partial\,s},\quad X_2=t\,\frac{\partial}{\partial\,t}+w\,\frac{\partial}{\partial\,w}. \tag{3.3.21}$$

Proof

We begin the classification with the two-dimensional commutative algebras. Making the change of variables $(x,\,u)\to(\tau,\,s)$, straightening X_1, we get the representation

$$X_1=\frac{\partial}{\partial\,s},\qquad X_2=a(\tau,s)\frac{\partial}{\partial\,w}+b(\tau,w)\frac{\partial}{\partial\,\tau}.$$

Using the commutative relation $[X_1,\,X_2]=0$, we conclude that $a_s=b_s=0$, in other words,

$$X_2=a(\tau)\frac{\partial}{\partial\,s}+b(\tau)\frac{\partial}{\partial\,\tau}.$$

Next we perform the change of variables $w = s + h(\tau), \quad t = t(\tau)$ which leads to the representation

$$X_1\,|_{(t,\,w)} = \tfrac{\partial}{\partial\,w},$$

$$X_2\,|_{(t,\,w)} = a(\tau)\tfrac{\partial}{\partial\,w} + b(\tau)\left[h'(\tau)\tfrac{\partial}{\partial\,w} + t'(\tau)\tfrac{\partial}{\partial\,t}\right] = [a(\tau) + b(\tau)\,h'(\tau)]\,\tfrac{\partial}{\partial\,w} + b(\tau)\,t'(\tau)\,\tfrac{\partial}{\partial\,t}.$$

If $b(\tau) = 0$, then putting $t = a$, we obtain the operators

$$X_1 = \frac{\partial}{\partial\,w}, \qquad X_2 = t\,\frac{\partial}{\partial\,w}. \tag{3.3.22}$$

If, in turn, $b \neq 0$, then we put $t' = 1/b$, $h' = -a/b$ and obtain the representation

$$X_1 = \frac{\partial}{\partial\,w}, \qquad X_2 = \frac{\partial}{\partial\,t}. \tag{3.3.23}$$

Now let us address the non-commutative case $[X_1,\,X_2] = X_1$. We again use the change of coordinates $(x,\,u) \to (\tau,\,s)$ "straightening" the operator X_1, such that

$$X_1\,|_{(\tau,\,s)} = \frac{\partial}{\partial\,s}, \qquad X_2\,|_{(\tau,\,s)} = a(\tau,\,s)\,\frac{\partial}{\partial\,\tau} + b(\tau,\,s)\,\frac{\partial}{\partial\,s}.$$

From the commutating relations we have:

$$[X_1,\,X_2] = a_s\,\frac{\partial}{\partial\,\tau} + b_s\,\frac{\partial}{\partial\,s} = \frac{\partial}{\partial\,s}.$$

Hence, $a = a(\tau)$, while $b = s + \beta(\tau)$. First we consider the case $X_1 \vee X_2 = -a = 0$. Let us make the change of coordinates $t = \tau,\;\; w = b = s + \beta(\tau)$. In new coordinates the generators are expressed as follows:

$$X_1 = \frac{\partial}{\partial\,s}[t]\frac{\partial}{\partial\,t} + \frac{\partial}{\partial\,s}[w]\frac{\partial}{\partial\,w} = \frac{\partial}{\partial\,w},$$

$$X_2 = [s + \beta(\tau)]\tfrac{\partial}{\partial\,s}[t]\tfrac{\partial}{\partial\,t} + [s + \beta(\tau)]\tfrac{\partial}{\partial\,s}[w]\tfrac{\partial}{\partial\,w} = w\,\tfrac{\partial}{\partial\,w},$$

so our goal is achieved and the result is true for any function $\beta(\tau)$, which it is convenient to set equal to zero.

Let us consider now the case $X_1 \vee X_2 = -a \neq 0$. We are going to find the change of variables $t = t(\tau)$, $w = w(\tau,\,s)$ such that the generators take the form (3.3.21). Calculating the first generator in new variables we have:

$$X_1 = \tfrac{\partial}{\partial\,s}[w(\tau,\,s)]\tfrac{\partial}{\partial\,w} = \tfrac{\partial}{\partial\,w}.$$

From this we conclude that $w = s + \gamma(\tau)$. Calculating the second generator and equating it to the RHS of Eq. (3.3.21), we obtain:

$$X_2 = [a(\tau)\tfrac{\partial}{\partial\,\tau} + (s + \beta(\tau))\tfrac{\partial}{\partial\,s}][t(\tau)]\tfrac{\partial}{\partial\,t} + +[a(\tau)\tfrac{\partial}{\partial\,\tau} + (s + \beta(\tau))\tfrac{\partial}{\partial\,s}][s + \gamma(\tau)]\tfrac{\partial}{\partial\,w} = t\tfrac{\partial}{\partial\,t} + w\tfrac{\partial}{\partial\,w} =$$

$$= t(\tau)\tfrac{\partial}{\partial\,t} + [s + \gamma(\tau)]\tfrac{\partial}{\partial\,w}.$$

Equating the coefficients of $\partial/\partial\,t$ we get the equation

$$a\,\dot{t}(\tau) = t(\tau),$$

having the solution

$$t = C_1\,\exp\left\{\int \frac{d\tau}{a(\tau)}\right\}.$$

Equating the coefficients of $\partial/\partial w$ we obtain the equation

$$a\,\gamma' - \gamma = -\beta,$$

In accordance with the *method of variation of constant*, we are looking first for the solution of the associated homogeneous equation

$$a\,\tilde{\gamma}' - \tilde{\gamma} = 0,$$

which is satisfied by the function

$$\tilde{}\gamma = Ce^{\int \frac{d\,\tau}{a(\tau)}}.$$

Now looking for the solution to the inhomogeneous equation in the form

$$\gamma = C(\tau)e^{\int \frac{d\,\tau}{a(\tau)}},$$

we obtain the following equation for $C(\tau)$:

$$C'(\tau) = -\frac{\beta(\tau)}{a(\tau)}\,e^{-\int \frac{d\,\tau}{a(\tau)}}$$

This equation has the solution

$$C(\tau) = C_2 - \int \frac{\beta(\tau)}{a(\tau)}\,e^{-\int \frac{d\,\tau}{a(\tau)}}\,d\,\tau,$$

so finally we obtain:

$$\gamma = \exp\left\{\int \frac{d\,\tau}{a(\tau)}\right\}\left[C_2 - \int\left\{\frac{\beta(\tau)}{a(\tau)}\,\exp\left(-\int \frac{d\,\tau}{a(\tau)}\right)\right\}\,d\,\tau\right].$$

Thus,

$$t = C_1\,\exp\left\{\int \frac{d\tau}{a(\tau)}\right\},\qquad C_1\,\neq\,0,$$

$$w = s + \exp\left\{\int \frac{d\,\tau}{a(\tau)}\right\}\left[C_2 - \int\left\{\frac{\beta(\tau)}{a(\tau)}\,\exp\left(-\int \frac{d\,\tau}{a(\tau)}\right)\right\}\,d\,\tau\right]$$

and with this substitution we achieve our goal.

Now we are ready to classify all the second-order ODEs admitting the two-dimensional Lie algebras.

Theorem 3.3.4. *The following statements hold true:*

1. *The most general scalar second-order ODE admitting the Lie algebra* $X_1 = \frac{\partial}{\partial\,w},\quad X_2 = \frac{\partial}{\partial\,t}$ *has the form*

$$w'' = F(w'). \tag{3.3.24}$$

2. *The most general scalar second-order ODE admitting the Lie algebra* $X_1 = \frac{\partial}{\partial\,w},\quad X_2 = t\,\frac{\partial}{\partial\,w}$*, has the form*

$$w'' = G(t). \tag{3.3.25}$$

3. *The most general scalar second-order ODE admitting the Lie algebra* $X_1 = \frac{\partial}{\partial w}, \quad X_2 = w\,\frac{\partial}{\partial w}$, *has the form*

$$w'' = w'\,L(t). \tag{3.3.26}$$

4. *The most general scalar second-order ODE admitting the Lie algebra* $X_1 = \frac{\partial}{\partial s}, \quad X_2 = t\,\frac{\partial}{\partial t} + w\,\frac{\partial}{\partial w}$, *has the form*

$$w'' = \frac{H(w')}{t}. \tag{3.3.27}$$

Proof We begin our proof by remarking that a general second-order equation that is solvable with respect to the second derivative can be written in the form

$$w'' - F\,(t,\,w,\,w') = 0.$$

Since in all of the cases the second-order ODE is assumed to admits the non-prolongable operator $X_1 = \partial/\partial\,w$, then the following equation holds:

$$\frac{\partial}{\partial\,w}\left[w'' - F\,(t,\,w,\,w')\right] = -F_w\,(t,\,w,\,w') = 0.$$

From this we conclude that the most general second-order ODE admitting $X_1 = \partial/\partial\,w$ takes the form

$$w'' - F\,(t,\,w') = 0. \tag{3.3.28}$$

In the first case the second-order ODE admits in addition a non-prolongable operator $X_2 = \frac{\partial}{\partial t}$, hence

$$\frac{\partial}{\partial\,t}\left[w'' - F\,(t,\,w')\right] = -F_t\,(t,\,w') = 0.$$

So the most general equation admitting both the operator $\partial\,w$ and $\partial\,t$ has the form (3.3.24).

In the second case the equation admits in addition the operator $X_2 = t\,\frac{\partial}{\partial w}$. Acting with two times prolonged operator on the equation, we have:

$$\left[\frac{\partial}{\partial\,w} + \frac{\partial}{\partial\,w_t}\right]\,\left[w'' - F\,(t,\,w')\right] = -F_{w_t}\,(t,\,w') = 0.$$

So the most general equation admitting both the operator $\partial\,w$ and $t\,\partial\,w$ has the form (3.3.25).

In the third case the equation admits in addition the operator $X_2 = w\,\frac{\partial}{\partial w}$. Acting with two times prolonged operator on the equation, we have:

$$\left[w\frac{\partial}{\partial\,w} + w_t\frac{\partial}{\partial\,w_t} + w_{tt}\frac{\partial}{\partial\,w_{tt}}\right]\,\left[w'' - F\,(t,\,w')\right]|_{(3.3.28)} = F - w_t\,F_{w_t} = 0.$$

Solving the rightmost equation, we obtain that $F = w_t L(t)$ and the most general equation admitting both of the operators has the form (3.3.26).

In the last case the equation admits in addition the operator $X_2 = t\,\frac{\partial}{\partial t} + w\,\frac{\partial}{\partial w}$. Acting with two times prolonged operator $X_2^{(2)}$ on the equation, we have:

$$\left[t\frac{\partial}{\partial\,t} + w\frac{\partial}{\partial\,w} - w_{tt}\frac{\partial}{\partial\,w_{tt}}\right]\,\left[w'' - F\,(t,\,w')\right]|_{(3.3.28)} = -F - t\,F_t = 0$$

Solving the rightmost equation, we obtain that $F = \frac{1}{t}H(w')$ and the most general equation admitting both of the operators has the form (3.3.27).

Remark 3.3.1. In all the cases enumerated in the above theorem there exist strategies enabling to find the general solutions to corresponding equations in the form of quadratures (see e.g. [4]).

Bibliography

[1] Bluman G., Kumei S., *Symmetries and differential eqations*, Springer, NY 1989.

[2] Olver P., *Application of Lie groups to differential equations*, Springer, NY, 1994.

[3] Ibragimov N., *Selected works, Vol.1, Paper 21, Ch. 1,2.*

[4] Stephani H., *Differential equations: their solutions using symmetryies*, Cambridge Univ. Press, NY, 1989.

[5] Barenblatt G., *Similarity, Self-Similarity and Intermediate Asymptotics*, Academic Press, NY, 1996.

[6] Elsgolts L., *Differential Equations and the Calculus of Variables,* Mir Publ., Moscow, 1977.