Compacton-like solutions to some nonlocal hydrodynamic-type model[1]

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Abstract. We study the appearance of compacton-like solutions within the hydrodynamic-type model taking into account effects of spatial non-locality

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1 Introduction

In this paper evolutionary PDEs describing wave patterns with compact support are studied. Very often wave patterns play key rules in nonlinear transport phenomena [1, 2, 3]. One of the most advanced mathematical theory dealing with wave patterns' formation and evolution is the soliton theory. The well-known Korteveg – de Vries equation (KdV)

$$u_t + \beta\, u\, u_x + u_{xxx} = 0, \tag{1}$$

possesses a one-parameter family of solutions describing exponentially localized wave patterns called *solitons* [4]:

$$u(t,\ x) = \frac{12\, a^2}{\beta}\, sech^2\big[a(x - 4\, a^2\, t)\big]. \tag{2}$$

Solitons demonstrate many outstanding features. Being the solutions to non-linear evolutionary equation (1), they manifest "elastic properties" during the collisions. Besides, any sufficiently smooth initial data possessing a finite energy norm creates a chain of solitons moving with different velocities. All this is recognized as a consequence of the complete integrability of equation (1) and existence of infinite hierarchy of conservation laws.

Actually the solitons have been associated with a number of phenomena observed in science and technics. However, solution (2) cannot be fully identified with any

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kind of solitary waves occurring in natural phenomena since the wave pattern described by this formula at fixed t extends to $\pm\infty$.

In recent years there have been discovered another type of solitary waves supported by the following generalization of the KdV hierarchy [5]:

$$K(m,\,n) = u_t + \left(u^m\right)_x + \left(u^n\right)_{xxx} = 0, \quad m,\; n\,\geq\,2. \tag{3}$$

In case when $m = 2,\; n = 2$ a generalized solution to this equation is described by the following formula [5]:

$$u(t,\,x) = \begin{cases} \frac{4\,D}{3}\,\cos^2\left[(x - D\,t)/4\right] \text{ when } |x - D\,t| \leq 2\,\pi, \\ 0 \text{ when } |x - D\,t| \geq\,2\,\pi. \end{cases} \tag{4}$$

Solutions of this sort are rather typical to the whole $K(m,\,n)$ hierarchy. All of them vanish outside some compact domain and hence they are referred to as *compactons*. Study of $K(m,\,n)$ hierarchy is actually in progress and there has already been realized that most of these equations are not completely integrable and do not possess an infinite set of conservation laws. Nevertheless, compacton-supporting equations inherit many features of equations belonging to the KdV hierarchy. In particular, a sufficiently smooth perturbations taken as Cauchý data give rise to a chain of compactons [5, 6, 7, 8]. Besides, compactons manifest almost perfectly elastic features during the mutual collisions [5, 7].

It should be emphasized, yet, that most papers dealing with the subject in question, are concerned with the compactons being the solutions to either completely integrable equations, or those which produce a completely integrable ones when being reduced onto subset of a traveling wave (TW) solutions [5, 9, 10].

In this paper compacton-like solutions to the hydrodynamic-type model taking into account the effects of spatial non-locality are considered. The model occurs to share the compacton-like solutions with a Hamiltonian system but only accidentally, i.e. at certain values of the parameters. In spite of such restriction, existence of this type of solutions seems to be significant for their appearance is connected with the presence of non-local effects and rather cannot be manifested in any local hydrodynamic model. Besides, localized invariant solutions sometimes manifest attractive features and when this is the case, they can be treated as some universal mechanism of the energy transfer in media with internal structure leading to the given type of the hydrodynamic-type system.

The structure of the paper is following. In section 2 we introduce dynamic equations of state (DES) taking into account effects of spatial and temporal non-localities. In section 3 we perform the qualitative analysis of a hydrodynamic system of balance equations closed by DES taking into account effects of spatial non-locality and state the conditions leading to the existence of compacton-like TW solutions. In the last section we summarize the results obtained, discuss open problems and prospects of further developing the actual study.

2 Non-local hydrodynamic-type models

We are going to analyze the existence of compacton-like solutions within the hydrodynamic-type models taking into account non-local effects. These effects are manifested when an intens pulse loading (impact, explosion, etc.) is applied to media possessing internal structure on mesoscale. Description of the non-linear waves propagation in such media depends in essential way on the ration of a characteristic size d of elements of the medium structure to the characteristic length λ of the wave pack. If d/λ is of the order of $O(1)$ then the basic concepts of continuum mechanics are not applicable and one should use the description based e.g. on the element dynamics methods. The models studied in this paper apply when the ratio d/λ is much less than unity and therefore the continual approach is still valid, but it is not as small that we can ignore the presence of internal structure.

As it is shown in [11], in the long wave approximation the balance equations for momentum and mass retain their classical form, which in the one-dimensional case can be written as follows:

$$u_t + p_x = 0, \tag{5}$$

$$\rho_t + \rho^2 u_x = 0. \tag{6}$$

Here u is mass velocity, ρ is density, p is pressure, t is time, x is mass (Lagrangean) coordinate. So the whole information about the presence of structure in this approximation is contained in DES which should be added to system (5)-(6) in order to make it closed.

It is of common knowledge that a unified dynamic equation of state well enough describing the behavior of condensed media in a wide range of changes of pressure, density and regimes of load (unload) actually does not exist. Various particular equations of state describing dynamical behavior of structural media are derived by means of different techniques. There exist, for example, several generally accepted DES in mechanics of heterogeneous media, derived on pure mechanical ground (see e.g. [12, 13]). Contrary, in papers [14, 15, 16, 17] derivations of DES are based on the non-equilibrium thermodynamics methods. Both, mechanical and thermodynamical approaches under the resembling assumptions give rise to similar DES, stating the functional dependencies between p, ρ and their partial derivatives.

There is also a number of works aimed at deriving the DES on the basis of the statistical theory of irreversible processes (see [18, 19] and references therein). It is rather firmly stated within this approach that DES for complicated condensed matter, being far from the state of thermodynamic equilibrium, takes the form of integral equations, linking together generalized thermodynamical fluxes I_n and generalized thermodynamical forces L_m, causing these fluxes:

$$I_n = f_n(L_k) + \int_{-\infty}^{t} d\,t' \int_{-\infty}^{+\infty} K_{m\,n}\,(t,\,t';x,x')\;g_m\left[L_k(t',\,x')\right]. \tag{7}$$

Here $K_{m\,n}\,(t,\,t';x,x')$ is the kernel of non-locality, which can be calculated, in principle, by solving dynamic problem of structure's elements interaction. Yet, such calculations are extremely difficult and very seldom are seen through to the end. Therefore we follow a common practice [20, 21] and use some model kernels describing well enough the main properties of the non-local effects, in particular, the fact that these effects vanish rapidly as $|t - t'|$ and $|x - x'|$ grow.

DES derived here are based on the following relation between the pressure and density:

$$p(t,\, x) = f\, \left[\rho\,(t,\, x)\right] + \int_{-\infty}^{t} \left\{ \int_{-\infty}^{+\infty} K\,(t - t',\; x - x')\; g\,[\rho\,(t',\, x')]\; d\,x' \right\} d\,t'. \quad (8)$$

Let us first assume that effects of spatial non-locality are unimportant. In this case the kernel of non-locality can be presented as $K\,(t - t')\;\delta\,(x - x')$ and the flux-force relation (8) takes the following form:

$$p(t,\, x) = f\, \left[\rho\,(t,\, x)\right] + \int_{-\infty}^{t} K\,(t - t')\; g\,[\rho\,(t',\, x)]\; d\,t'. \quad (9)$$

Since we rather do not want to deal with the system of integro-differential equations, our next step is to extract some acceptable kernels enabling to pass from (9) to pure differential relations. Differentiating equation (9) with respect to the temporal variable we obtain:

$$p_t = \dot{f}\; [\rho]\; \rho_t + K(0)\, g\,[\rho] + \int_{-\infty}^{t} K_t\,(t - t')\; g\,[\rho\,(t',\, x)]\; d\,t'. \quad (10)$$

Equation(10) is equivalent to a pure differential one provided that function $K(z)$ satisfies equation $\dot{K}(z) = c\,K(z)$. In this case we can obtain from (9), (10) the following differential equation:

$$\tau \left\{ p_t - \dot{f}\,[\rho]\,\rho_t \right\} = \tau\, A\, g\,[\rho] + f\,[\rho] - p. \quad (11)$$

This equation corresponds to fading memory kernel $K(z) = A\,\exp\,[-\frac{z}{\tau}]$. For $A = 1,\; f\,[\rho] = \chi \rho^{n+1},\; g\,[\rho] = -\sigma \rho^{n+1}$, we get the following DES:

$$\tau\,\{p_t - \chi\,(n + 1)\,\rho^n\,\rho_t\} = \kappa\,\rho^{n+1} - p. \quad (12)$$

where $\kappa = \chi - \sigma\,\tau$. Equations coinciding with (12) under certain additional assumptions are widely used to describe nonlinear processes in multi-component media with one relaxing process in the elements of structure [12, 16].

In more complicated cases, e.g. when more than one relaxing process should be taken into account, another kernels are to be applied. Let us look for the kernels leading from (9) to pure differential equation under the assumption that such passage is possible with the help of two differentiations. Taking derivative of (10) with respect to the temporal variable we get:

$$p_{tt} = \left[\dot{f}\; [\rho]\; \rho_t\right]_t + K(0)\,\dot{g}\,[\rho]\;\rho_t + \dot{K}[0]\,g\,[\rho] + \int_{-\infty}^{t} K_{tt}\,(t - t')\; g\,[\rho\,(t',\, x)]\; d\,t'. \quad (13)$$

So now we are interested in whether the linear combination $h\,p_{tt} + s\,p_t + p$ can be expressed as a pure differential equation. It is easy to see that the integral term corresponding to this combination is as follows:

$$\int_{-\infty}^{t} \{h\,K_{tt}\,(t-t') + s\,K_t\,(t-t') + K\,(t-t')\}\; g\,[\rho\,(t',\,x)]\;d\,t'.$$

Our requirement will be addressed if $K[z]$ satisfies the equation

$$h\,K_{zz}\,(z) + s\,K_z\,(z) + K\,(z) = 0. \tag{14}$$

If this is so, then DES takes on the form

$$\begin{aligned} h\,p_{tt} + s\,p_t + p = f(\rho) + [s\,K(0) + h\,K_t(0)]\;L(\rho) + \\ + \left[s\,\dot{f}(\rho) + h\,K(0)\dot{g}(\rho)\right]\rho_t + h\left[\dot{f}(\rho)\rho_{tt} + \ddot{f}(\rho)\rho_t^2\right]. \end{aligned} \tag{15}$$

Their are two cases for which requirement of fading memory holds true:

$$K[z] = \alpha\,e^{-\frac{z}{\tau_1}} + \beta\,e^{-\frac{z}{\tau_2}}, \quad h = \tau_1\,\tau_2, \quad s = \tau_1\,+\,\tau_2,$$

and

$$K[z] = e^{-\frac{z}{\tau_1}}\,\sin\left[\frac{z}{\tau_2} + \gamma\right], \quad h = \frac{\tau_1^2\,\tau_2^2}{\tau_1^2\,+\,\tau_2^2}, \quad s = \frac{2\,\tau_1\,\tau_2^2}{\tau_1^2\,+\,\tau_2^2}.$$

In the first case we have the kernel describing media with two relaxing processes, while in the second case the kernel describes relaxing media with oscillating component in the elements of structure. Note that in both cases we get formally identical DES (15), yet the parameters corresponding to them satisfy quite different inequalities. In the first case the parameters $h,\ s$ obey the inequalities $s^2 > 4\,h > 0$, while in the second case the inequalities are as follows: $0 < s^2 < 4\,h$. So the parameters lie in the distinct sets of parameter space and this fact occurs to be crucial since the behavior of the system of balance equation (5)-(6) closed by DES (15) is extremely sensible on the values of these parameters. [17, 22].

Now let us address the case of pure spatial non-locality. Following [21], we shall use the kernel of the form $K\,[t-t',\,x-x'] = \hat{\sigma}\,\exp\left[-\left(\frac{x-x'}{l}\right)^2\right]\cdot\delta\,[t-t']$ giving the equation

$$p = f\;(\rho) + \hat{\sigma}\int_{-\infty}^{+\infty} e^{-\left[\frac{x-x'}{l}\right]^2}\,g\,[\rho\,(t,\,x')]\;d\,x'. \tag{16}$$

Since the function $\exp\left[-\left(\frac{x-x'}{l}\right)^2\right]$ extremely quickly approaches zero as $|x - x'|$ grows, we use the following approximation for function $g\,[\rho\,(t,\,x')]$ inside the inner integral:

$$g\,[\rho\,(t,\,x')] = g\,[\rho\,(t,\,x)] + \left\{g\,[\rho\,(t,\,x)]\right\}_x \frac{(x'-x)}{1\,!} + \left\{g\,[\rho\,(t,\,x)]\right\}_{xx} \frac{(x'-x)^2}{2\,!} + O(|x'-x|^3). \tag{17}$$

Dropping out the term $O(|x'-x|^3)$ and integrating over $d\,x'$, we get

$$p = f\left[\rho(t,\,x)\right] + \sigma_0\, g\left[\rho\,(t,\,x)\right] + \sigma_2\, \left\{g\left[\rho\,(t,\,x)\right]\right\}_{xx} \tag{18}$$

where

$$\sigma_0 = \hat{\sigma}\int_{-\infty}^{+\infty} e^{-\left[\frac{x-x'}{l}\right]^2}\,d\,x', \quad \sigma_2 = \hat{\sigma}\int_{-\infty}^{+\infty} \frac{(x'-x)^2}{2\,!}\,e^{-\left[\frac{x-x'}{l}\right]^2}\,d\,x'.$$

It is obvious, that one can easily combine two types of non-localities by taking the kernel in the form of the product

$$\bar{K}\left[t-t',\;x-x'\right] = K\left[t-t'\right]\cdot \exp\left[-\left(\frac{x-x'}{l}\right)^2\right],$$

with $K\left[t-t'\right]$ satisfying (14). Employment of technics used in the last two items enables to obtain the DES taking into account both effects of spatial and temporal non-localities:

$$\begin{aligned} h\,p_{tt} + s\,p_t + p = f[\rho] + \left[s\,K[0] + h\,K_t[0]\right]\,\left[\sigma_0\,g[\rho] + \sigma_2\left(\dot{g}[\rho]\,\rho_x\right)_x\right] + \\ +s\,\dot{f}[\rho]\,\rho_t + h\,K[0]\left[\sigma_0\,g[\rho] + \sigma_2\left(\dot{g}[\rho]\,\rho_x\right)_x\right]_t + h\left[\dot{f}[\rho]\rho_t\right]_t. \end{aligned} \tag{19}$$

Let us note that DES with higher derivatives were employed in many papers dealing with the structured media. Qualitative and numerical investigations undertaken in [22, 23, 24] show that the modeling system taking into account both types of non-localities possesses a rich set of the TW solutions, including periodic, quasi-periodic, multi-periodic, soliton-like and chaotic regimes.

3 Compacton-like solutions within the hydrodynamic-type model taking into account spatial non-localitiy

Before we start to "capture" compacton-like solutions within the hydrodynamic model, it is desired to get a clear idea on what is a compactons from geometric point of view. In paper [25] such analysis has been performed for the equations belonging to Rosenau-Hyman $K(n,\,m)$ hierarchy and this is the way things stand. Being treated as function of a single value $\xi - x - D\,t$, solution (4) satisfies some second-order ODE which is obtained from (3) by means of the ansatz $u(t,\,x) = U(\xi)$. Substituting this ansatz into the source equation one gets, after one integration and some manipulation, a second-order Hamiltonian system [25]. This system occurs to have a one-parameter family of periodic phase trajectories and the homoclinic trajectory surrounding them. The last one just corresponds to compacton.

In the case of KdV the similar procedure leads to the Hamiltoinan system with the same geometry of the phase trajectories. The difference between two phase portraits arises from the fact that in the latter case the homoclinic loop is the bi-asymptotic trajectory of a simple saddle while the trajectory representing the compacton is the separatrix of a saddle settled on the line of singular points of the vector field. Therefor the homoclinic trajectlry is penetrated in a finite "time" and the compacton solution (4) consists of non-zero part with compact support, glued with zero solution corresponding to the saddle point.

Let us note at the end of this short introduction, that we do not distinguish solutions having a compact support and those which can be made so by proper change of variables.

Now let us consider the system (5), (6) closed by DES

$$p = f[\bar{\rho}] + \kappa\, \bar{\rho}^{n+2} + \sigma\, \left[\bar{\rho}^{n+1} \bar{\rho}_x\right]_x \tag{20}$$

corresponding to pure spatial non-locality. Here $\bar{\rho} = \rho - \rho_0$, where $0 < \rho_0$ is a constant, $f[\bar{\rho}]$ is a function which will be defined later on. We use the ansatz

$$u(t, x) = U(x - D\,t) \equiv U(\xi), \qquad \bar{\rho}(t,\, x) = R(\xi), \tag{21}$$

enabling to factorize system (6), (5), closed by the DES (20). Inserting (21) into equation (6), we get the following quadrature:

$$U = \frac{D}{\rho_0} - \frac{D}{R[\xi] + \rho_0}. \tag{22}$$

Constant of integration have been chosen in such a way that $u(t, \pm\infty) = 0$.

Using the ansatz (21), we express DES in new invariant variables. Inserting (20) into (5), using the formula (22) and integrating once the expression obtained this way we pass, after some manipulation, to the following second order ODE

$$\frac{D^2}{R + \rho_0} + f[R] + \kappa\, R^{n+2} + \sigma\, \left[R^{n+1}\, R'\right]' = E = \frac{D^2}{\rho_0} + f[0]. \tag{23}$$

It is obvious that above equation can be re-written as an equivalent dynamical system. To do this, we define a new function $W = -R'$. Next we introduce new independent variable T such that $\frac{d}{dT} = \sigma\, R^{n+1}\, \varphi[R] \frac{d}{d\omega}$, where $\varphi[R]$ is an integrating factor which is incorporated in order to make the system Hamiltonian. With this notation we get the following dynamical system equivalent to (23):

$$\begin{cases} \frac{dR}{dT} = -\sigma\, \varphi[R]\, R^{n+1}\, W = -\frac{\partial H[R,W]}{\partial W}, \\ \frac{dR}{dT} = \varphi[R] \left[\sigma(n+1) R^n\, W^2 + f(R) - f(0) + \kappa R^{n+2} - \frac{D^2 R}{\rho_0 (R+\rho_0)}\right] = \frac{\partial H[R,W]}{\partial R}. \end{cases} \tag{24}$$

Solving the first equation of system (24) with respect to function $H[R,\, W]$ we obtain:

$$H[R,\, W] = \sigma\, \varphi[R]\, R^{n+1} \frac{W^2}{2} + \theta[R]. \tag{25}$$

Now, comparing the RHS of second equation of system (24) with partial derivative of (25) with respect to R, we get the system

$$R\,\varphi'[R] = (n+1)\varphi[R],$$
$$\theta'[R] = \varphi[R]\left\{f(R) - f(0) + \kappa R^{n+2} - \frac{D^2 R}{\rho_0\,(R+\rho_0)}\right\}.$$

The first equation is satisfied by the function $\varphi[R] = C\,R^{n+1}$. For convenience we put $C = 2$. Inserting $\varphi[R]$ in the second equation we obtain the quadrature

$$\theta(R) = 2\int R^{n+1}\left\{f(R) - f(0) + \kappa R^{n+2} - \frac{D^2 R}{\rho_0\,(R+\rho_0)}\right\}d\,R.$$

So the Hamiltonian function is finally expressed as follows:

$$H[R,\,W] = \sigma\,R^{2(n+1)}W^2 + 2\int R^{n+1}\left\{f(R) - f(0) + \kappa R^{n+2} - \frac{D^2 R}{\rho_0\,(R+\rho_0)}\right\}d\,R. \tag{26}$$

The Hamiltonian function (26) is a first integral of the system (24), so every phase trajectory can be presented in the form

$$W^2 = \frac{1}{\sigma\,R^{2(n+1)}}\left\{K - 2\int R^{n+1}\left[f(R) - f(0) + \kappa R^{n+2} - \frac{D^2 R}{\rho_0\,(R+\rho_0)}\right]d\,R\right\}, \tag{27}$$

where K is the constant value of the Hamiltonian on a particular trajectory. Now let us analyze formula (27). If we want to have a closed trajectory approaching the origin, then we must properly choose the constant K and "suppress" all the singular terms by the proper choice of function $f(R)$. It can be easily shown by induction that the following decomposition for the last term inside the integral takes place:

$$\frac{R^{n+2}}{R+\rho_0} = R^{n+1} - \rho_0\,R^n + ... + (-1)^k\,\rho_0^k\,R^{n+1-k} + ... + (-1)^{n+1}\,\rho_0^{n+1} + (-1)^{n+2}\frac{\rho_0^{n+2}}{R+\rho_0}.$$

Hence

$$W^2 = \frac{1}{\sigma\,R^{2(n+1)}}\left\{K - 2\int R^{n+1}\left[f(R) - f(0) + \kappa R^{n+2}\right]d\,R - \right.$$
$$\left. -\frac{2\,D^2}{\rho_0}\left(\frac{R^{n+2}}{n+2} - \rho_0\frac{R^{n+1}}{n+1} + ... + (-\rho_0)^{n+1}\,R + (-\rho_0)^{n+2}\log\left(R+\rho_0\right)\right)\right\}.$$

From this we conclude that the last term in (27) always produces singularities. More precisely, singularities are connected with monomials R^m when $m < 2(n+1)$ and with the logarithmic term. Therefore the last term in (27) should be rather removed by the proper choice of $f(R)$.

A simple analysis shows that function

$$f(R) = f_0 + \frac{A\,R}{\rho_0\;(R+\rho_0)} + g_1\,R^{n+1} + g_2\,R^{n+2}$$

with $A > 0$ will suppress singularity provided that $D = \pm\sqrt{A}$. In fact, in this case

$$W^2 = \frac{1}{\sigma\, R^{2(n+1)}}\left\{K - R^{2(n+1)}\left[\frac{2 g_1\, R}{2n+3} + \frac{\bar{g}_2\, R^2}{n+2}\right]\right\}, \tag{28}$$

where $\bar{g}_2 = g_2 + \kappa$. With such a choice the only trajectory approaching the origin corresponds to $K = 0$. If in addition $g_1 = -\alpha_1 < 0$ and $\bar{g}_2 = \alpha_2 > 0$, i.e.

$$f(R) = f_0 + \frac{A\, R}{\rho_0\, (R + \rho_0)} - \alpha_1\, R^{n+1} + \alpha_2\, R^{n+2} \tag{29}$$

then

$$W = \pm\frac{1}{\sqrt{\sigma}}\sqrt{\frac{2\,\alpha_1}{2n+3}\, R - \frac{\alpha_2}{n+2} R^2} \tag{30}$$

and there is the point $R_* = \frac{2(n+2)\alpha_1}{(2n+3)\alpha_2}$ in which the trajectory intersects the horizontal axis.

In fact, under the above assumptions we get the geometry which is similar to that obtained when the member of $K(n, m)$ hierarchy is reduced to an ODE describing the set of TW solutions [25]. To show that, let us consider the system arising from (23) under the above assumptions:

$$\begin{aligned} \frac{d\,R}{d\,T} &= -2\,\sigma\, W\, R^{2(n+1)} \\ \frac{d\,W}{d\,T} &= 2\,(n+1)\,\sigma\, W^2\, R^{2n+1} + 2\,\alpha_2\, R^{2n+3} - 2\,\alpha_1\, R^{2n+2}. \end{aligned} \tag{31}$$

System (31) possesses two stationary points lying on the horizontal axis: the point (0, 0), and the point $(R_1,\, 0)$, where $R_1 = \alpha_1/\alpha_2$. Analysis of the Jacobi matrix shows that $(R_1,\, 0)$ is a center. Moreover, $R_1 < R_*$ when $n > -2$ so the closed trajectory corresponding to $K = 0$ encircles the domain filled with the periodic trajectories. Using the asymptotic decomposition of the solution represented by the homoclinic trajectory near the origin, it can be shown that the trajectory reaches the stationary point (0, 0) in finite "time" so it does correspond to the compacton.

Besides, we can get more direct evidence of the existence of compacton-like solutions by integrating equation (30). In fact, taking in mind that $W = -d\,R/d\,\xi$, we obtain the following equation:

$$\frac{1}{\gamma}\frac{d\,R}{\sqrt{1 - \left(\frac{R}{\gamma} - 1\right)^2}} = \delta\, d\,\xi,$$

where $\gamma = \frac{\alpha_1 (n+2)}{\alpha_2 (2n+3)}$, $\delta = \sqrt{\frac{\alpha_2}{\sigma(n+2)}}$. Solving this equation we get:

$$R[\xi] = \begin{cases} \gamma\, [1 + \sin(\delta\,(\xi - \xi_0))] & \text{when} - \pi \leq\, 2\,\delta\,(\xi - \xi_0) \leq\, 3\,\pi \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

Using the formulae (20)–(22) and (32), one can easily reproduce the functions u, ρ and p.

4 Discussion

In this work we have shown that hydrodynamic system of balance equations (5)-(6) closed by DES (20) possesses the compacton-like solutions. In contrast to analogous solutions to most of the compacton-supporting equations, the presented solutions do not form a one-parameter family. More precisely, for fixed values of the parameters A, α_1 and α_2 in (29) there exists exactly one pair of compactons moving with velocity $\sqrt{A}$ in the opposite directions. Whether these solutions are of interest from the point of view of applications or not, depends on their stability and behavior during the mutual collisions. Discussion of these topics goes beyond the scope of this paper. Let us mention, however, that the mere fact of existence of compactons is the consequence of the non-local effects incorporation. To our best knowledge, this type of solutions does not exist in any local hydrodynamic-type model, i.e. the system of balance equations (5)-(6) closed by the functional state equation $p = \Phi(\rho)$. Let us note that invariant TW solutions very often play rule of intermediate asymptotics [26, 27], attracting near-by, not necessarily invariant, solutions. This feature demonstrate compacton-like solutions of another non-local model obtained when the system of balance equations (5)-(6) is closed by the DES (12), describing relaxing media [25]. Solutions with compact supports appear in this model merely in presence of the mass force. Exactly one compacton-like solution occurs to exist for the given set of the parameters, yet this solution serves as an attractor for the wave packs created by the wide class of the initial value problems [25]. In contrast to the above mentioned relaxing model, incorporation of the effects of spatial non-locality leads to the existence of a pair of compacton-like solutions in absence of an external force. In fact, it is shown for the first time that a non-local hydrodynamic-type model possesses more than one compacton-like solution and from now on there exists the opportunity to investigate their behavior during the collisions.

There are some evidences in favor of the stable behavior of these compactons and their elastic collisions. The first is connected with the fact that by proper choice of parameters A, α_1 and α_2 the evolutionarity conditions [28] $\partial p/\partial\rho > 0$, $\partial^2 p/\partial^2\rho > 0$ can be fulfilled for DES (20). Besides, the system of balance equations (5)-(6) closed by the DES (20) possesses at least two conservation laws. Though actually it is not known how many conserved quantities assure the stability of the wave patterns during the collisions, it is almost certain that this number should not be infinite. For example, the members of the Rosenau-Hyman $K(n, m)$ hierarchy, possessing with certain four conserved quantities, collide almost elastically and so is with the compacton-compacton and kovaton-compacton interactions in the Pikovsky-Rosenau model [7].

Another open question connected with the non-local hydrodynamic-type models is as follows. It is unknown as yet whether the compacton-like solutions exist within the hydrodynamic-type models closed by the DES (19) taking into account both the effects of spatial and temporal non-localities. Our preliminary analysis shows that incorporation of terms connected with temporal non-locality makes the model dissipative. So the Hamiltonian formalism is of no use in this case and investigations of the compacton-like regimes appearance should be based on some other technics. Study of these questions is actually in progress.

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