LINEARIZABILITY AND NONLOCAL SUPERPOSITION FOR NONLINEAR TRANSPORT EQUATION WITH MEMORY

W. RZESZUT

Faculty of Mathematics and Computer Sciences, Jagiellonian University,
S. Łojasiewicz Street 6, 30-348 Kraków, Poland
(e-mail: wojciech.rzeszut@im.uj.edu.pl)

O. TERTYSHNYK, V. TYCHYNIN

Prydniprovs'ka State Academy of Civil Engineering and Architecture,
Chernyshevsky Street 24a, 49005 Dnipropetrovsk, Ukraine
(e-mails: OlesyaTNik@yandex.ru, tychynin@ukr.net)

and

V. VLADIMIROV*

Faculty of Applied Mathematics, AGH University of Science and Technology,
Mickiewicz Avenue 30, 30-059 Kraków, Poland
(e-mail: vsevolod.vladimirov@gmail.com)

(*Received April 2, 2013 – Revised May 21, 2013*)

Potential symmetry of a class of nonlinear transport equations taking into account the effects of memory is studied. For a specific transport coefficient the symmetry is shown to be infinite. This fact is used for constructing nonlocal transformation linearizing the transport equation. New formulae of nonlocal nonlinear superposition and generation of solutions are proposed. Additional Lie symmetries of the corresponding linear equations are used to construct nonlocal symmetries of the source equation. The formulae derived are used for the construction of exact solutions.

Keywords: nonlinear transport equation, effects of memory, potential symmetry, nonlocal linearization, exact solutions, nonlinear superposition principle.

1. Introduction

In this work we consider the following modification of nonlinear transport equation

$$\tau\, u_{tt} + u_t = \kappa \left[u^n\, u_x\right]_x . \tag{1}$$

Here $\tau > 0$, $\kappa > 0$, n is an integer number. Eq. (1) can be formally introduced [1, 2] if one changes in the balance equation the convenient Fick law

$$J(t,\, x) = -\nabla\, q(t,\, x),$$

*Supported by the Ministry of Science and Higher Education of Poland.

describing the thermodynamical flow-force relations, with the Cattaneo equation

$$\tau \frac{\partial J(t,\, x)}{\partial t} + J(t,\, x) = -\nabla q(t,\, x),$$

which takes into account the effects of memory [1–4].

The aim of this work is two-fold. In the first part (Section 2) we are focusing on the study of the potential symmetry [5] admitted by Eq. (1). For a generic value of the parameter n it is shown to be trivial. Yet for $n = -2$ the symmetry is infinite. In this case Eq. (1) is linearized [5] by means of the nonlocal change of variables.

Then, in Section 3 and the following sections we use the linearization of Eq. (1) to generate exact solutions and to construct new formulae of nonlinear superposition principle. Besides, the Lie symmetry of connected linear PDE is used for constructing the corresponding induced nonlocal symmetry of the transport equation. In this approach we use the method of nonlocal transformations of variables [6–10] for the systematic construction of new formulae of nonlocal nonlinear superposition and generation of solutions for nonlinear transport (or telegraph) equation.

In Section 4 we apply the nonlocal transformation for mapping a special case of nonlinear transport equation into itself (in nonlocal sense), and use this transformation to generate its solutions.

A Lie symmetry generator admitted by a linear homogeneous equation allows for the generation of new solutions from the known ones. This property enables us to construct the formulae for generation of solutions to Eq. (1) (Section 5).

2. Potential symmetry and nonlocal linearization of equation (1)

First of all, let us note that for positive τ and κ one can, using the scaling transformation

$$t = \mathrm{e}^{\alpha}\, \bar{t}, \qquad x = \mathrm{e}^{\beta}\, \bar{x}, \qquad \alpha = \log \tau, \quad \beta = \log \sqrt{\tau\, \kappa}, \tag{2}$$

rewrite Eq. (1) in the dimensionless form

$$u_{\bar{t}\bar{t}} + u_{\bar{t}} = \left[u^n\, u_{\bar{x}}\right]_{\bar{x}} \tag{3}$$

(for simplicity we omit bars in the forthcoming formulae). It is evident, that Eq. (3) can be presented as a local conservation law

$$\frac{\partial}{\partial t}\bigl(u_t + u\bigr) - \frac{\partial}{\partial x}\left[u^n\, u_x\right] = 0.$$

So, following [5], we present Eq. (3) in the form of the potential system

$$\begin{cases} v_x = u_t + u, \\ v_t = u^n\, u_x, \end{cases} \tag{4}$$

and look for the Lie symmetry generator [5, 11]

$$X = \xi(x,t,u,v)\frac{\partial}{\partial x} + \eta(x,t,u,v)\frac{\partial}{\partial t} + \varphi(x,t,u,v)\frac{\partial}{\partial u} + \psi(x,t,u,v)\frac{\partial}{\partial v}$$

for this system. If any of the coefficients ξ, η or φ depends explicitly upon the potential variable v, then the infinitesimal symmetry generator X (IFG to abbreviate) induces the nonlocal *potential* symmetry of Eq. (3).

Acting with the operator X on the system (4), and performing the splitting [5, 11], one can obtain the following system of determining equations:

$$\eta_u - \xi_v = 0, \tag{5}$$

$$\xi_u - u^n \eta_v = 0, \tag{6}$$

$$\xi_t - u^n \eta_x - u^{n+1}\eta_v = 0, \tag{7}$$

$$u\xi_u + u^n \varphi_v - \psi_u = 0, \tag{8}$$

$$\xi_x - \eta_t + 2u\xi_v + \varphi_u - \psi_v = 0, \tag{9}$$

$$n\varphi - u[\xi_x - \eta_t - \varphi_u + \psi_v] = 0, \tag{10}$$

$$u\xi_x + u^2\xi_v + \varphi + \varphi_t - \psi_x - u\psi_v = 0, \tag{11}$$

$$u\xi_t + u^n\varphi_x + u^{n+1}\varphi_v - \psi_t = 0. \tag{12}$$

Let us note, that Eqs. (5)–(7) form the autonomous subsystem containing merely the functions ξ, η. In order to facilitate its solution, we use the Rosenfeld–Groebner algorithm (running under the "Mapple"), which splits a self-consistent overdetermined system of differential equations into a list of regular equivalent systems. Since this list is different for different values of the parameter n, we present the results in the form of the table.

No.	$n = -2$	$n = -1$	$n = 0$	$n \notin \{-2, -1, 0\}$
1	$\eta_{vvv} = \eta_{vtt} + \eta_{vt}$	$\eta_{vvv} = 0$	$\eta_{vvv} = 0$	$\eta_{vvv} = 0$
2	$\eta_{vvx} = 0$	$u\eta_{uu} = \eta_{vv}$	$\eta_{vvt} = \eta_{vv}$	$\eta_{uu} = \eta_{vv}u^n$
3	$u^2\eta_{uu} = \eta_{vv}$	$\eta_{uv} = 0$	$\eta_{vvx} = 0$	$\eta_{uv} = 0$
4	$u^2\eta_{uv} = \eta_{vt}u + \eta_v u + \eta_x$	$\eta_{ut} = \eta_{vv}$	$\eta_{vtt} = \eta_{vxx} + \eta_{vt}$	$\eta_{ut} = \eta_{vv}u^{n+1}$
5	$u^2\eta_{ut} = \eta_{vv}u + \eta_{vx}$	$u\eta_{ux} = \eta_x$	$\eta_{uu} = \eta_{vv}$	$u\eta_{ux} = \eta_x$
6	$u\eta_{ux} = \eta_x$	$\eta_{vt} = 0$	$\eta_{uv} = 0$	$\eta_{vt} = \frac{n+1}{u}\{\eta_v u + \eta_x\}$
7	$\eta_{tx} = 0$	$\eta_{vx} = 0$	$\eta_{ut} = \eta_{vv}u + \eta_{vx}$	$\eta_{vx} = 0$
8	$\eta_{xx} = 0$	$\eta_{tx} = 0$	$\eta_{ux} = \eta_{vt} - \eta_v$	$\eta_{xt} = 0$
9	–	–	–	$\eta_{xx} = 0$

We begin with the generic case. Solving step by step the equations placed in the rows 1–9 of the last column, we obtain that

$$\eta = C_1 e^{(n+1)t}\left(\frac{u^{n+2}}{(n+1)(n+2)} + \frac{v^2}{2}\right) + v\left(C_2 e^{(n+1)t} - C_3\right) + u\left(C_3 x + C_4\right) + g(t). \tag{13}$$

Solving next Eqs. (5)–(7), we find that

$$\xi = \frac{u^{n+1}}{n+1}\left(e^{(n+1)t}\left(C_1 v + \frac{C_2}{2}\right) - C_3\right) + v\,(C_3 x + C_4) + f(x). \tag{14}$$

Now let us address the system (8)–(12). Subtracting equation (10) from (9), we get

$$\varphi = \frac{2}{n}u(\xi_x - \eta_t + u\xi_v). \tag{15}$$

Using this formula, we express the derivatives of the function ψ in terms of the already known functions:

$$\psi_x = -u^2\xi_v + u\eta_t + \varphi + \varphi_t - u\varphi_u, \tag{16}$$

$$\psi_t = u^{n+1}\varphi_v + u^n\varphi_x + u\xi_t, \tag{17}$$

$$\psi_u = u\xi_u + u^n\varphi_v, \tag{18}$$

$$\psi_v = \xi_x - \eta_t + 2u\xi_v + \varphi_u. \tag{19}$$

Comparison of the mixed derivatives $\psi_{\mu\,v} = \psi_{v\,\mu}$ shows that $C_i = 0$, $i = 0, \ldots, 4$, $g(t) = C_5 = \text{const}$, and $f(x) = C_6\,x + C_7$. The remaining coefficients of the IFG can now be easily calculated:

$$\varphi = C_6\,\frac{2\,u}{n}, \qquad \psi = \left\{\left(1 + \frac{2}{n}\right) C_6\,v + d\right\}.$$

The results obtained can be summarized as follows.

STATEMENT. *For* $n \neq \{-2, -1, 0\}$ *the system* (4) *admits symmetry generators*

$$X_1 = x\frac{\partial}{\partial x} + \frac{2}{n}\,u\,\frac{\partial}{\partial u} + \left(1 + \frac{2}{n}\right) v\,\frac{\partial}{\partial v}, \tag{20}$$

$$X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial t}, \qquad X_4 = \frac{\partial}{\partial v}.$$

Thus, in the generic case Eq. (3) does not possess the potential symmetry.

Now let us discuss the particular cases that were excluded in the above consideration. We omit the linear case $n = 0$, falling out of the context of this article. In the case $n = -1$, following the steps outlined above, one easily gets convinced that the system (5)–(12) admits just the same set of generators as the generic one, and no another local symmetry generator appears. The situation is completely different when $n = -2$. General solution of equations placed in the rows 1–9 of the first column of the table is as follows,

$$\eta = C_1\left(xuv + \ln|u| - \frac{v^2}{2}\right) + C_2(ux - v) + f_1(t) + e^{-t}F(X_1, X_2), \tag{21}$$

where $X_1 = u\,e^t$, $X_2 = v$, $f_1(t)$ is an arbitrary function. To obtain ξ, we insert (21) into the formulae (5)–(6). Equating the mixed derivatives $\xi_{u\,v} = \xi_{v\,u}$, we get that $C_1 = 0$, while $F(X_1, X_2)$ satisfies the equation

$$F_{X_2 X_2}(X_1, X_2) = X_1^2 F_{X_1 X_1}(X_1, X_2).$$

Now, solving Eqs. (5)–(7), under the above restrictions, we get the formula

$$\xi = C_2\left(vx + \frac{1}{u}\right) + G(u\,e^t,\ v) + f_3(x), \tag{22}$$

where $f_3(x)$ is an arbitrary function, and $G(X_1,\ X_2)$ satisfies the equation

$$G_{X_2}(X_1,\ X_2) = F_{X_1}(X_1,\ X_2).$$

From (15) we get

$$\varphi = u\left[f_1'(t) - f_3'(x) - C_2\,(v + x\,u) - e^{-t}F(X_1,\ X_2)\right].$$

Using the formulae (16)–(19) and comparing the mixed derivatives $\psi_{\mu\,\nu} = \psi_{\nu\,\mu}$, we obtain the expressions

$$f_1(t) = C_3 = \text{const}, \qquad f_3(x) = C_4 + C_5\,x,$$

and an extra equation

$$X_1^2\,G_{X_1}(X_1,\ X_2) = F_{X_2}(X_1,\ X_2).$$

Returning now to the system (16)–(19), we find out the coefficient ψ,

$$\psi = C_6 - 2\,C_2\,(\log u + t)\,.$$

Thus, the following statement holds true.

THEOREM 1. *For $n = -2$ the potential system* (4) *admits an infinite-dimensional Lie algebra spanned by the following operators:*

$$\begin{aligned}
&X_1 = \frac{\partial}{\partial t}, \qquad X_2 = \frac{\partial}{\partial x}, \qquad X_3 = \frac{\partial}{\partial v}, \qquad X_4 = x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u},\\
&X_5 = (u^{-1} + xv)\frac{\partial}{\partial x} + (xu - v)\frac{\partial}{\partial t} - u(v + ux)\frac{\partial}{\partial u} - 2(\log u + t)\frac{\partial}{\partial v},\\
&X_\infty = G(X_1, X_2)\frac{\partial}{\partial x} + e^{-t}F(X_1, X_2)\frac{\partial}{\partial t} - ue^{-t}F(X_1, X_2)\frac{\partial}{\partial u}.
\end{aligned} \tag{23}$$

As can be checked, the generator X_∞ fulfills the theorem from chapter VI of [5], which guarantees the existence of a linearizing change of variables. In our case the corresponding formulae are as follows:

$$p = X_1 = e^t u, \qquad q = X_2 = v, \qquad w = \psi^1 = x, \qquad z = \psi^2 = e^t. \tag{24}$$

The substitution (24) transforms (4) into the system

$$\begin{cases} w_q(p, q) = z_p(p, q), \\ w_p(p, q) = \frac{1}{p^2} z_q(p, q) \end{cases} \tag{25}$$

(note that an extra condition $z > 0$ should be fulfilled). The system (25), in turn, is equivalent to the second-order linear PDE

$$p^2 z_{pp}(p,q) - z_{qq}(p,q) = 0, \tag{26}$$

which, after the change of variables

$$q = q, \qquad r = \log p, \qquad F = \frac{z}{\sqrt{p}},$$

turns into the Klein–Gordon equation

$$F_{qq}(r,q) - F_{rr}(r,q) + \frac{1}{4}F(r,q) = 0. \tag{27}$$

Solutions of this equation can be obtained by standard methods, described e.g. in [12].

3. Linearization and nonlinear superposition principle

In order to present Eq. (3) in the form of conservation law, a potential function v determined by the auxiliary system

$$v_x = \phi^t(x,t,u_{(r)}), \qquad v_t = -\phi^x(x,t,u_{(r)}) \tag{28}$$

should be introduced. Integrating, for example, the first equation with respect to x, we get the expression

$$v = \int \phi^t(x,t,u_{(r)})dx. \tag{29}$$

Suppose that a nonlocal transformation

$$\mathcal{T}: \quad x^i = h^i(y, v_{(k)}), \qquad i = 1,\ldots,n, \qquad u^K = H^K(y, v_{(k)}), \qquad K = 1,\ldots,m,$$

including the dependence of integral variables of the type (29), maps the given equation

$$F_1(x, u_{(n)}) = 0 \tag{30}$$

into an integro-differential equation $\Phi(y, v_{(q)}) = 0$ of order $q = n + k$, which admits a factorization to another equation

$$F_2(y, v_{(s)}) = 0, \tag{31}$$

i.e. $\Phi(y, v_{(q)}) = \lambda F_2(y, v_{(s)})$. The symbol $u_{(r)}$ denotes the tuple of derivatives of the function u from order zero up to order r, λ is an integro-differential operator of order $n + k - s$. Then we say that equations $F_1(x, u_{(n)}) = 0$ and $F_2(y, v_{(s)}) = 0$ are connected by the nonlocal transformation $\mathcal{T}$.

If Eq. (31) coincides with the initial equation (30), then $\mathcal{T}$ is a nonlocal symmetry transformation of Eq. (30) and we can directly use $\mathcal{T}$ for the construction of formulae for generating solutions of this equation. Otherwise, there exist two possibilities.

Suppose that Eq. (31) possesses a Lie symmetry generator X which is not associated with a Lie symmetry generator of Eq. (30). Then we construct the one-parameter group of Lie symmetries generated by X and the corresponding formula for generating of solutions of (31). This formula is extended onto Eq. (30) by inverting the transformation $\mathcal{T}$. If an additional Lie symmetry generator is admitted by Eq. (30), then we apply the above procedure in the inverse direction.

If Eq. (31) is linear, then we can construct the formulae of nonlinear nonlocal superposition of solutions for Eq. (30) [13, 14]. Such formulae allow us to construct a new solution to Eq. (30) from known ones.

3.1. Formula of superposition of solutions for the potential system

Let us present the nonlinear transport equation

$$u_{tt} + u_t - \partial_x\left(u^{-2}u_x\right) = 0 \tag{32}$$

in the potential form:

$$v_x - u_t - u = 0, \qquad v_t - u^{-2}u_x = 0. \tag{33}$$

The transformation

$$x = p(r, s), \qquad t = \ln|q(r, s)|, \qquad u(x, t) = \frac{r}{q(r, s)}, \qquad v(x, t) = s. \tag{34}$$

maps the system (33) into the linear system

$$p_s - q_r = 0, \qquad p_r - r^{-2}q_s = 0, \tag{35}$$

which can be interpreted as the Bäcklund transformation, connecting the pair of linear partial differential equations

$$q_{ss} - r^2 q_{rr} = 0 \tag{36}$$

and

$$p_{ss} - \partial_r\left(r^2 \ p_r\right) = 0. \tag{37}$$

The homogeneous linear system (35) admits a linear superposition of solutions. We use this fact for the construction of a new solution from two known ones, supposing that

$$p^{\mathrm{III}} = p^{\mathrm{I}} + p^{\mathrm{II}}, \qquad q^{\mathrm{III}} = q^{\mathrm{I}} + q^{\mathrm{II}}. \tag{38}$$

And since the systems (33) and (35) are connected by the transformation (34), the corresponding principle of nonlinear superposition of solutions can be constructed for Eq. (33).

THEOREM 2. *The nonlinear superposition formula of solutions for the system* (33) *has the form:*

$$u^{\mathrm{III}}(x, t) = u^{\mathrm{I}}(\tau^{\mathrm{I}}, \eta^{\mathrm{I}}) \ \mathrm{e}^{(\eta^{\mathrm{I}} - t)}, \qquad v^{\mathrm{III}}(x, t) = v^{\mathrm{I}}(\tau^{\mathrm{I}}, \eta^{\mathrm{I}}), \tag{39}$$

where the functional parameters $\tau^{\mathrm{I}} = \tau^{\mathrm{I}}(t, x)$ *and* $\eta^{\mathrm{I}} = \eta^{\mathrm{I}}(t, x)$ *are defined by the system*

$$v^{\mathrm{I}}(\tau^{\mathrm{I}}, \eta^{\mathrm{I}}) = v^{\mathrm{II}}(x - \tau^{\mathrm{I}}, \ln|\mathrm{e}^t - \mathrm{e}^{\eta^{\mathrm{I}}}|), \tag{40}$$

$$\mathrm{e}^{\eta^{\mathrm{I}}}\ u^{\mathrm{I}}(\tau^{\mathrm{I}}, \eta^{\mathrm{I}}) = u^{\mathrm{II}}(x - \tau^{\mathrm{I}}, \ln|\mathrm{e}^t - \mathrm{e}^{\eta^{\mathrm{I}}}|)(\mathrm{e}^t - \mathrm{e}^{\eta^{\mathrm{I}}}). \tag{41}$$

Really, let there exist two known solutions $u^{\mathrm{K}}(\tau^{\mathrm{K}}, \eta^{\mathrm{K}})$ $(\mathrm{K} = \mathrm{I}, \mathrm{II})$ for two systems (33) depending on their own independent variables. Applying the transformation

$$\tau^{\mathrm{K}} = p^{\mathrm{K}}(r, s), \quad \eta^{\mathrm{K}} = \ln|q^{\mathrm{K}}(r, s)|, \quad u^{\mathrm{K}}(\tau^{\mathrm{K}}, \eta^{\mathrm{K}}) = \frac{r}{q^{\mathrm{K}}(r, s)}, \quad v^{\mathrm{K}}(\tau^{\mathrm{K}}, \eta^{\mathrm{K}}) = s \tag{42}$$

to each of them, we get two linear systems equivalent to (35):

$$\begin{aligned} p_s^{\mathrm{K}}(r, s) - q_r^{\mathrm{K}}(r, s) = 0, \\ p_r^{\mathrm{K}}(r, s) - r^{-2} q_s^{\mathrm{K}}(r, s) = 0. \end{aligned} \tag{43}$$

As the initial system (33) admits linearization to (35), we can apply the linear superposition principle (38) to the solutions of the corresponding linear system

$$p(r, s) \equiv p^{\mathrm{III}} = p^{\mathrm{I}} + p^{\mathrm{II}}, \qquad q(r, s) \equiv q^{\mathrm{III}} = q^{\mathrm{I}} + q^{\mathrm{II}}. \tag{44}$$

This way, using the transformation (34), one can construct a new solution to the nonlinear system (33).

In order to present this result in a more convenient form, we consider the transformations inverse to (42) and (34), accordingly:

$$r = \mathrm{e}^{\eta^{\mathrm{K}}} u^{\mathrm{K}}(\tau^{\mathrm{K}}, \eta^{\mathrm{K}}), \qquad s = q^{\mathrm{K}}(r, s), \qquad p^{\mathrm{K}}(r, s) = \tau^{\mathrm{K}}, \qquad q^{\mathrm{K}}(r, s) = \mathrm{e}^{\eta^{\mathrm{K}}}, \tag{45}$$

$$r = \mathrm{e}^t u(x, t), \quad s = v(x, t), \quad p\,(r, s) = p^{\mathrm{I}} + p^{\mathrm{II}} = x, \quad q(r, s) = q^{\mathrm{I}} + q^{\mathrm{II}} = \mathrm{e}^t, \tag{46}$$

Comparing these formulae with (34), we find the equalities

$$\tau^{\mathrm{II}} = x - \tau^{\mathrm{I}}, \qquad \eta^{\mathrm{II}} = \ln|\mathrm{e}^t - \mathrm{e}^{\eta^{\mathrm{I}}}| \tag{47}$$

and

$$v(x, t) \equiv v^{\mathrm{III}}(x, t) = v^{\mathrm{I}}(\tau^{\mathrm{I}}, \eta^{\mathrm{I}}) = v^{\mathrm{II}}(\tau^{\mathrm{II}}, \eta^{\mathrm{II}}) = s.$$

Second equation of the formulae (39) and equality (40) follow from these equations. One can easily see, that first equations in (45), (46) allow for obtaining the first formula of (39). Applying next (47), one gets the condition (41).

Given the solutions u^{I}, v^{I}, u^{II} and v^{II}, the values of the functional parameters τ^{I} and τ^{II} are found by solving the system of Eqs. (40), (41). The last step is the specialization of the new functions $u^{\mathrm{III}}(x, t)$ and $v^{\mathrm{III}}(x, t)$ attained via the substitution of the expressions of τ^{I} and η^{I} into (39).

We illustrate the usage of the superposition formula (39) for the generation of solutions of the system (33). Note first that the chosen initial solutions

$$u^{\mathrm{I}} = \frac{1}{x}, \qquad v^{\mathrm{I}} = \ln|x| - t + c_3, \qquad u^{\mathrm{II}} = -\frac{\mathrm{e}^{-t}}{c_4\ x + c_5}, \qquad v^{\mathrm{II}} = c_4 \mathrm{e}^t + c_6 \tag{48}$$

do not generate the resulting solutions $u^{\mathrm{III}}, v^{\mathrm{III}}$ of the system (33) via the straightforward application of the superposition algorithm.

For another choice of the initial solutions we get the following cases:

1. $u^{\mathrm{I}} = \dfrac{h}{x},\ v^{\mathrm{I}} = h\ln|x| - th^{-1} + c_3,\ u^{\mathrm{II}} = h,\ v^{\mathrm{II}} = hx + c_4$ generates the implicit solutions

$$\begin{aligned} &h^3 - h^2x\ \ u^{\mathrm{III}} + ((c_3 - c_4 - h)h + \ln|h| - t)\ \ u^{\mathrm{III}} \\ &\qquad\qquad + (h^2-1)\ \ u^{\mathrm{III}}\ln|h - u^{\mathrm{III}}| - h^2\ \ u^{\mathrm{III}}\ln|u^{\mathrm{III}}| = 0, \\ &c_3h - h^2\ln|h| - t - h\ \ v^{\mathrm{III}} + (h^2-1)\ln|v^{\mathrm{III}} - h\ \ x - c_4| \\ &\qquad\qquad + \ln|v^{\mathrm{III}} - h\ \ x - h - c_4| = 0. \end{aligned} \tag{49}$$

2. $u^{\mathrm{I}} = \dfrac{h}{x},\ v^{\mathrm{I}} = h\ln|x| - th^{-1} + c_3,\ u^{\mathrm{II}} = -\frac{\mathrm{e}^{-t}}{c_4\ \ x + c_5},\ v^{\mathrm{II}} = c_4\ \ \mathrm{e}^t + c_6$ generates the implicit solutions

$$\begin{aligned} &-1 + (c_6 - c_3)h - \ln|h| + (h^2-1)\ln|c_4| + h^2t + (c_4h - u^{\mathrm{III}})(c_4x + c_5)\mathrm{e}^t \\ &\qquad + h^2\ln|u^{\mathrm{III}}| - (h^2-1)\ln|u^{\mathrm{III}}\mathrm{e}^t(c_4x + c_5) + 1| = 0, \\ &-\ln|c_4| - c_3h + h^2(\ln|c_4| - \ln|h|) - h^2\ln|c_4x + c_5| + h\ \ v^{\mathrm{III}} \\ &\qquad - (h^2-1)\ln|-v^{\mathrm{III}} + c_4\mathrm{e}^t + c_6| + h^2\ln|h(-v^{\mathrm{III}} + c_4\mathrm{e}^t + c_6) - 1| = 0. \end{aligned} \tag{50}$$

3. $u^{\mathrm{I}} = \dfrac{c_1 + c_2\mathrm{e}^{-t}}{x},\ v^{\mathrm{I}} = c_1\ln|x| - \frac{1}{c_1}\ln|c_1 + c_2\mathrm{e}^{-t}| - \frac{1}{c_1}t + c_3,\ u^{\mathrm{II}} = -\dfrac{\mathrm{e}^{-t}}{c_4\ \ x + c_5},$ $v^{\mathrm{II}} = c_4\mathrm{e}^t + c_6 \to u^{\mathrm{III}} = \mathrm{e}^{w-t},\ v^{\mathrm{III}} = c_1^{-1}(-\mathrm{e}^w(c_4\ \ x + c_5) + c_4\ \ (c_1\mathrm{e}^t + c_2) + c_1\ \ c_6 - 1),$ were w satisfies the equation

$$\begin{aligned} &c_1^2w - x\ \ \mathrm{e}^w(c_4\ \ x + c_5) + c_4\ \ (c_1\mathrm{e}^t + c_2) + c_1(c_6 - c_3) - 1 \\ &\qquad\qquad\qquad + (1 -\ \ c_1^2)\,(\ln|1 + c_4\ \ x(\mathrm{e}^w + c_5)| - \ln|c_4|) = 0. \end{aligned}$$

Here $c_1, \ldots, c_6$ are constants.

3.2. The formula of superposition of solutions for nonlinear transport equation

Integrating the first equation of the linear system (35) and substituting the result

$$q(r,s) = \int p_s(r,s)\ \ dr$$

into the formulae of the transformation (34), we obtain the corresponding nonlocal integro-differential transformation

$$x = p(r,s), \qquad t = \ln\left|\int p_s(r,s)dr\right|, \qquad u(x,t) = r\left(\int p_s(r,s)dr\right)^{-1}. \tag{51}$$

THEOREM 3. *The nonlocal integro-differential transformation* (51) *leaves Eq.* (32) *invariant on the manifold defined by this equation and its appropriate integro-differential consequences.*

To prove this statement we just insert righthand sides of the integro-differential equations

$$\int p_{sss}(r,s)\,dr = \partial_s\left(r^2\partial_r p(r,s)\right),$$

$$\int p_{ss}(r,s)\,dr = r^2\partial_r p(r,s),$$

and

$$p_{ss}(r,s) = \partial_r\left(r^2\partial_r p(r,s)\right)$$

into the result of mapping equation (32) by the transformation (51), which, after some algebraic manipulation, gets zero.

Transformation (51) can be applied for the construction of the nonlocal superposition principle for Eq. (32).

ALGORITHM 1. (General principle of superposition.) *Let* $u^{\mathrm{J}}(\tau^{\mathrm{J}},\eta^{\mathrm{J}})$, $\mathrm{J}=\mathrm{I},\mathrm{II}$ *be two known (initial) solutions of Eq.* (32). *Applying transformations* (51) *in the form*

$$\begin{aligned}\tau^{\mathrm{J}} &= p^{\mathrm{J}}(r,s), \quad \eta^{\mathrm{J}} = \ln\left|\int p_s^{\mathrm{J}}(r,s)\,dr\right|,\\ u^{\mathrm{J}}(\tau^{\mathrm{J}},\eta^{\mathrm{J}}) &= r\left(\int p_s^{\mathrm{J}}(r,s)\,dr\right)^{-1},\end{aligned} \tag{52}$$

and solving (52) *with respect to* $\int p_s^{\mathrm{J}}(r,s)dr$, *we get two possibilities of deriving the corresponding differential equation. The first—differentiating both sides of the equality with respect to* r. *The second—differentiating both sides of the same equality with respect to* s *and then using the integro-differential consequence of Eq.* (32)

$$\int p_{ss}^{\mathrm{J}}dr = r^2 p_r^{\mathrm{J}}. \tag{53}$$

Solving the equations obtained with respect to $p^{\mathrm{I}}(r,s) = P^{\mathrm{I}}(r,s)$ *and* $p^{\mathrm{II}}(r,s) = P^{\mathrm{II}}(r,s)$, *we find solutions containing arbitrary functions* $f^{\mathrm{J}}(r,s)$, *which are specified by Eq.* (37). *Next we apply the linear superposition principle to functions defined above*

$$p(r,s) = P^{\mathrm{I}}(r,s) + P^{\mathrm{II}}(r,s), \tag{54}$$

obtaining this way corresponding differential equation. Application of the inverse transformation

$$r = \mathrm{e}^t u(x,t), \qquad s = \int (u_t(x,t) + u(x,t))\,dx, \qquad p(r,s) = x \tag{55}$$

to Eq. (54) *allows us to get an equation for the integral* $\int u_t(x,t)\ dx$. *Solving it we obtain the relation in the form*

$$\int u_t(x,t)\ dx = H(x,t,u\dots).$$

Again, there are two ways of obtaining the corresponding differential equation on $u(x,t)$. *The first—differentiating both sides of the equality with respect to* x. *The second—differentiating both sides of the same equality with respect to* t *and then using the integro-differential consequence of* (32) *in the form*

$$u^{-2}u_x = \partial_t H(x,t,u\dots) + H(x,t,u\dots). \tag{56}$$

Solving the appropriate differential equation with respect to $u(x,t)$, *we get the nonlocal ansatz for Eq.* (32), *which must be specialized by substituting it into this equation.*

Using the transformations (51), (52) and the inverse ones, then taking advantage of the equalities

$$\begin{aligned} r &= u^{\mathrm{I}}(\tau^{\mathrm{I}},\eta^{\mathrm{I}})e^{\eta^{\mathrm{I}}} = u^{\mathrm{II}}(\tau^{\mathrm{II}},\eta^{\mathrm{II}})e^{\eta^{\mathrm{II}}},\\ s &= \int \left(u_{\eta^{\mathrm{I}}}{}^{\mathrm{I}}(\tau^{\mathrm{I}},\eta^{\mathrm{I}}) + u^{\mathrm{I}}(\tau^{\mathrm{I}},\eta^{\mathrm{I}})\right)\ d\tau^{\mathrm{I}}\\ &= \int \left(u_{\eta^{\mathrm{II}}}{}^{\mathrm{II}}(\tau^{\mathrm{II}},\eta^{\mathrm{II}}) + u^{\mathrm{II}}(\tau^{\mathrm{II}},\eta^{\mathrm{II}})\right)\ d\tau^{\mathrm{II}}, \end{aligned}$$

we find another way of constructing the superposition principle for Eq. (32).

ALGORITHM 2. ("Light" principle of superposition.) *The nonlinear superposition formula of solutions for equation* (32) *has the form:*

$$u^{\mathrm{III}}(x,t) = u^{\mathrm{I}}(\tau^{\mathrm{I}},\eta^{\mathrm{I}})\ e^{(\eta^{\mathrm{I}}-t)}, \tag{57}$$

where the functional parameters $\tau^{\mathrm{I}} = \tau^{\mathrm{I}}(t,x)$ *and* $\eta^{\mathrm{I}} = \eta^{\mathrm{I}}(t,x)$ *are defined by the system*

$$\int \left(\partial_{\eta^{\mathrm{I}}}u^{\mathrm{I}} + u^{\mathrm{I}}\right)\ d\tau^{\mathrm{I}} + s_1(\eta^{\mathrm{I}}) = \int \left(\partial_{\eta^{\mathrm{II}}}u^{\mathrm{II}} + u^{\mathrm{II}}\right)\ d\tau^{\mathrm{II}} + s_2(\eta^{\mathrm{II}}), \tag{58}$$

$$\tau^{\mathrm{II}} = x - \tau^{\mathrm{I}}, \qquad \eta^{\mathrm{II}} = \ln|e^t - e^{\eta^{\mathrm{I}}}|, \tag{59}$$

$$e^{\eta^{\mathrm{I}}}\ u^{\mathrm{I}}(\tau^{\mathrm{I}},\eta^{\mathrm{I}}) = u^{\mathrm{II}}(x-\tau^{\mathrm{I}}, \ln|e^t - e^{\eta^{\mathrm{I}}}|)(e^t - e^{\eta^{\mathrm{I}}}), \tag{60}$$

containing two arbitrary functions of t *which are specified by the equations*

$$\partial_{\eta^{\mathrm{J}}}\left(\int (\ \partial_{\eta^{\mathrm{J}}}u^{\mathrm{J}} + u^{\mathrm{J}})\ d\tau^{\mathrm{J}} + s_J(\eta^{\mathrm{J}})\right) - \frac{\partial_{\tau^{\mathrm{J}}}u^{\mathrm{J}}}{(u^{\mathrm{J}})^2} = 0, \qquad \mathrm{J} = \mathrm{I}, \mathrm{II}. \tag{61}$$

Note that both of these algorithms are useful for generating solutions of Eq. (32). Applying the superposition formula (57) to the initial constant solutions $u^{\mathrm{I}} = k$, $u^{\mathrm{II}} =$

h, we get a new solution

$$u^{\mathrm{III}} = \frac{kh}{k+h}. \tag{62}$$

The same initial solutions produce less trivial solutions via application of the general superposition principle (Algorithm 1):

$$u^{\mathrm{III}} = \frac{c_1 + c_2}{\mathrm{e}^t (x + c_3)},$$
$$u^{\mathrm{III}} = \frac{kh}{2(k+h)^2 \ \mathrm{e}^t} \left(G \pm \sqrt{G^2 - 4(k+h)^2 c_1^2} \right),$$

where

$$G = kh(x + c_3)c_1 + (k+h) \ \mathrm{e}^t,$$

c_i are arbitrary constants. For $c_1 = 0$ this solution turns into (62).

If the construction of new solutions via the Algorithm 1 is too complicated, then the superposition formula (57) may be useful for the construction of more simple solutions.

The usage of the superposition formula (57) for generating solutions of equation (32) allows one to get a new solution from the initial solutions without any knowledge of appropriate potential function. We illustrate the application of this superposition formula by means of several examples.

1. $u^{\mathrm{I}} = k, \quad u^{\mathrm{II}} = -\dfrac{\mathrm{e}^{-t}}{c_4 \ x + c_5} \to u^{\mathrm{III}} = k \ \mathrm{e}^{-t} \ \dfrac{w \pm \sqrt{w^2 - 4c_4^2}}{2c_4^2}, \quad w = c_4^2 \mathrm{e}^t - k(c_4 x + c_5) - c_4 c_6.$

2. $u^{\mathrm{I}} = \dfrac{1}{x}, \quad u^{\mathrm{II}} = -\dfrac{\mathrm{e}^{-t}}{c_4 x + c_5} \to u^{\mathrm{III}} = \dfrac{\mathrm{e}^{-t}(-1 + c_4 \mathrm{e}^g)}{c_4 \ x + c_5}$, were g is implicitly defined by the equation

$$-c_3 - c_4 \ (\mathrm{e}^g - \mathrm{e}^t) + c_6 + \ln \left| \frac{-1 + c_4 \ \mathrm{e}^g}{c_4 \ x + c_5} \right| = 0. \tag{63}$$

There exists a nonlocal integro-differential transformation inverse to (51), connecting equations (37) and (32):

$$r = \mathrm{e}^t u(x,t), \qquad s = \int (\partial_t \ u(x,t) + u(x,t))\, dx, \qquad p(r,s) = x, \tag{64}$$

and the similar transformation

$$r = \mathrm{e}^t \ u(x,t), \qquad s = \int (\partial_t \ u(x,t) + u(x,t))\, dx, \qquad q(r,s) = \mathrm{e}^t, \tag{65}$$

which connects Eq. (36) with (32). So, the exact solutions

$$q(r,s) = c_2\sqrt{r}\left(c_3 r^{1/2\ \sqrt{1+4c_1}} + c_4 r^{-1/2\ \sqrt{1+4c_1}}\right)\mathrm{e}^{\sqrt{c_1}s},$$

$$p(r,s) = c_7 c_5 r^{1/2\ \sqrt{1+4c_1}-1/2}\mathrm{e}^{-\sqrt{c_1}s},$$

of the linear equations (36) and (37) allow us to construct corresponding solutions of nonlinear Eq. (32):

$$(t+\ln|u|)\sqrt{1+4c_1} + \ln\left|-\frac{c_3\left(-2xu\sqrt{c_1}\pm(\sqrt{1+4c_1}+1)+2hu\sqrt{c_1}\right)}{c_4\left(-2xu\sqrt{c_1}\pm(\sqrt{1+4c_1}-1)+2hu\sqrt{c_1}\right)}\right| = 0,$$

$$u = -\frac{(1+\sqrt{1+4c_1})}{2\ x\sqrt{c_1}}\left(\mathrm{e}^{\frac{(k-t)(-1-2c_1+\sqrt{1+4c_1})-\ln|x|(\sqrt{1+4c_1}+1)}{\left(-1-2c_1+\sqrt{1+4c_1}\right)}}x^{\frac{-2}{-1+\sqrt{1+4c_1}}}-1\right).$$

Here c_i, k and h are arbitrary constants.

4. The formulae generating solutions of an equation invariant with respect to nonlocal transformations

The existence of transformation (34) enables one to map any Lie symmetry generator of the system (35) to the corresponding Lie symmetry generator of the system (33). It is easy to check that the linear system admits the generator

$$Y = rs\ \partial_r + \ln|r|\partial_s - \left(\frac{psr+q}{2r} - P\right)\partial_p + \left(\frac{qs-pr}{2} + Q\right)\partial_q, \tag{66}$$

where $P = P(r,s)$, $Q = Q(r,s)$ are arbitrary solutions of (35).

Applying to this generator transformation (34), we get an equivalent Lie symmetry generator of the system (33),

$$X = \Big(-\frac{xuv+1}{2u} + P(\mathrm{e}^t u, v)\Big)\partial_x + \Big(\frac{(v-xu)}{2} + \mathrm{e}^{-t}Q(\mathrm{e}^t u, v)\Big)\partial_t$$
$$+ u\Big(\frac{(v+xu)}{2} - \mathrm{e}^{-t}Q(\mathrm{e}^t u, v)\Big)\partial_u + (t+\ln|u|)\partial_v, \tag{67}$$

determining the potential symmetry of the initial equation (32).

Just in the similar way other generators of the Lie invariance algebra of the linear system (35) can be obtained by adding the $P(r,s)\ \partial_u$, $Q(r,s)\ \partial_v$ components to any generator of the corresponding finite Lie invariance algebra of this system. Any such symmetry can be transformed by (34) into the equivalent Lie symmetry generator of the nonlinear system (33), and consequently admits its presentation in the form of a nonlocal formula for generating solutions.

Consider, for instance, the Lie symmetry generator

$$X_6 = v\ \partial_x + u\ \partial_t - u^2\ \partial_u, \tag{68}$$

admitted by the potential system (33). Obviously, it determines the corresponding potential symmetry for (32). The finite group transformation corresponding to X_6 is as follows:

$$x^{\mathrm{I}} = x + \varepsilon v(x,t), \quad t^{\mathrm{I}} = \ln|u(x,t)\varepsilon + 1| + t,$$
$$u^{\mathrm{I}}(x^{\mathrm{I}}, t^{\mathrm{I}}) = \frac{u(x,\ t)}{u(x,\ t)\varepsilon + 1}, \quad v^{\mathrm{I}}(x^{\mathrm{I}}, t^{\mathrm{I}}) = v(x,\ t). \tag{69}$$

According to these formulae, having the known solution $u^{\mathrm{I}}(x^{\mathrm{I}}, t^{\mathrm{I}}), v^{\mathrm{I}}(x^{\mathrm{I}}, t^{\mathrm{I}})$, we can find a new solution $u(x,t), v(x,t)$ of the system (33).

Such additional symmetry allows for the generation of new solutions to Eq. (32). Integrating the first equation of the system (33), we obtain the corresponding nonlocal integro-differential transformation

$$v(x,t) = \int (u_t(x,t) + u(x,t))\ dx.$$

Substituting this into (69), we exclude the variable $v(x,t)$ from these equations. Following this way, one can obtain the formulae for the generation of new solutions of Eq. (32) from the known ones.

THEOREM 4. *If* $u^{\mathrm{I}}(\tau,t)$ *(respectively* $u(x,t)$*) is a known (initial) solution and* $u(x,t)$ *(resp.* $u^{\mathrm{I}}(\tau,t)$*) is a new (constructed) solution of Eq.* (32) *then the generation of new solution by the transformation* (51) *is carried out by the formula*

$$u^{\mathrm{I}}(x^{\mathrm{I}}, t^{\mathrm{I}}) = \frac{u(x,t)}{u(x,t)\varepsilon + 1}, \qquad x^{\mathrm{I}} = x + \varepsilon \int (u_t(x,t) + u(x,t))\ dx,$$
$$t^{\mathrm{I}} = \ln|u(x,t)\varepsilon + 1| + t. \tag{70}$$

There are two ways of implementing the above formulae. Assume that we have an initial solution $u(x,t)$. Substituting it into (70) and solving the second and the third equations with respect to the variables x and t, we can express them as functions of $x^{\mathrm{I}}, t^{\mathrm{I}}$. Substituting the expressions $x = x(x^{\mathrm{I}}, t^{\mathrm{I}})$ and $t = t(x^{\mathrm{I}}, t^{\mathrm{I}})$ into the known function $u(x,t)$ in the first equation of (70), we obtain a new solution $u^{\mathrm{I}}(\tau,t)$ of the initial equation. For instance

1. $u(x,t) = \mathrm{e}^{-t} \rightarrow u^{\mathrm{I}} = \mathrm{e}^{-t^{\mathrm{I}}}$,

2. $u(x,t) = \pm\dfrac{\mathrm{e}^{-t}}{c_1\ x + c_2} \rightarrow u^{\mathrm{I}} = \pm\dfrac{\mathrm{e}^{-t^{\mathrm{I}}}}{c_1\ (x^{\mathrm{I}} - \varepsilon\, c_3) + c_2}$.

Another algorithm based on Theorem 4 can be realized in the following way: we substitute x^{I} and t^{I} determined by (70) into the initial solution $u^{\mathrm{I}}(x^{\mathrm{I}}, t^{\mathrm{I}})$ as follows

$$u^{\mathrm{I}}\left(x + \varepsilon \int (u_t(x,t) + u(x,t))\ dx, \ln|u(x,t)\varepsilon + 1| + t\right) = \frac{u(x,t)}{u(x,t)\varepsilon + 1}. \tag{71}$$

In order to find a new solution $u(x,t)$, it is necessary to solve Eq. (71) with respect to $\int (u_t(x,t) + u(x,t))\, dx$ and differentiate both sides of the obtained equality with respect to x. Solving the partial differential equation with respect to $u(x,t)$, we can find for Eq. (32) a non-Lie ansatz.

Applying the above algorithm to the solutions $u^{\mathrm{I}} = \mathrm{e}^{-t^{\mathrm{I}}}$ and $u^{\mathrm{I}} = -\mathrm{e}^{-t^{\mathrm{I}}}(c_1 x^{\mathrm{I}} + c_2)^{-1}$ considered as the initial ones, we obtain, respectively, the following exact solutions of Eq. (32):

1. $u(x,t) = \mathrm{e}^{-t}$, so, this solution is invariant with respect to the proposed algorithm.

2. The second initial solution leads to the pair of equations with respect to $u(x,t)$,

$$\left(\mathrm{e}^{-t}\varepsilon - c_1\ c_2(u\ \ \varepsilon + 1)\right) u + c_1 x u(u\ \ \varepsilon + 1) + \mathrm{e}^{-t} = 0,$$

$$\left(c_1^2 \varepsilon(u\varepsilon + 1)^2 \mathrm{e}^{t} - c_1 c_2(u\varepsilon + 1) + \varepsilon \mathrm{e}^{-t}\right) u + c_1 x u(u\varepsilon + 1) + \mathrm{e}^{-t} = 0.$$

This gives us two extra solutions:

$$u = -\frac{1}{\varepsilon}, \qquad u = -\frac{\mathrm{e}^{-t}}{c_1(x - c_2)},$$

and

$$u = -\frac{1}{\varepsilon}, \qquad u = -\mathrm{e}^{-t}\frac{w \pm \sqrt{w^2 - 4\varepsilon^2}}{2c_1\ \ \varepsilon^2}, \qquad w = c_1\, \varepsilon \mathrm{e}^{t} + x - c_2.$$

Here all c_i are arbitrary constants, ε is the group parameter.

5. The generation of solutions via the nonlocal symmetries of linear equation

Another algorithm allowing to generate solutions for a linearizable equation may be constructed by combining a symmetry of the corresponding linear equation and the nonlocal linearizing transformation [6–10].

Assume that L is a linear differential equation, F is a nonlinear one, and τ is a nonlocal transformation that maps equation F into L. If L admits a Lie symmetry generator X (being a vector field on the space of independent and dependent variables) then the corresponding differential operator Q acting in the space of unknown functions of L maps any solution z^{I} of L to another solution $z^{\mathrm{II}} = Qz^{\mathrm{I}}$. So, nonlocal mapping τ allows to construct a new solution $u^{\mathrm{II}} = \tau^{-1} z^{\mathrm{II}}$ of F from its known solution $u^{\mathrm{I}} = \tau^{-1} z^{\mathrm{I}}$. Here τ^{-1} is treated in a certain way. Therefore, in the case of direct application of transformation τ for the generation of new solutions of F we should perform the following series of transformations: $u^{\mathrm{I}} \to z^{\mathrm{I}} \to z^{\mathrm{II}} \to u^{\mathrm{II}}$.

It is possible to derive a formula generating of new solutions of F without involving solutions of the linear equation. This way is more preferable since it allows to avoid unnecessary (as a rule cumbersome) calculations and produce solutions in one step.

Let us return to the linear system (35). There exists the point transformation

$$t = s, \qquad x = \ln|r|, \qquad w(x,t) = q\ \ r^{-\frac{1}{2}}, \tag{72}$$

connecting equation (36) with the well-known Klein–Gordon equation

$$w_{tt} - w_{xx} + w/4 = 0. \tag{73}$$

The Lie algebra admitted by Eq. (73) consists of the following operators:

$$X_1 = \partial_t, \qquad X_2 = \partial_x, \qquad X_3 = w\ \partial_w, \qquad X_4 = t\ \partial_x + x\ \partial_t, \qquad X_5 = \beta(x,\ t)\ \partial_x, \tag{74}$$

where the function $\beta(x,\ t)$ runs through the set of solutions of Eq. (73). We use the linear combination $k_1X_1 + k_2X_2$ and the generator X_4 for the construction of a new solution $w(x,t) = w^{\mathrm{II}}(x,t)$ of equation (73) from the known one $w^{\mathrm{I}}(x,t)$, using the relations

$$w^{\mathrm{I}}(x,t) = k_1\partial_x w^{\mathrm{II}}(x,t) + k_2\partial_t w^{\mathrm{II}}(x,t) \tag{75}$$

or

$$w^{\mathrm{I}}(x,t) = t\partial_x w^{\mathrm{II}}(x,t) + x\partial_t w^{\mathrm{II}}(x,t). \tag{76}$$

5.1. Generation of solutions

Nonlocal symmetries of the Klein–Gordon equation allow us to construct formulae generating solutions for the initial equation (32). The point transformation (72) maps the Klein–Gordon equation into the linear equation (36). So, application of this transformation to the nonlocal symmetries (75) or (76) of the Klein–Gordon equation generates the corresponding nonlocal symmetries of (36). As the nonlocal integro-differential transformation (65) maps equation (36) into the initial nonlinear equation (32), the derivation of the formulae generating its solutions is possible.

THEOREM 5. *The formula for the generating of solutions of Eq.* (32), *corresponding to the generator a)* $k_1\ X_1 + k_2\ X_2$, *b)*X_4 *belonging to the invariance algebra* (74), *is as follows,*

$$\mathrm{e}^t = w\left\{t,\ \int [u_t(x,t) + u(x,t)]\mathrm{d}x\right\}, \tag{77}$$

where w *is defined by the procedure presented below.*

1. *Given a solution* $u^{\mathrm{I}}(x^{\mathrm{I}}, t^{\mathrm{I}})$ *of Eq. (32), we solve the overdetermined system of PDEs*

$$\frac{r}{Q(r,s)} = u^{\mathrm{I}}\left(\int \partial_r Q(r,s)\ ds, \ln|Q(r,s)|\right), \qquad Q(r,s)_{ss} - r^2\ Q(r,s)_{rr} = 0$$

with respect to the function $Q = Q(r,s)$.

2. *Then the equations of the overdetermined system*

$$\text{a)}\quad k_1 w_X + k_2 w_T = Q(\mathrm{e}^{-X}, T), \qquad w_{TT} - w_{XX} + w/4 = 0,$$

or

$$\text{b)}\quad T w_X + X w_T = Q(\mathrm{e}^{-X}, T), \qquad w_{TT} - w_{XX} + w/4 = 0,$$

define the function $w(X,\ T)$.

3. *Solving Eq.* (77) *with respect to* $\int (u_t(x,t) + u(x,t))\,dx$ *and differentiating it with respect to* x*, we find the resulting PDE, which, after integration, gives the appropriate ansatz for Eq.* (32).

Applying Theorem 5 to the initial solutions $u = c = \text{const}$ and $u = \frac{1}{x}$, we obtain in both cases the same exact solution of Eq. (32),

$$u = -\frac{\mathrm{e}^{-t}}{c_1 x + c_2},$$

where c_1, c_2 are arbitrary constants. And further

$$u(x,t) = -\frac{\mathrm{e}^{-t}}{c_5\ x + c_6} \rightarrow$$
$$u^{\mathrm{I}} = \exp\left(-\frac{1}{2}\,\mathrm{LambertW}\left[-c_1^{-2}(k-x)^2 \exp\left(\frac{8\,c_1\mathrm{e}^t - c_2^2}{4c_1^2}\right) - c_1^{-1}(c_1 t - \mathrm{e}^t)\right]\right).$$

Note that one can construct a new solution $w(x,t) = w^{\mathrm{II}}(x,t)$ of Eq. (73) from the known one $w^{\mathrm{I}}(x,t)$ using the relations

$$w^{\mathrm{II}}(x,t) = k_1 \partial_x w^{\mathrm{I}}(x,t) + k_2 \partial_t w^{\mathrm{I}}(x,t) \tag{78}$$

or

$$w^{\mathrm{II}}(x,t) = t\ \partial_x w^{\mathrm{I}}(x,t) + x\ \partial_t w^{\mathrm{I}}(x,t) \tag{79}$$

and substituting the function $w^{\mathrm{I}}(x,t)$ obtained before into the appropriate equality (note that the second PDE from item 2 of Theorem 5 can be omitted this way).

So using the operators X_1 and X_2 belonging to the algebra (74) we get within the given approach the following cases:

1. $u(x,t) = -\dfrac{\mathrm{e}^{-t}}{c_5\ x + c_6} \rightarrow u^{\mathrm{I}} = \dfrac{\mathrm{e}^{-t}\left(c_3^2 \pm \sqrt{c_3^4 - 4\,x^2(\mathrm{e}^t + 2c_2)^2}\right)}{2\,x^2}.$
2. $u(x,t) = h \quad \rightarrow \quad u^{\mathrm{I}} = \mathrm{e}^{-t}\sqrt{\dfrac{-h^2(\mathrm{e}^{-t} - c_2)^2}{\mathrm{LambertW}\left[-h^2(\mathrm{e}^{-t} - c_2)^2 \mathrm{e}^{2h(c_3 - x)}\right]}},$

where $c_1, \ldots, c_6$ and h are constants.

More solutions can be constructed by additional application of the Lie symmetry transformations or any other formula generating solutions.

6. Conclusion

Thus, it is shown in this paper that the potential symmetry of the generalized transport equation (1) in the case $n = -2$ is infinite and this equation can be linearized. Basing on this property, we propose new algorithms for superposition and generation of solutions. The employment of point and nonlocal transformations allows for construction of a number of new exact solutions. Some of them are obtained in an explicit form, while the other have parametrical representations with functional parameters given in implicit form. Their relations between additional Lie symmetry, potential and nonlocal symmetries for pairs of differential equations are established by means of appropriate nonlocal transformation. All solutions found can be naturally extended to multi-parameter families of solutions by means of Lie symmetry transformations or any other formula enabling to generate new solutions. Note that the results obtained in the present paper for equation (32) can be extended to similar classes of equations using appropriate point transformations to the formulae employed to generate of new solutions. We hope that the methods presented in this paper can be applied to different classes of nonlinear telegraph-type or transport equations.

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